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LINEAR ABSTRACT CAUCHY PROBLEM ASSOCIATED WITH AN EXPONENTIALLY BOUNDED C-SEMIGROUP IN A BANACH SPACE*

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1. Introduction

The purpose of this paper is to consider the inhomogeneous initial value problem

$$(1)\qquad \begin{cases} \dfrac{d}{dt}u(t)=zu(t)+f(t) \\ u(0)=x \end{cases}$$

in a Banach space X, where Z is the generator of an exponentially bounded C-semigroup in $X, f(t)\ :\ [0,\ T)\rightarrow X$ and $x\in X$. Davies-Pang[1] showed the corresponding homogeneous equation, that is, the equation with $f(t)\equiv 0$, has a unique solution depending continuously on the initial value $x\in CD(Z)$ in the C^{-1}-graph norm on $CD(Z)$ when $T=\infty$.

2. Preliminaries

We recall here definitions and characterizations for an exponentially bounded C-semigroup given by Davies-Pang [1]. Besides [1], one can refere to [2], [3], [4], [5] and [6] for an exponentially bounded C-semigroup in a Banach space.

Let X be a Banach space and let C be an injective bounded linear operator from X into itself with dense range $R(C)$ in X. We say that $\{S(t)\,|\,t\geqq 0\}$ is an exponentially bounded C-semigroup in X if $\{S(t)\,|\,t\geqq 0\}$ is a strongly continuous family of bounded linear operator from X into itself satisfying

(a_1) $S(0)=C$,

(a_2) $S(t+s)C=S(t)S(s)$ for $t, s\geqq 0$,

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(a_3) there exists constants $M \geqq 0$ and $a \geqq 0$ such that $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$.

Letting $s \to 0+$ in (a_2), we have $S(t)C = CS(t)$, $S(t)x \in R(C)$ and $C^{-1}S(t)x = S(t)C^{-1}x$ for $x \in R(C)$.

Let $T(t)$ be the closed linear operator defined by

$$T(t)x = C^{-1}S(t)x \tag{2}$$

for $x \in D(T(t)) = \{x \in X \mid S(t)x \in R(C)\}$. Then $R(C) \subset D(T(t))$ and

(b_1) $T(0)x = x$ for $x \in X$,

(b_2) $T(t+s)x = T(t)T(s)x$ for $x \in R(C^2)$,

(b_3) $T(t)x$ is continuous in $t \geqq 0$ for $x \in R(C^2)$.

Let $\lambda > a$. We define the bounded linear operator L_λ from X into itself by

$$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t) x dt$$

for $x \in X$. The operator L_λ with $\lambda > a$ will be called the C-resolvent of $\{S(t) \mid t \geqq 0\}$. L_λ is injective and

$$(\lambda - L_\lambda^{-1}C)x = (\mu - L_\mu^{-1}C)x$$

for $x \in X$ with $Cx \in R(L_\lambda) = R(L_\mu)$ for $\lambda, \mu > a$. Therefore the closed linear operator Z defined by

$$Zx = (\lambda - L_\lambda^{-1}C)x$$

for $x \in D(Z) = \{x \in X \mid Cx \in R(L_\lambda)\}$, is independent of $\lambda > a$. The operator Z will be called the generator of $\{S(t) \mid t \geqq 0\}$ with $\|S(t)\| \leqq Me^{at}$. We have

$$L_\lambda^{-1}Cx = CL_\lambda^{-1}x \text{ and } ZCx = CZx$$

for $x \in CD(Z)$. The generator Z is densely defined in X and $S(t)x \in D(Z)$,

$$\frac{d}{dt} S(t)x = ZS(t)x = S(t)Zx \tag{3}$$

for $x \in D(Z)$. Furthermore $T(t)x \in D(Z)$,

$$\frac{d}{dt} T(t)x = ZT(t)x = T(t)Zx \tag{4}$$

for $x \in CD(Z)$.

We define the linear operator G by

$$Gx = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x)$$

for $x \in D(G) = \{x \in R(C) \mid \lim_{t \to 0+} \frac{1}{t}(T(t)x - x) \text{ exists}\}$. The operator G is also densely defined in X and $G \subset Z$. Furthermore $C^4 D(Z) \subset D(G)$ and $Gx = Zx$ for $x \in C^4 D(Z)$

Here, $C^0 = I, C^k = CC^{k-1}$ and $C^k D(Z) = \{C^k x \in X \mid x \in D(Z)\}$ for $k = 1, 2, \cdots$.

3. Abstract Cauchy problem

Throughout this section, let $\{S(t) \mid t \geqq 0\}$ be an exponentially bounded C-semigroup with $\|S(t)\| \leqq Me^{at}$ in a Banach space X and let Z be its generator. Let $T(t) = C^{-1}S(t)$ be the operator defined by (2) and $T < \infty$.

Definition 1. A function $u : [0, T) \to X$ is called a solution of (1) on $[0, T)$ if the following conditions $(c_1) - (c_4)$ are satisfied:

(c_1) u is continuous on $[0, T)$,

(c_2) u is continuously differentiable on $[0, T)$,

(c_3) $u(t) \in D(Z)$ for $t \in (0, T)$,

(c_4) (1) is satisfied.

We give some properties of a solution of (1) on $[0, T)$.

Proposition 2. *Let $f(t) \in R(C)$ for $t \in [0, T)$ with $C^{-1}f \in L^1(0, T; X)$. If u is a solution of* (1) *on $[0, T)$ for $x \in CD(Z)$, then*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds \tag{6}$$

for $t \in [0, T)$.

Proof. The X-valued function $S(t-s)u(s)$ is differentiable for $0 < s < t$ and from (3)

$$\begin{aligned} \frac{d}{ds}S(t-s)u(s) &= -ZS(t-s)u(s) + S(t-s)\frac{d}{ds}u(s) \\ &= -ZS(t-s)u(s) + S(t-s)Zu(s) + S(t-s)f(s) \\ &= S(t-s)f(s). \end{aligned} \tag{7}$$

Since $f \in L^1(0, T; X)$, $S(t-s)f(s)$ is integrable and integrating (7) from 0 to t yields

$$Cu(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{8}$$

Since $S(t-s)f(s) \in L^1(0, T; X)$ and

$$C^{-1}S(t-s)f(s) = S(t-s)C^{-1}f(s) \in L^1(0, T; X),$$

it follows from (8) that

$$\begin{aligned} u(t) &= C^{-1}S(t)x + \int_0^t C^{-1}S(t-s)f(s)\,ds \\ &= T(t)x + \int_0^t T(t-s)f(s)\,ds. \end{aligned}$$

PROPOSITION 3. *Let $f(t) \in R(C)$ for $t \in [0, T)$ with $C^{-1}f \in L^1(0, T; X)$ and let u be a solution of* (1) *on* $[0, T)$ *for* $x \in C^2D(Z)$. *Put*

$$v(t) = \int_0^t T(t-s)f(s)\,ds \tag{9}$$

for $t \in [0, T)$. *Then*

(i) *v is continuously differentiable on* $(0, T)$,

(ii) $v(t) \in D(Z)$ *for* $t \in (0, T)$ *and if f' is continuous on* $(0, T)$, $Zv(t)$ *is continuous on* $(0, T)$.

Proof. Let $x \in C^2D(Z)$. (i) From (6) and (9), $v(t) = u(t) - T(t)x$. Thus v is continuous on $[0, T)$ and differentiable on $(0, T)$. From (4) and Definition 1,

$$\begin{aligned} \frac{d}{dt}v(t) &= \frac{d}{dt}u(t) - \frac{d}{dt}T(t)x \\ &= \frac{d}{dt}u(t) - T(t)Zx \end{aligned}$$

is continuous in $t \in (0, T)$. Thus (i) follows.

(ii) From (4) and Definition 1,

$$v(t) = u(t) - T(t)x \in D(Z)$$

for $t \in (0, T)$ and

$$\begin{aligned} Zv(t) &= Zu(t) - ZT(t)x \\ &= \frac{d}{dt}u(t) - f(t) - T(t)Zx \end{aligned}$$

is continuous in $t \in (0, T)$.

Now we consider the existence of solutions of (1) on $[0, T)$.

THEOREM 4. *Let $f(t) \in R(C)$ for $t \in [0, T)$ with $C^{-1}f \in L^1(0, T; X)$. Let f be continuous on $[0, T)$ and put*

$$v(t) = \int_0^t T(t-s)f(s)\,ds$$

for $t \in [0, T)$.

(i) *If v is continuously differentiable on $(0, T)$ with $v(t) \in R(C)$ for $t \in [0, T)$, then u defined by* (6) *is a unique solution of* (1) *on $[0, T)$ for $x \in C^2 D(Z)$.*

(ii) *If $v(t) \in C^4 D(Z)$ and $Zv(t)$ is continuous in $t \in (0, T)$, then u defined by* (6) *is a unique solution of* (1) *on $[0, T)$ for $x \in C^2 D(Z)$.*

Proof. *Let $x \in C^2 D(Z)$.* The uniqueness follows from Proposition 2. (i) Since $T(t-s)f(s) \in L^1(0, T;\ X)$, v is continuous on $[0, T)$ and thus $u(t) = T(t)x + v(t)$ is continuous in $t \in [0, T)$. From assumption and (4),

$$\frac{d}{dt}u(t) = \frac{d}{dt}T(t)x + \frac{d}{dt}v(t)$$
$$= T(t)Zx + \frac{d}{dt}v(t)$$

is continuous in $t \in (0, T)$. Thus u is continuously differentiable on $(0, T)$. From the differentiability of $v(t)$ with $v(t) \in R(C)$, we have

$$\text{(10)} \quad \frac{1}{\tau}(T(\tau) - I)v(t) = \frac{1}{\tau}(v(t+\tau) - v(t)) - \frac{1}{\tau}\int_{\tau}^{t+\tau} T(t+\tau-s)f(s)\,ds$$

for $\tau \in (0, T-t)$ with $t \in (0, T)$ and thus

$$\lim_{\tau \to 0+} \frac{1}{\tau}(T(\tau) - I)v(t) = \frac{d}{dt}v(t) - f(t).$$

Hence $v(t) \in D(G)$ antd $Gv(t) = \frac{d}{dt}v(t) - f(t)$ for $t \in (0, T)$. Since $G \subset Z$, $v(t) \in D(Z)$ and $Gv(t) = Zv(t)$ for $t \in (0, T)$. Thus

$$\text{(11)} \qquad \frac{d}{dt}v(t) = Zv(t) + f(t)$$

for $t \in (0, T)$. From (4) and (11), $u(t) = T(t)x + v(t) \in D(Z)$ and

$$\frac{d}{dt}u(t) = \frac{d}{dt}T(t)x + \frac{d}{dt}v(t)$$
$$= ZT(t)x + Zv(t) + f(t)$$
$$= Z(T(t)x + v(t)) + f(t)$$
$$= Zu(t) + f(t)$$

for $t \in (0, T)$ and $u(0) = x$. Thus u is a solution of (1) on $[0, T)$.

(ii) From (5), $v(t) \in C^4 D(Z) \subset D(G) \subset D(Z)$ and $Gv(t) = Zv(t)$ for $t \in (0, T)$. Since f is continuous on $[0, T)$, from (10),

$$\frac{d^+}{dt}v(t) = Gv(t) + f(t)$$
$$= Zv(t) + f(t)$$

is continuous in $t \in (0, T)$. Thus v is continuously dfferentiable on $(0, T)$ and

$$\frac{d}{dt}v(t) = Zv(t) + f(t)$$

for $t \in (0, T)$. As in the proof of (1), u is a solution of (1) on $[0, T)$.

COROLLARY 5. *Let $f(t) \in R(C^2)$ for $t \in [0, T)$ and let $C^{-1}f(t)$ be continuously differentiable for $t \in [0, T)$ with $f'(t) \in R(C)$ for $t \in [0, T)$. Then u defined by (6) is a unique solution of (1) on $[0, T)$ for $x \in C^2 D(Z)$.*

Proof. Let $x \in C^2 D(Z)$. From the assumptions, $C^{-1}f \in L^1(0, T; X)$ and $f(t)$ is continuously differentiable for $t \in [0, T)$. Since $f'(t) \in R(C)$ and $C^{-1}f'(t) = (C^{-1}f(t))'$ for $t \in [0, T)$, $C^{-1}f'(t)$ is continuous for $t \in [0, T)$. Thus

$$v(t) = \int_0^t T(t-s)f(s)\,ds = \int_0^t T(s)f(t-s)\,ds \tag{12}$$

is continuous and differentiable for $t \in [0, T)$. From (12),

$$\begin{aligned} v'(t) &= T(t)f(0) + \int_0^t T(s)f'(t-s)\,ds \\ &= T(t)f(0) + \int_0^t T(t-s)f'(s)\,ds \end{aligned}$$

is continuous for $t \in [0, T)$. Thus $v(t)$ is continuously differentiable for $t \in [0, T)$. From $f(t) \in R(C^2)$ and (12), $v(t) \in R(C)$ for $t \in [0, T)$. The result therefore from Theorm 4, (i).

COROLLARY 6. *Let $f(t) \in C^5 D(Z)$ for $t \in [0, T)$ and let $C^{-1}f \in L^1(0, T; X)$ be continuous on $[0, T)$. If $C^{-1}Zf \in L^1(0, T; X)$, then u defined by (6) is a unique solution of (1) on $[0, T)$ for $x \in C^2 D(Z)$.*

Proof. From $f(t) \in C^5 D(Z)$, $T(t-s)f(s) \in D(Z)$ and

$$v(t) = \int_0^t T(t-s)f(s)\,ds \in C^4 D(Z)$$

for $t \in [0, T)$. Since $C^{-1}Zf \in L^1(0, T; X)$, $T(t-s)Zf(s)$ is integrable and

$$Zv(t) = Z\int_0^t T(t-s)f(s)\,ds$$

$$=\int_0^t ZT(t-s)f(s)\,ds$$

$$=\int_0^t T(t-s)Zf(s)\,ds$$

is continuous for $t\in[0, T)$. The result follows from Theorem 4, (ii).

THEOREM 7. *Let $f(t)\in R(C^2)$ for $t\in[0, T)$ with $C^{-2}f\in L^1(0, T;X)$. Then for every $T'<T$, u defined by* (6) *is the uniform limit of solutions of* (1) *on* $[0, T')$ *for* $x\in C^2D(Z)$.

Proof. Let $x_n\in C^2D(Z)$ such that $x_n\to x$ in the C^{-1}-graph norm. Let $g_n\in C^1([0, T'];X)$ satisfying $g_n\to C^{-2}f$ in $L^1(0, T';X)$. Put $f_n=C^2g_n$. Then $f_n(t)\in R(C^2)$ for $t\in[0, T')$ and $C^{-2}f_n\in C^1([0, T'];X)$. Thus $C^{-1}f_n\in C^1([0, T'];X)$, $f_n'(t)=C(C^{-1}f_n(t))'\in R(C)$ for $t\in[0, T')$ and $C^{-1}f_n\to C^{-1}f$ in $L^1(0, T';X)$. From Corollary 5, the equation

$$\begin{cases} \dfrac{d}{dt}u_n(t)=Zu_n(t)+f_n(t) \\ u_n(0)=X_n \end{cases}$$

has a unique solution u_n on $[0, T')$ and

$$u_n(t)=T(t)x_n+\int_0^t T(t-s)f_n(s)\,ds. \tag{13}$$

From (6) and (13),

$$\|u_n(t)-u(t)\|\leqq Me^{at}(\|C^{-1}x_n-C^{-1}x\|+\int_0^t\|C^{-1}f_n(s)-C^{-1}f(s)\|ds)$$

and the result follows.

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