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AN ACCELERATING MONOTONE ITERATIVE TECHNIQUE FOR NONLINEAR BOUNDARY VALUE PROBLEMS*

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1. Introduction

Consider the problem of the type

(1) $$Lx+N(x)=0,$$

where L is an $n\times n$ linear matrix, $N:\mathrm{Dom}(N)\subset R^n\to R^n$ a nonlinear transformation. Such problems arise for example as finite difference approximations to nonlinear differential equations of the type

(2) $$\Delta u+f(u)=g(x, y, u)$$

or

(3) $$u_t-\Delta u=g(x, y, u)$$

with boundary conditions.

The study of problem (1) by a monotone method has been extensively studied in [7] as a generalization of the ideas in [4]. The problem has also been studied in [5] even L is singular. Numerical computation has been applied to (1) by [6, 8, 9, 10].

In this paper, we study a method which speeds up the convergence of monotone iterations and give some numerical examples of the types (2) and (3). In [2] this method has been applied to a system of hyperbolic differential equations with initial and boundary conditions.

2. An accelerating monotone method

We consider the nonlinear problem (1) R^n in the form

(4) $$F(x):=Lx+N(x)=0.$$

In order to investigate the existence and approximations of solutions of this problem, we state the theorem of convergence of monotone iterative sequences due to [5]. We call this monotone method the usual monotone method.

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THEOREM 1. *Let* $F: Dom(F) \subset R^n \to R^n$ *be continuous and let there exist* $x^0, y^0 \in Dom(F)$ *such that*

$$x^0 \leq y^0, \quad [x^0, y^0] \subset Dom(F), \quad F(x^0) \leq 0 \leq F(y^0),$$

where $[x^0, y^0] = \{x \in R^n \mid x^0 \leq x \leq y^0\}$. *Further let there exist a nonsingular M-matrix A such that*

$$F(y) - F(x) \leq A(y-x) \text{ for all } x^0 \leq x \leq y \leq y^0.$$

Then the sequences $\{x^k\}$ *and* $\{y^k\}$ *defined by*

$$y^{k+1} = y^k - A^{-1}F(y^k) \tag{5}$$

and if A is isotone,

$$x^{k+1} = x^k - A^{-1}F(x^k) \tag{6}$$

converge monotonically to y^* *and* x^*, *respectively, so that*

$$F(x^*) = 0 = F(y^*)$$

where $x^0 \leq x^* \leq y^* \leq y^0$.

Proof. See [5].

In the followings, we study a method of speeding up the convergence of monotone iterative sequences $\{x^k\}$ and $\{y^k\}$, since the usual monotone method is often slowly convergent. To accelerate the convergence of these sequences, we combine elements of both $\{x^k\}$ and $\{y^k\}$ by convex combinations. This method will be called the accelerating monotone method. This idea comes from [3].

Consider a nonlinear function $F: \mathrm{Dom}(F) \subset R^n \to R^n$ defined as $F(x) = Lx + N(x)$, where L is an M-matrix and $N(x)$ is a continuously differentiable nonlinear convex function. Let there exist $x^0, y^0 \in \mathrm{Dom}(F)$ such that

$$x^0 \leq y^0, \quad [x^0, y^0] \subset \mathrm{Dom}(F), \quad F(x^0) \leq 0 \leq F(y^0).$$

We assume that there exists a diagonal matrix $D \geq 0$ such that

$$\begin{aligned} N(y) - N(x) &= N'(\xi)(y-x) \\ &\leq D(y-x) \end{aligned}$$

for all x, y such that $x^0 \leq x \leq y \leq y^0$ where $\xi \in [x, y]$.

Since L is an M-matrix and D is a nonnegative diagonal matrix, $L+D$ is also an M-matrix. So we can define new vectors

$$x^1 = x^0 - (L+D)^{-1}F(x^0) = (L+D)^{-1}(Dx^0 - N(x^0)) \tag{7}$$

$$y^1 = y^0 - (L+D)^{-1}F(y^0) = (L+D)^{-1}(Dy^0 - N(y^0)) \tag{8}$$

and

$$p^1 = (1-a_1)x^1 + a_1 y^1 \tag{9}$$

(10) $$q^1=(1-b_1)y^1+b_1x^1$$

where a_1 and b_1 are nonnegative constants such that

$$a_1(x^1-x^0+y^0-y^1)\leq x^1-x^0.$$
$$b_1(x^1-x^0+y^0-y^1)\leq y^0-y^1,$$

respectively. Then we have $p^1\geq x^1$ and $q^1\leq y^1$.

From (7) and (8) we obtain

$$\begin{aligned}p^1&=(1-a_1)x^1+a_1y^1\\&=(1-a_1)(L+D)^{-1}\{Dx^0-N(x^0)\}\\&\quad+a_1(L+D)^{-1}\{Dy^0-N(y^0)\}\end{aligned}$$

Multiplying both sides by $(L+D)$, we get

$$(L+D)p^1=(1-a_1)\{Dx^0-N(x^0)\}+a_1\{Dy^0-N(y^0)\}.$$

Hence we have

$$\begin{aligned}F(p^1)&=Lp^1+N(p^1)\\&=(1-a_1)D(x^0-x^1)+a_1(y^0-y^1)-(1-a_1)N(x^0)\\&\quad-a_1N(y^0)+N\{(1-a_1)x^1+a_1y^1\}\\&\leq D\{(1-a_1)(x^0-x^1)+a_1(y^0-y^1)\}\\&\quad-N\{(1-a_1)x^0+a_1y^0\}+N\{(1-a_1)x^1+a_1y^1\}\\&=D\{(1-a_1)(x^0-x^1)+a_1(y^0-y^1)\}\\&\quad-N'(\xi)\ \{(1-a_1)(x^0-x^1)+a_1(y^0-y^1)\}\\&=(D-N'(\xi))\{(1-a_1)(x^0-x^1)+a_1(y^0-y^1)\}\\&\leq 0,\end{aligned}$$

since $D-N'(\xi)\geq 0$ and $(1-a_1)(x^0-x^1)+a_1(y^0-y^1)\leq 0$ where ξ is a vector between $(1-a_1)x^0+a_1y^0$ and $(1-a_1)x^1+a_1y^1$. This shows that p^1 is a lower solution of $F(x)=0$.

Similarly, we can show that q^1 is an upper solution.

If we continue these processes by defining

(11) $$x^{k+1}=p^k-(L+D)^{-1}F(p^k)=(L+D)^{-1}\{Dp^k-N(p^k)\}$$

(12) $$y^{k+1}=q^k-(L+D)^{-1}F(q^k)=(L+D)^{-1}\{Dq^k-N(q^k)\}$$

for $k=1, 2, \ldots$ and

(13) $$p^k=(1-a_k)x^k+a_ky^k$$

(14) $$q^k=(1-b_k)y^k+b_kx^k$$

for $k=2, 3, \ldots$, where a_k and b_k are constants such tha

$$a_k(x^k-p^{k-1}+q^{k-1}-y^k)\leq x^k-p^{k-1}$$
$$b_k(x^k-p^{k-1}+q^{k-1}-y^k)\leq q^{k-1}-y^k,$$

we have the following inequalities;

$$x^0\leq x^1\leq p^1\leq x^2\leq p^2\leq\ldots\leq q^2\leq y^2\leq q^1\leq y^1\leq y^0.$$

Then $\{p^k\}$ and $\{x^k\}$ are sequences of lower solutions and $\{q^k\}$ and $\{y^k\}$ are sequences of upper solutions. Furthermore, these sequences converge monotonically.

Therefore we have the following theorem.

THEOREM 2. *Let $F: Dom(F) \subset R^n \to R^n$ be defined as $F(x) = Lx + N(x)$, where L is an M-matrix and $N(x)$ is a continuously differentiable convex function. Let there exist $x^0, y^0 \in Dom(F)$ such that*

$$x^0 \leq y^0, \quad [x^0, y^0] \subset Dom(F), \quad F(x^0) \leq 0 \leq F(y^0)$$

Furthermore we assume that there exists a diagonal matrix $D \geq 0$ such that

$$N(y) - N(x) = N'(\xi)(y-x) \leq D(y-x)$$

for all x, y such that $x^0 \leq x \leq y \leq y^0$, where $\xi \in [x, y]$. Then the sequences $\{u_k\}$ and $\{v_k\}$ defined as

$$u_0 = x^0, \quad u_{2k-1} = x^k, \quad u_{2k} = p^k$$

and

$$v_0 = y^0, \quad v_{2k-1} = y^k, \quad v_{2k} = q^k,$$

where $\{x^k\}$, $\{p^k\}$, $\{y^k\}$ and $\{q^k\}$ are generated as in (7)–(14), converge monotonically to x^* and y^* respectively. And we get

$$F(x^*) = 0 = F(y^*),$$

where $x^0 \leq x^* \leq y^* \leq y^0$.

3. Numerical examples

In this section we will give some numerical examples of nonlinear boundary value problems of the type (2) and (3). We approximate the given differential equations by finite difference methods and use the centered difference scheme for the second derivatives. We compute approximate solutions using the usual monotone method and the accelerating monotone method and count numbers of iterations to compare the speed of convergence for both methods.

EXAMPLE 1. We first consider a nonlinear heat conduction problem

$$u''=u+u^2/4, \quad 0<x<1$$

with boundary conditons $u'(0)=0$ and $u(1)=1$. This problem has been studied in [1] where analytical pointwise bounds for the lower and upper solutions have been obtained.

From the physical point of view, we use the centered difference scheme for the Neumann boundary condition and we can take $u_0=0$, $v_0=1$ and a diagonal matrix with entries 1.5 [1]. We stopped computing when $|u_n-v_n|\leq 10^{-5}$ and applied the accelerating monotone method if $|u_{n+1}-u_n+v_n-v_{n+1}\leq 10^{-2}$. For the usual monotone method we needed 7781 iterations and for the accelerating monotone method 5098 iterations and 561 accelerations were needed. The following table shows the numerical solutions obtained by the accelerating monotone method.

x	x^*	y^*
0.0	0.60850	0.60851
0.2	0.62259	0.62260
0.4	0.66557	0.66558
0.6	0.73976	0.73976
0.8	0.84918	0.84918
1.0	1.00000	1.00000

EXAMPLE 2. We consider a nonlinear elliptic boundary value problem

$$\begin{aligned} \Delta u+2u&=\tan^{-1}(u)+\sin x \ \sin y, && (x, y)\in\Omega \\ u&=0, && (x, y)\in\partial\Omega \end{aligned}$$

where $\Omega=[0,\pi]\times[0,\pi]$. This type of problem has been studied in [5] by the usual monotone method.

Using $u_0=0$, $v_0=1$, $h=1/50$ and a diagonal matrix D with entries 1, we obtained the following table which shows the lower and upper soluitons at $x=1.50796$. We stopped computing when iterative sequences met conditions in Example 1. For the usual monotone method we obtained 3258 iterations and for the accelerating monotone method 2232 iterations and 369 accelerations were needed.

x	y	x^*	y^*
	0. 000000	0. 000000	0. 000000
	0. 314159	0. 157598	0. 157601
	0. 628319	0. 300289	0. 300295
	0. 942478	0. 414175	0. 414183
	1. 256637	0. 487691	0. 487701
1. 570796	1. 570796	0. 513105	0. 513115
	1. 884956	0. 487691	0. 487701
	2. 199115	0. 414175	0. 414183
	2. 513274	0. 300289	0. 300295
	2. 827433	0. 157598	0. 157601
	3. 141593	0. 000000	0. 000000

EXAMPLE 3. For the last example we treat a nonlinear periodic parabolic problem

$$\begin{aligned} &u_t - u_{xx} = c_1(c_2^4 - u^4) + c_3(t, x) \\ &u_x = 0 \text{ at } x = a, \quad x = b \\ &u(t, x) = u(t+p, x) \end{aligned}$$

where $a=0$, $b=5$, $p=0.85\times10^{-2}$, $c_1=0.1958291075\times10^{-2}$, $c_2=0.1\times10^2$, $\hat{c}=0.2493389252\times10^5$, and

$$c_3(t, x) = \begin{cases} \hat{c}\exp(-x^2/2), & 0 \le t \le 0.0005 \\ 0 & 0.0005 < t < p. \end{cases}$$

This problem has been studied in [10] to model the temperature distribution in a device used to measure properties of the meson beam at Los Alamos Meson Physics Facility.

We used $u_0=10.0$, $v_0=1900$ and a diagonal matrix D with diagonal elements $4c_1v_0^3$. And we used grids size $\Delta t=p/350$ and $\Delta x=b/16$.

Choosing a centered difference scheme, we computed both the lower and the upper solution. Solutions u_n and v_n were accepted when $|u_n - v_n| \le 10^{-6}|v_n|$, and we applied the accelerating monotone method when $|u_n - v_n| > 10^{-2}|v_n|$. We obtained 3960 iterations for the usual mono-

t	x	x^*	y^*
	0. 0000	893. 98071	893. 98071
	0. 3125	883. 74265	883. 74266
	0. 6250	854. 18422	854. 18422
	0. 9375	808. 64629	808. 64631
	1. 2500	752. 20224	752. 20226
	1. 5625	690. 78229	690. 78232
	1. 87500	630. 00395	630. 00401
	2. 1875	574. 13169	574. 13178
0. 0	2. 5000	525. 59694	525. 59707
	2. 8125	485. 16906	485. 16922
	3. 1250	452. 51224	452. 51245
	3. 43750	426. 77900	426. 77925
	3. 75000	407. 03215	407. 03244
	4. 06250	392. 46214	392. 46247
	4. 3750	382. 45916	382. 45952
	4. 6875	376. 61302	376. 61339
	5. 0000	374. 69080	374. 69117

tone method and 1940 iterations and 13 accelerations were obtained for the accelerating monotone method.

The following table shows approximate soluitons at $t=0$.

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