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DUAL OPERATOR ALGEBRAS AND C_0-OPERATORS

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1. Introduction

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. A *dual algebra* is a subalgebra of $\mathscr{L}(\mathscr{H})$ that contains $I_{\mathscr{H}}$ and is closed in the weak* topology on $\mathscr{L}(\mathscr{H})$. The theory of dual algebras have been applied to the topics of invariant subspaces, dilation theory, and reflexivity (cf. [6]). In [5], Bercovici-Foias-Pearcy studied the problem of solving systems of simultaneous equations in the predual of a dual algebra. The theory of dual algebras is deeply related to the study of the classes $\boldsymbol{A}_{m,n}$ (to be defined below), where m and n are any cardinal numbers with $1\leq m, n\leq \aleph_0$. That is the main topic of this paper. Jung [16] showed that the classes $\boldsymbol{A}_{m,n}$ are distinct one from another. Apostol-Bercovici-Foias-Pearcy [1] obtained an abstract geometric criterion for membership in $\boldsymbol{A}_{\aleph_0}$. Bercovici-Brown-Chevreau-Exner-Pearcy [7][10][11][12] obtained some relationship between dual algebras and Fredholm theory, and established topological criteria for membership in $\boldsymbol{A}_{\aleph_0}$ or $\boldsymbol{A}_{1,\aleph_0}$. Bercovici [3] and Chevreau [9] proved independently that $\boldsymbol{A}=\boldsymbol{A}_1$. Recently several functional analysists have studied sufficient conditions for membership in the class $\boldsymbol{A}_{1,\aleph_0}$, $\boldsymbol{A}_{\aleph_0}$ or $\boldsymbol{A}$ (cf. [13], [15], [17]). As a sequel to this study, we shall obtai a sufficient condition for membership in the classes $\boldsymbol{A}_{1,n}$ in this paper. Also, we study C_0-operators and the unilateral shift $S^{(n)}$ of multiplicity n concerning the classes $\boldsymbol{A}_{m,n}$.

2. Notation and preliminaries

The notation and terminology employed herein agree with those in

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[2], [6], [8], and [19]. The class $\mathscr{C}_1(\mathscr{H})$ is the Banach space of trace-class operators on $\mathscr{H}$ equipped with the trace norm. The dual algebra $\mathscr{A}$ can be identified with a dual space of $Q_{\mathscr{A}}=\mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, where ${}^{\perp}\mathscr{A}$ is the preannihilator in $\mathscr{C}_1(\mathscr{H})$ of $\mathscr{A}$, under the pairing

(2.1) $$\langle T, [L]_{\mathscr{A}}\rangle = tr(TL), \quad T\in\mathscr{A}, \quad [L]\in Q_{\mathscr{A}}.$$

We write $[L]$ for $[L]_{\mathscr{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathscr{H}$, we denote a rank one operator $(x\otimes y)(u)=(u, y)x$ for all u in $\mathscr{H}$. As noted above, we write $[x\otimes y]$ for $[x\otimes y]_{\mathscr{A}}$ when there is no possibility of confusion.

Definition 2.1 [6]. Suppose m and n are cardinal numbers such that $1\leq m, n\leq\aleph_0$. A dual algebra $\mathscr{A}$ will be said to have property $(\boldsymbol{A}_{m,n})$ if every $m\times n$ system of simultaneous equations of the form

(2.2) $$[x_i\otimes y_j]=[L_{ij}], \quad 0\leq i<m, \quad 0\leq j<n,$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m\\0\leq j<n}}$ is an arbitrary $m\times n$ array from $Q_{\mathscr{A}}$, has a solution $\{x_i\}_{0\leq i<m}$, $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathscr{H}$. Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r\geq 1$, a dual algebra $\mathscr{A}$ (with property $(\boldsymbol{A}_{m,n})$) is said to have property $(\boldsymbol{A}_{m,n}(r))$ if for every $s>r$ and every $m\times n$ array $\{[L_{ij}]\}_{\substack{0\leq i<m\\0\leq j<n}}$ from $Q_{\mathscr{A}}$, there exist sequences $\{x_i\}_{0\leq i<m}$, and $\{y_j\}_{0\leq j<n}$ that satisfy (2.2) and also satisfy the following conditions:

(2.3a) $$\|x_i\|^2\leq s\sum_{0\leq j<n}\|[L_{ij}]\|, \quad 0\leq i<m$$

and

(2.3b) $$\|y_j\|^2\leq s\sum_{0\leq i<m}\|[L_{ij}]\|, \quad 0\leq j<n.$$

Finally, a dual algebra $\mathscr{A}\subset\mathscr{L}(\mathscr{H})$ has property $(\boldsymbol{A}_{m,\aleph_0}(r))$ (for some real number $r\geq 1$) if for every $s>r$ and every array $\{[L_{ij}]\}_{\substack{0\leq i<m\\0\leq j<\infty}}$ from $Q_{\mathscr{A}}$ with summable rows, there exist sequences $\{x_i\}_{0\leq i<m}$ and $\{y_j\}_{0\leq j<\infty}$ of vectors from $\mathscr{H}$ that satisfy (2.2) and (2.3a, b) with the replacement of n by $\aleph_0$. Properties $(\boldsymbol{A}_{\aleph_0,n}(r))$ and $(\boldsymbol{A}_{\aleph_0,\aleph_0}(r))$ are defined similarly.

For a brief notation, we shall denote $(\boldsymbol{A}_{n,n})$ by $(\boldsymbol{A}_n)$. A contraction operator $T\in\mathscr{L}(\mathscr{H})$ is *absolutely continuous* if in the canonical decomposition $T=T_1\oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0). We write $\boldsymbol{D}$ for the open unit disc in the complex plane $\boldsymbol{C}$ and $\boldsymbol{T}$ for the boundary of $\boldsymbol{D}$. The space $L^p = L^p(\boldsymbol{T})$, $1 \le p \le \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\boldsymbol{T}$. The space $H^p = H^p(\boldsymbol{T})$, $1 \le p \le \infty$, is the usual Hardy space. It is well-known (cf. [14]) that the space H^∞ is the dual space of L^1/H_0^1, where

(2.4) $$H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(e^{it}) e^{int} dt = 0, \text{ for } n = 0, 1, 2, \ldots\}$$

and the duality is given by the pairing

(2.5) $$\langle f, [g] \rangle = \int_T fg\, dm, \text{ for } f \in H^\infty, \ [g] \in L^1/H_0^1.$$

We denote by $\mathcal{A}_T$ the dual algebra generated by T and denote by $Q_{\mathcal{A}}$ the predual space $Q_{\mathcal{A}_T}$ of $\mathcal{A}_T$.

THEOREM 2.2 [6, Theorem 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there exists a functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ for every f in H^∞. The mapping Φ_T is a norm-decreasing, weak* continuous algebra homomorphism, and the range of Φ_T is weak* dense in $\mathcal{A}_T$. Furthermore, there exists a bounded, linear, one-to-one map ϕ_T of Q_T into L^1/H_0^1 such that $\Phi_T = \phi_T^*$.*

DEFINITION 2.3 [5]. We define by $\boldsymbol{A} = \boldsymbol{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, if m and n are any cardinal numbers such that $1 \le m, n \le \aleph_0$, we define by $\boldsymbol{A}_{m,n} = \boldsymbol{A}_{m,n}(\mathcal{H})$ the set of all T in $\boldsymbol{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\boldsymbol{A}_{m,n})$.

If follows from [5] that if $T \in \boldsymbol{A}$, then Φ_T is a weak* homeomorphism and ϕ_T is an isometry of Q_T onto L^1/H_0^1. Let $\mathcal{K}$ be a Hilbert space and let $T_1, T_2 \in \mathcal{L}(\mathcal{K})$. Then we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2. Throughout this paper, $\boldsymbol{N}$ is the set of all natural numbers. For an invariant subspace $\mathcal{M}$ for an operator $T \in \mathcal{L}(\mathcal{K})$, we write $T|\mathcal{M}$ for the restriction to $\mathcal{M}$.

3. A sufficient condition for membership in the classes $A_{1,n}$

We start this section as the following lemma.

LEMMA 3.1. *Suppose* $T, S \in \boldsymbol{A}(\mathcal{H})$. *Let* $[L]_T \in Q_T$ *and let* $[M]_S \in Q_S$. *Then* $\phi_T([L]_T) = \phi_S([M]_S)$ *if and only if* $\langle T^n, [L]_T \rangle = \langle S^n, [M]_S \rangle$, *for* $n = 0, 1, 2, \ldots$

Proof ($\Rightarrow$) For $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \langle T^n, [L]_T \rangle &= \langle T^n, \phi_T^{-1}\phi_S([M]_S) \rangle \\ &= \langle \Phi_T(\xi^n), \phi_T^{-1}\phi_S([M]_S) \rangle \\ &= \langle \xi^n, \phi_S([M]_S) \rangle \\ &= \langle S^n, [M]_S \rangle. \end{aligned} \tag{3.1}$$

($\Leftarrow$) Since $\langle p(T), [L]_T \rangle = \langle p(S), [M]_S \rangle$ for any polynomial p, we have

$$\langle h(T), [L]_T \rangle = \langle h(S), [M]_S \rangle \tag{3.2}$$

for any $h \in H^\infty$. Then

$$\begin{aligned} \langle h, \phi_T([L]_T) \rangle &= \langle h(T), [L]_T \rangle = \langle h(S), [M]_S \rangle \\ &= \langle h, \phi_S([M]_S) \rangle, \end{aligned} \tag{3.3}$$

for any $h \in H^\infty$. Hence $\phi_T([L]_T) = \phi_S([M]_S)$. The proof is complete.

The following proposition is a generalization of [15, Proposition 2.4].

PROPOSITION 3.2. *Suppose* $\mathcal{H}_i$ *is a separable, infinite dimensional, complex Hilbert space, for* $i = 1, 2, \cdots, m$. *Let* $n_i \in \boldsymbol{N}$, *and let* $T_i \in \boldsymbol{A}_{1, n_i}(\mathcal{H}_i)(r_i)$ *and* $r_i \geq 1$ *for* $i = 1, 2, \cdots, m$. *Then*

$$\bigoplus_{i=1}^{m} T_i \in \boldsymbol{A}_{1,N}\left(\bigoplus_{i=1}^{m} \mathcal{H}_i\right)(r), \tag{3.4}$$

where $r = \max\{r_i \mid 1 \leq i \leq m\}$ *and* $N = n_1 + \cdots + n_m$.

Proof. Let us denote $\hat{T} = \bigoplus_{i=1}^{m} T_i$. Let $s > r$ and let

$$\{[L_k]_{\hat{T}}\}_{1 \leq k \leq N} \subset Q_{\hat{T}}. \tag{3.5}$$

For a convenient calculation, we correspond $\{[L_j^{(i)}]\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}}$ with $\{[L_k]\}_{1 \leq k \leq N}$. Let us denote $\phi_i = \phi_{T_i}^{-1}\phi_{\hat{T}}$. Then it is obvious that ϕ_i is an isometric isomorphic weak* homeomorphism from $Q_{\hat{T}}$ onto Q_{T_i}. Since $\phi_i([L_j^{(i)}]_{\hat{T}}) \in Q_{T_i}$ and $s > r_i$, for $1 \leq j \leq n_i$, $1 \leq i \leq m$, there exist vectors $x^{(i)}$ and $y_j^{(i)}$ in $\mathcal{H}_i$, $1 \leq i \leq m$, $1 \leq j \leq n_i$ such that

$$\phi_i([L_j^{(i)}]_{\hat{T}}) = [x^{(i)} \otimes y_j^{(i)}]_{T_i}, \tag{3.6a}$$

$$\|x^{(i)}\|^2 \leq s \sum_{j=1}^{n_i} \|[L_j^{(i)}]_{\hat{T}}\|, \text{ and } \|y_j^{(i)}\|^2 \leq s \|[L_j^{(i)}]_{\hat{T}}\|. \tag{3.6b}$$

Now let $\tilde{x}=x^{(1)}\oplus\cdots\oplus x^{(m)}$ and $\tilde{y}_j^{(i)}=\underbrace{0\oplus\cdots\oplus 0}_{(i-1)}\oplus y_j^{(i)}\oplus 0\oplus\cdots\oplus 0$, for $1\leq j\leq n_i$, $1\leq i\leq m$. Then it follows from Lemma 3.1 and (3.6a) that we have

$$\phi_i([\tilde{x}\otimes\tilde{y}_j^{(i)}]_{\hat{T}})=[x^{(i)}\otimes y_j^{(i)}]_{T_i}=\phi_i([L_j^{(i)}]_{\hat{T}}) \tag{3.7}$$

since

$$\begin{aligned}<\hat{T}^n, [\tilde{x}\otimes\tilde{y}_j^{(i)}]_{\hat{T}}> &= (\hat{T}^n\tilde{x}, \tilde{y}_j^{(i)})\\ &= (T_i^n x^{(i)}, y_j^{(i)})\\ &= <T_i^n, [x^{(i)}\otimes y_j^{(i)}]_{T_i}>, \text{ for } n=0, 1, 2, \cdots.\end{aligned} \tag{3.8}$$

Hence $[\tilde{x}\otimes\tilde{y}_j^{(i)}]_{\hat{T}}=[L_j^{(i)}]_{\hat{T}}$. Furthermore, since

$$\|\tilde{x}\|^2=\sum_{i=1}^{m}\|x^{(i)}\|^2\leq s\sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n_i}}\|[L_j^{(i)}]_{\hat{T}}\| \tag{3.9}$$

and

$$\|\tilde{y}_j^{(i)}\|^2=\|y_j^{(i)}\|^2\leq s\|[L_j^{(i)}]_{\hat{T}}\|,\quad 1\leq j\leq n_i,\quad 1\leq i\leq m, \tag{3.10}$$

we have $\hat{T}\in \boldsymbol{A}_{1,N}(\bigoplus_{i=1}^{m}\mathcal{H}_i)(r)$. Thus the proof is complete.

It is well known that $T\in\boldsymbol{A}_{1,n}(\mathcal{H})$ if and only if for any weak*-continuous functional φ_i on $\mathcal{A}_T$, $0\leq i<n$, there exist vectors x and y_i in $\mathcal{H}$, $0\leq i<n$ such that

$$\varphi_i(h(T))=(h(T)x, y_i). \tag{3.11}$$

Using the above statement, we obtain a sufficient condition for membership in $\boldsymbol{A}_{1,n}$.

THEOREM 3.3. *Suppose $T\in\boldsymbol{A}(\mathcal{H})$ and $n\in\boldsymbol{N}$. Assume that for any $\tilde{x}$ in $\underbrace{\mathcal{H}\oplus\cdots\oplus\mathcal{H}}_{(n)}$, there exists a vector x in $\mathcal{H}$ such that*

$$\tilde{T}|(\vee_{k\geq 0}\tilde{T}^k\tilde{x})\cong T|(\vee_{k\geq 0}T^k x), \tag{3.12n}$$

where $\tilde{T}=\underbrace{T\oplus\cdots\oplus T}_{(n)}$. Then $T\in\boldsymbol{A}_{1,n}$.

Proof. Let $\zeta_i:\mathcal{A}_T\longrightarrow\boldsymbol{C}$ be a weak*-continuous functional, for $i=1, 2, \ldots, n$. Then it follows from [6, Proposition 1.7] that there exists $[L_i]_T\in Q_T$, $i=1, 2, \ldots, n$, such that

$$\zeta_i(h(T))=<h(T), [L_i]_T>,\quad {}^{\forall}h\in H^\infty. \tag{3.13}$$

Furthermore, since $\boldsymbol{A}=\boldsymbol{A}_1$ (cf. [3], [9]), there exist vectors x_i and

y_i in $\mathcal{H}$, $i=1, 2, \ldots, n$, such that $[L_i]_T=[x_i \otimes y_i]_T$. It is equivalent to

$$(3.14) \qquad \zeta_i(h(T)) = (h(T)x_i, y_i), \quad {}^{\forall}h \in H^{\infty}, \quad i=1, 2, \ldots, n.$$

Moreover, we have $\tilde{T} \in \boldsymbol{A}_{1,n}$ by Proposition 3. 2. Since

$$\begin{aligned}(3.15) \qquad \zeta_i(T^k) &= (T^k x_i, y_i) \\ &= (\tilde{T}^k(x_i \underbrace{\oplus 0 \oplus \cdots \oplus 0}_{(n-1)}), \ (y_i \underbrace{\oplus 0 \oplus \cdots \oplus 0}_{(n-1)})) \\ &= \langle \tilde{T}^k, [(x_i \oplus \underbrace{0 \oplus \cdots \oplus 0}_{(n-1)}) \otimes (y_i \oplus \underbrace{0 \oplus \cdots \oplus 0}_{(n-1)})]_{\tilde{T}} \rangle,\end{aligned}$$

for $i=1, 2, \ldots, n$, $k=0, 1, 2, \ldots$, there exist vectors $\tilde{x}, \tilde{y}_i \in \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{(n)}$, $i=1, 2, \ldots, n$ such that

$$(3.16) \qquad \zeta_i(T^k) = (\tilde{T}^k \tilde{x}, \tilde{y}_i), \quad i=1, 2, \ldots, n, \quad k=0, 1, 2, \ldots.$$

If we set $\mathfrak{M} = \bigvee_{0 \le k} \tilde{T}^k \tilde{x}$, we get from the hypothesis that there exist a vector u in $\mathcal{H}$ and a unitary operator

$$(3.17) \qquad U : \mathfrak{M} \longrightarrow \bigvee_{0 \le k} T^k u$$

such that $TU = U(\tilde{T}|\mathfrak{M})$. Hence if we denote $v_i = UP_{\mathfrak{M}}\tilde{y}_i$, $i=1, 2, \ldots, n$, where $P_{\mathfrak{M}}$ is the orthogonal projection whose range is $\mathfrak{M}$, then

$$\begin{aligned}(3.18) \qquad (\tilde{T}^k \tilde{x}, \tilde{y}_i) &= (\tilde{T}^k \tilde{x}, P_{\mathfrak{M}} \tilde{y}_i) = (\tilde{T}^k \tilde{x}, U^* v_i) \\ &= (U\tilde{T}^k \tilde{x}, v_i) = (T^k U \tilde{x}, v_i), \quad \text{for } i=1, 2, \ldots, n, \\ & \qquad\qquad\qquad\qquad\qquad\qquad k=0, 1, 2, \ldots.\end{aligned}$$

According to (3. 16), we have

$$(3.19) \qquad \zeta_i(T^k) = (T^k U \tilde{x}, v_i), \quad i=1, 2, \ldots, n, \quad k=0, 1, 2, \ldots.$$

Therefore it is obvious that $T \in \boldsymbol{A}_{1,n}$. Therefore the proof is complete.

The following example shows that there exists an operator satisfying the condition (3. 12n) in Theorem 3. 3.

EXAMPLE 3. 4. The (forward) unilateral shift operators S of multiplicity one satisfies (3. 12n) in Theorem 3. 3 for any $n \in \boldsymbol{N}$, since S has a cyclic vector.

REMARK 3. 5. Since the backward unilater shift $S^{(n)*}$ of multiplicity $n \ge 2$ has a cyclic vector, $S^{(n)*}$ cannot have the property (3. 12n) in Theorem 3. 3 for $n \ge 2$. Hence the converse implication of Theorem 3. 3 is not always true.

4. C_0-operators and the classes $\mathbf{A}_{m,n}$

Before starting this section, we recall that a set $\{e_i\}_{0\le i<n}$ of vectors in a Hilbert space $\mathcal{K}$ is an *n-cyclic set* for an operator A in $\mathcal{L}(\mathcal{K})$ if $\mathcal{K}$ is the smallest invariant subspace for A containing $\{e_i\}_{0\le i<n}$. If $T \in \mathcal{L}(\mathcal{K})$ and $\mathfrak{M}$ is a semi-invariant subspace for T, we shall write $T_{\mathfrak{M}} = P_{\mathfrak{M}}T|\mathfrak{M}$ for the compression of T to $\mathfrak{M}$, where $P_{\mathfrak{M}}$ is the orthogonal projection whose range is $\mathfrak{M}$. If X is a (unbounded) linear transformation, we write $\mathcal{D}(X)$ for the domain of X and $\mathcal{R}(X)$ for the range of X. Recall that a completely nonunitary contraction $T\in\mathcal{L}(\mathcal{H})$ is said to be of class C_0 if there exists $u\in H^\infty$, $u\not\equiv 0$, such that the functional calculus $u(T)=0$. The following is an intresting dilation theorem concerning C_0-operators and the classes $\mathbf{A}_{m,n}$.

THEOREM 4.1. *Suppose $T\in\mathbf{A}_{m,n}(\mathcal{H})$ for some positive integers m and n. Let A be any absolutely continuous contraction on a Hilbert space $\mathcal{K}$ and let $A\notin C_0$. If A possesses an m-cyclic $\{e_1, \cdots e_m\}$ of vectors in $\mathcal{K}$ and its adjoint operator A^* has an n-cyclic set $\{f_1, \cdots, f_n\}$ of vectors in $\mathcal{K}$, then there exist $\mathfrak{M}, \mathfrak{N}\in \mathrm{Lat}(T)$ with $\mathfrak{M}\supset\mathfrak{N}$ and a closed one-to-one linear transformation*

$$X : \mathcal{D}(X) \longrightarrow \mathfrak{M}\ominus\mathfrak{N} \tag{4.1}$$

such that

(*a*) *the linear manifold $\mathcal{D}(X)$ is dense in $\mathcal{K}$ and contains $\{e_1, \cdots, e_m\}$,*
(*b*) *the range $\mathcal{R}(X)$ of X is dense in $\mathfrak{M}\ominus\mathfrak{N}$, and*
(*c*) *$A\mathcal{D}(X)\subset\mathcal{D}(X)$ and $T_{\mathfrak{M}\ominus\mathfrak{N}}Xz = XAz$ for all z in $\mathcal{D}(X)$*
(*d*) *$T_{\mathfrak{M}\ominus\mathfrak{N}}\notin C_0$.*

Proof. (a), (b) and (c) follow from [16, Theorem 3.1]. Hence we shall claim (d). Let us denote $\tilde{T} = T_{\mathfrak{M}\ominus\mathfrak{N}}$ and suppose $\tilde{T}\in C_0$. Then there exists a bounded analytic function $h\in H^\infty$ such that $h(\tilde{T})=0$. Let $p_n(\lambda)$ be polynomials such that $p_n\longrightarrow h$ in H^∞. Since $\tilde{T}Xz=XAz$ for all z in $\mathcal{D}(X)$, we have

$$\tilde{T}^k Xz = XA^k z \tag{4.2}$$

for all z in $\mathcal{D}(X)$, $k=0, 1, 2, \cdots$. Thus

$$p_n(\tilde{T})Xz = Xp_n(A)z \tag{4.3}$$

for all z in $\mathcal{D}(X)$. Moreover, since

$$Xp_n(A)z = p_n(\tilde{T})Xz \longrightarrow h(\tilde{T})Xz \ (n\longrightarrow\infty) \tag{4.4}$$

and since

$$p_n(A)z \longrightarrow h(A)z \quad (n \longrightarrow \infty), \tag{4.5}$$

it follows from the closedness of X that

$$Xh(A)z = h(\tilde{T})Xz \tag{4.6}$$

for all z in $\mathcal{D}(X)$. Hence $Xh(A)z=0$ for all z in $\mathcal{D}(X)$. So $h(A)z=0$ for all z in $\mathcal{D}(X)$ because X is one-to-one. Since $\mathcal{D}(X)$ is dense in $\mathcal{K}$, we have $h(A)=0$. Therefore $A \in C_0$ and this contradiction proves the theorem.

We now give two simple propositions before suggesting a conjecture.

PROPOSITION 4.2 [18, Theorem 1]. *Let $S^{(n)}$ be a unilateral shift of multiplicity n. Assume that*

$$S^{(n)} \cong \begin{pmatrix} S^{(n)} & * \\ 0 & E \end{pmatrix}. \tag{4.7}$$

Then $E \in C_0$.

For a given contraction T in $\mathcal{L}(\mathcal{H})$, we recall (cf. [19]) that

$$D_T = (I - T^*T)^{\frac{1}{2}}, \quad D_{T^*} = (I - TT^*)^{\frac{1}{2}} \tag{4.8}$$

$$\mathcal{D}_T = \overline{\text{Range}(D_T)}, \quad \mathcal{D}_{T^*} = \overline{\text{Range}(D_{T^*})}. \tag{4.9}$$

$$d_T = \dim \mathcal{D}_T, \text{ and } d_{T^*} = \dim \mathcal{D}_{T^*}. \tag{4.10}$$

PROPOSITION 4.3. *Let $S^{(n)}$ be a unilateral shift of multiplicity n. Suppose*

$$S^{(n)} \cong \begin{pmatrix} S^{(n)} & * \\ 0 & E \end{pmatrix}. \tag{4.11}$$

Then $S^{(n)} \oplus E \in \boldsymbol{A}_{n,\aleph_0}(1) \setminus \boldsymbol{A}_{n+1,1}$.

Proof. By Proposition 4.2, we have $E \in C_0$. Furthermore, it is easy to show that $d_E = d_{E^*} \leq n$. According to [16, Corollary 5.4], we get $S^{(n)} \oplus E \in \boldsymbol{A}_{n,\aleph_0}(1) \setminus \boldsymbol{A}_{n+1,1}$. Hence this proposition is proved.

Finally according to Proposition 4.3 and [16, Corollary 5.4], we can conjecture the following statement.

CONJECTURE 4.4. We conjecture that if B is an operator of class C_0 acting on a separable Hilbert space, then we have

$$S^{(n)} \oplus B \in \boldsymbol{A}_{n,\aleph_0}(1) \setminus \boldsymbol{A}_{n+1,1}, \quad n \in \boldsymbol{N}. \tag{4.12}$$

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