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A CHARACTERIZATION OF DIRICHLET SETS

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The notion of a Dirichlet set has been studied for several decades. Such sets are named in honour of Dirichlet's Theorem [4, p. 235] which, in modern terminology, simply says that every finite set in $\boldsymbol{R}$ is a Dirichlet set.

In this paper, we present a structure theorem which characterizes all D-sets on the real line. We also use our structure theorem to give a new proof of a known criterion for proving that a set fails to be a D-set.

Definition 1. [2, p. 1] A bounded set $A\subset\boldsymbol{R}$ is called a *Dirichlet set* (in short, D-set) if there exists a sequence $(\alpha_k)_{k=1}^{\infty}$ in $\boldsymbol{R}$ such that $\lim_{k\to\infty}\alpha_k=\infty$ and $\lim_{k\to\infty}(\sup_{x\in A}|\sin\alpha_k x|)=0$. (Define $\sup\phi=0$ for the empty set ϕ, so ϕ is a D-set).

Let us state a proposition which can easily be proved.

Proposition 2. *If $A\subset\boldsymbol{R}$ is a D-set and $\beta\in\boldsymbol{R}$, then there exists $(n_k)_{k=1}^{\infty}$ in $\boldsymbol{N}$ such that*

$$\lim_{k\to\infty} n_k=\infty \text{ and } \lim_{k\to\infty}(\sup_{x\in A}|\sin n_k\beta x|)=0.$$

In particular, $\beta A=\{\beta x : x\in A\}$ is a D-set for every $\beta\in\boldsymbol{R}$.

Remark 3. Proposition 2 shows that for any D-set A, we may choose a sequence in $\boldsymbol{N}$ satisfying the condition in the definition of D-set.

We will use the following notation throughout the rest of this paper.

Notation 4. Let $\boldsymbol{a}=(a_j)_{j=1}^{\infty}\subset\boldsymbol{N}\setminus\{1\}$ be given. Write $D_j=\{x\in\boldsymbol{Z} : 0\leq x<a_j\}$ for the set of "digits" in the j-th place and define

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$$b_j=\prod_{i=1}^{j} a_i \text{ for every } j\in \boldsymbol{N}.$$

Fix any sets F_j with $\phi\neq F_j\subset D_j$ for every $j\in \boldsymbol{N}$. Put $\boldsymbol{F}=(F_j)_{j=1}^{\infty}$. Write $E=E(\boldsymbol{a}, \boldsymbol{F})=\{\sum_{j=1}^{\infty}\frac{x_j}{b_j} : x_j\in F_j \text{ for every } j\geq 1\}$.

LEMMA 5. *Let* $x\in \boldsymbol{R} : x=x_0+\sum_{j=1}^{\infty}\frac{x_j}{b_j}$, *where* $x_j\in D_j$ *for all* $j\in \boldsymbol{N}$ *and* $x_0\in \boldsymbol{Z}$. *Suppose that there exist* $n\in \boldsymbol{N}$ *and* $z_n\in \boldsymbol{Z}$ $(1\leq z_n\leq a_n)$ *such that either* $0\leq x_n<z_n$ *or* $a_n-z_n\leq x_n<a_n$. *Then* $|\sin b_{n-1}\pi x|\leq\frac{\pi z_n}{a_n}$.

Proof. We consider two cases separately.

Case (1): $0\leq x_n<z_n$. Let

$$m_n\overset{\text{def}}{=}b_{n-1}x_0+b_{n-1}\sum_{j=1}^{n-1}\frac{x_j}{b_j}.$$

Then, $m_n\in \boldsymbol{Z}$ and $m_n\leq b_{n-1}x\leq m_n+\frac{z_n}{a_n}$. Thus,

$$\left|\frac{1}{\pi}\sin \pi b_{n-1}x\right|\leq \text{dist}(b_{n-1}, \boldsymbol{Z})\leq\frac{z_n}{a_n}.$$

Case (2): $a_n-z_n\leq x_n<a_n$. With m_n as above, we have

$$m_n+1\geq b_{n-1}x=m_n+\sum_{j=n}^{\infty}\frac{x_j}{a_na_{n+1}\cdots a_j}\geq m_n+1-\frac{z_n}{a_n}.$$

Thus,

$$\left|\frac{1}{\pi}\sin\pi b_{n-1}x\right|\leq \text{dist}(b_{n-1}, \boldsymbol{Z})\leq\frac{z_n}{a_n}.$$

Using this lemma, we next prove a theorem that is needed to prove Theorem 8, our main theorem of this paper.

THEOREM 6. *For each* $n\in \boldsymbol{N}$ *define*

$$z_n=\min\{k\in \boldsymbol{N} : F_n\subset\{0, 1, \ldots, k-1\}\cup\{a_n-k, \ldots, a_n-1\}\}.$$

If $\lim_{n\to\infty}\frac{z_n}{a_n}=0$, *then* $E=E(\boldsymbol{a}, \boldsymbol{F})$ *is a D-set.*

Proof. Let $x\in E : x=\sum_{j=1}^{\infty}\frac{x_j}{b_j}$, where $x_j\in F_j$ for every $j\in \boldsymbol{N}$. Let $(z_{n_k}, a_{n_k})_{k=1}^{\infty}$ be a double sequence such that

$$\frac{z_{n_k}}{a_{n_k}} \to 0 \text{ as } k \to \infty. \tag{1}$$

By Lemma 5, we have

$$|\sin b_{n_k-1}\pi x| \leq \frac{\pi z_{n_k}}{a_{n_k}} \text{ for all } k \geq 1. \tag{2}$$

The sequence $(n_k)_{k=1}^{\infty}$ does not depend on $x \in E$, so (1) and (2) yield that E is a D-set.

In order to give a simple statement of our main theorem, we make the following definition.

Definition 7. A set of the form $E = E(\boldsymbol{a}, \boldsymbol{F})$ is called a *special D-set* if

(i) $\overline{\lim_{n\to\infty}}\, a_{2n} = \infty$

(ii) $F_{2j} = \{0, a_{2j}-1\}$ and $F_{2j-1} = D_{2j-1}$ for all $j \in \boldsymbol{N}$.

Note that every special D-set is indeed a D-set by Theorem 6. This provides lots of examples of uncountable D-sets.

Now, we are ready for the main theorem which can be compared with Marcinkiewicz's [2, p. 3].

Theorem 8. *If $A \subset \boldsymbol{R}$ is a D-set, then $A \subset F_0 + E$ for some finite set $F_0 \subset \boldsymbol{Z}$ and some special D-set E.*

Proof. Choose any sequence $(s_j)_{j=1}^{\infty} \subset \boldsymbol{N} \setminus \{1\}$ such that $\overline{\lim_{j\to\infty}}\, s_j = \infty$. Put $a_0 = b_0 = 1$ and $a_{2j} = s_j$ for $j \geq 1$. If $j > 0$ and $a_1, a_3, \ldots, a_{2j-1} \in \boldsymbol{N}$ has been determined, use the fact that $b_{2j}\pi A$ is a D-set, to choose $a_{2j+1} \in \boldsymbol{N} \setminus \{1\}$ such that

$$|\sin a_{2j+1} b_{2j} \pi x| < \frac{2}{a_{2j+2}} \text{ for all } x \in A.$$

Then,

$$\operatorname{dist}(a_{2j+1} b_{2j} x, \boldsymbol{Z}) < \frac{1}{a_{2j+2}}$$

so,

$$\operatorname{dist}\left(x, \frac{\boldsymbol{Z}}{b_{2j+1}}\right) < \frac{1}{b_{2j+2}} \text{ for } x \in A. \tag{1}$$

This defines $\boldsymbol{a} = (a_i)_{i=1}^{\infty}$ inductively in such a way that (1) holds for all $j \geq 0$. Let $x \in A$ be given. Consider the $\boldsymbol{a}$-adic expansion, $x = x_0 + \sum_{j=1}^{\infty} \frac{x_j}{b_j}$, where $x_j \in D_j$ for every $j \in \boldsymbol{N}$, $x_0 = \max\{n \in \boldsymbol{Z} : n < x\}$ and $\sum_{j=1}^{\infty} x_j = \infty$ [3, p. 88]. Given $n \in \boldsymbol{Z}$ with $n \geq 0$, choose $p \in \boldsymbol{Z}$ such that

$$\frac{p}{b_{2n+1}} < x = \frac{p}{b_{2n+1}} + \sum_{j=2n+2}^{\infty} \frac{x_j}{b_j} \leq \frac{p+1}{b_{2n+1}} \tag{2}$$

Then (1) and (2) show either

$$0 < \sum_{j=2n+2}^{\infty} \frac{x_j}{b_j} = x - \frac{p}{b_{2n+1}} < \frac{1}{b_{2n+2}} \tag{3}$$

or

$$0 \leq \sum_{j=2n+2}^{\infty} \frac{(a_j - 1) - x_j}{b_j} = \frac{1}{b_{2n+1}} - \sum_{j=2n+2}^{\infty} \frac{x_j}{b_j} = \frac{p+1}{b_{2n+1}} - x < \frac{1}{b_{2n+2}} \tag{4}$$

If (3) holds, then $x_{2n+2} = 0$. If (4) holds, then $(a_{2n+2} - 1) - x_{2n+2} = 0$. In either case, we have $x_{2n+2} \in \{0, a_{2n+2} - 1\}$ for $n \geq 0$. Now define $\boldsymbol{F}$ by

$$F_{2n+2} = \{0, a_{2n+2} - 1\} \text{ and } F_{2n+1} = D_{2n+1} \text{ for } n \geq 0.$$

We have just proved that $x \in x_0 + E(\boldsymbol{a}, \boldsymbol{F})$ where $E(\boldsymbol{a}, \boldsymbol{F})$ is a special D-set. Finally to define $F_0 \subset \boldsymbol{Z}$, note that we can take $s, t\,(s \leq t)$ in $\boldsymbol{Z}$ such that $A \subset (s, t)$ since A is bounded. Now define $F_0 = [s, t] \cap \boldsymbol{Z}$. Then F_0 is finite and $x_0 \in F_0$. Since $x \in A$ was arbitrary, we have

$$A \subset F_0 + E(\boldsymbol{a}, \boldsymbol{F}).$$

The following theorem, which appears in [2, p. 2] with a very different proof, when combined with the fact that any translate of a D-set, affords our simplest way of proving that certain bounded sets of measure zero are not D-sets.

THEOREM 9. *Suppose that $A \subset \boldsymbol{R}$ and A contains a strictly decreasing sequence $(x_k)_{k=1}^{\infty}$ with*

$$\lim_{k \to \infty} x_k = 0 \text{ and } \varliminf_{k \to \infty} \frac{x_{k+1}}{x_k} > 0.$$

Then A is not a D-set.

Proof. Assume to the contrary that A is a D-set. Then $A \cap (0, 1]$ is a D-set so Theorem 8 provides a special D-set, $E = E(\boldsymbol{a}, \boldsymbol{F})$ with $A \cap (0, 1] \subset E$. Choose $\delta > 0$ and $l \in \boldsymbol{N}$ such that

$$k \geq l \Rightarrow x_k < 1 \text{ and } \frac{x_{k+1}}{x_x} \geq \delta. \tag{1}$$

Next fix $n\in \boldsymbol{N}$ such that

$$\frac{1}{b_{2n}}<x_l \text{ and } a_{2n}>\frac{1}{\delta}+2. \tag{2}$$

Define p by

$$p+1=\min\left\{k\in \boldsymbol{N}:\ x_k\leq\frac{1}{b_{2n}}\right\}.$$

Then

$$x_{p+1}\leq\frac{1}{b_{2n}}<x_p \tag{3}$$

so (2) shows that $p\geq l$ and hence (1) gives $x_p\in A\cap(0,1]\subset E$ and

$$x_p\leq\frac{x_{p+1}}{\delta}. \tag{4}$$

Let $x_p=\sum_{j=1}^{\infty}\frac{x_{p,j}}{b_j}$ be the $\boldsymbol{a}$-adic expansion of x_p having $x_{p,j}\in F_j$ for $j\geq 1$. Now we can choose $j_0\leq 2n$ with $x_{p,j_0}>0$ since otherwise we would have

$$x_p=\sum_{j=2n+1}^{\infty}\frac{x_{p,j}}{b_j}\leq\sum_{j=2n+1}^{\infty}\frac{a_j-1}{b_j}=\frac{1}{b_{2n}}.$$

If $j_0=2n$, then $x_{p,j_0}=a_{2n}-1$ so $x_p\geq\frac{a_{2n}-1}{b_{2n}}$. If $j_0<2n$, then $x_{p,j_0}\geq 1$ and $b_{2n}\geq b_{j_0}a_{2n}$ so $x_p\geq\frac{x_{p,j_0}}{b_{j_0}}\geq\frac{a_{2n}}{b_{2n}}$. In either case we obtain by use of (2), (3), and (4) that

$$\frac{1}{\delta b_{2n}}<\frac{a_{2n}-2}{b_{2n}}=\frac{a_{2n}-1}{b_{2n}}-\frac{1}{b_{2n}}$$
$$\leq x_p-x_{p+1}<\frac{x_{p+1}}{\delta}\leq\frac{1}{\delta b_{2n}}.$$

This contradiction completes the proof that A is not a D-set.

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