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# ULTRAPRODUCTS OF LOCALLY CONVEX SPACES

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## 0. Introduction

It is well known that the concept of ultraproducts plays very important role in various branches ([1], [2], [3], [4], [5], [7], [8], [9]). Recently among others, it has been employed to characterize finitely represented Banach spaces in [8].

In this paper, we try to generalize ultraproducts in the category of locally convex spaces.

To do so, we introduce $D$-ultracolimits.

It is known [7] that the topology on a non-trivial ultraproduct in the category $TVec$ of topological vector spaces and continuous linear maps is trivial.

To generalize the category $Ban_1$ of Banach spaces and linear contractions, we introduce the category $LC_1$ of vector spaces endowed with families of semi-norms closed underfinite joins and linear contractions (see Definition 1. 1) and its subcategory, $LC_2$ determined by Hausdorff objects of $LC_1$.

It is shown that $LC_1$ contains the category $LC$ of locally convex spaces and continuous linear maps as a coreflective subcategory and that $LC_2$ contains the category $Norm_1$ of normed linear spaces and linear contractions as a coreflective subcategory.

Thus $LC_1$ is a suitable category for the study of locally convex spaces.

In $LC_2$, we introduce $l_\infty(I, E_i)$ for a family $(E_i)_{i\in I}$ of objects in $LC_2$ and then for an ultrafilter $\mathcal{U}$ on $I$. we have a closed subspace $N_{\mathcal{U}}$. Using this, we construct ultraproducts in $LC_2$.

Using the relationship between $Norm_1$ and $LC_2$ and that between $Norm_1$ and $Ban_1$, we show thatour ultraproducts in $Norm_1$ and $Ban_1$

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are exactly those in the literatures.

For the terminology, we refer to [6] for the category theory and to [8] for ultraproducts in $Ban_1$.

## 1. Category of locally convex spaces

Let $LC$ denote the category of locally convex spaces and continuous linear maps between them.

We recall that a topological vector space $E$ is locally convex iff the topology on $E$ is generated by a family of semi-norms on $E$.

For a locally convex space $(E, \mathfrak{T})$, let $(d_i)_{i\in I}$ be a family of semi-norms on $E$ which generats $\mathfrak{T}$. Then for $d_i$ and $d_k$ in $(d_i)_I$, $d_j \vee d_k$ is again a semi-norm on $E$. Let $(d_j)_{j\in J}$ be the smallest set of semi-norms on $E$ containing $(d_i)_I$ which is closed under finite joins. Then $\mathfrak{T}$ is also generated by the family $(d_j)_J$ and hence we may assume that $(d_i)_I$ is closed under finite joins.

Definition 1. 1. Let $(d_i)_{i\in I}$ and $(e_j)_{j\in J}$ be families of semi-norms on vector spaces $E$ and $F$, respectvely. A map $f: E \longrightarrow F$ is said to be a *linear contraction* on $(E, (d_i)_I)$ to $(F, (e_j)_J)$ if $f$ is linear and for each $j \in J$ there exists an $i \in I$ such that $e_j(f(x)) \leq d_i(x)$ for all $x \in E$.

It is clear that for any family $(d_i)_{i\in I}$ of semi-norms on a vector space $E$, the identity map $1_E : (E, (d_i)_I) \longrightarrow (E, (d_i)_I)$ is a linear contraction and that the composite of two linear contractions is again a linear contraction. Now we define a category $LC_1$ as follows: objects of $LC_1$ are all pairs $(E, (d_i)_{i\in I})$, where $E$ is a vector spaces and $(d_i)_I$ is a family of semi-norms on $E$ which is closed under finite joins; morphisms of $LC_1$ are all linear contractions between them. Clearly the category $Ban_1$ is a full subcategory of $LC_1$.

Remark 1. 2. Using the fact that for any $(E, (d_i)_{i\in I}) \in LC_1$, $(d_i)_I$ is closed und er finite joins, one can easily show that $LC_1$ is finitely complete.

We define $T: LC \longrightarrow LC_1$ as follows: for any $E \in LC$, $T(E) = (E, (d_i)_{i\in I})$, where $(d_i)_I$ is the set of all continuous semi-norms on $E$ and for any morphism $f$ in $LC$, $T(f) = f$. Then it is well known

[10] that for any $E, F \in LC$, a linear map $f : E \longrightarrow F$ is continuous iff $T(f) : T(E) \longrightarrow T(F)$ is a contraction. Thus $T$ is a full faithful functor. Moreover, $T$ is 1–1 on objects; hence $T : LC \longrightarrow LC_1$ is an embedding i. e., we may consider $LC$ as a subcategory of $LC_1$.

THEOREM 1. 3. *The functor $T$ has a right adjoint and hence $LC$ is bicoreflective in $LC_1$.*

*Proof.* Take any $(E, (d_i)_{i\in I})$ in $LC_1$. Let $\mathcal{T}$ be the topology generated by $(d_i)_I$. Then $(E, \mathcal{T}) \in LC$, and one can easily show that the identity map $1_E : T((E, \mathcal{T})) \longrightarrow (E, (d_i)_I)$ is the $T$-couniversal map for $(E, (d_i)_I)$. Thus $T$ has a right adjoint.

COROLLARY 1. 4. *$LC$ is closed under the formation of colimits in $LC_1$.*

Let $LC_2$ denote the full subcategory of $LC_1$ determined by those objects whose topology generated by the given semi-norms is a Hausdorff topology. It is known [10] that an object $(E, (d_i)_{i\in I})$ in $LC_1$ belongs to $LC_2$ iff $(d_i)_{i\in I}$ is total i. e., $d_i(x)=0$ for all $i\in I$ imply $x=0$.

THEOREM 1. 5. *The category $LC_2$ is epireflective in $LC_1$.*

*Proof.* Take any $(E, (d_i)_{i\in I})$ in $LC_1$ and let $K=\bigcap_{i\in I} d_i^{-1}(0)$. Then $K$ is clearly a linear subspace of $E$. Let $hE=E/K$ and define $\bar{d}_i : hE \longrightarrow \boldsymbol{R}$ by $\bar{d}_i([x])=d_i(x)$ for all $[x]\in hE$. By a simple calculation, $\bar{d}_i$ is indeed a semi-norm on $hE$ and $(hE, (\bar{d}_i)_{i\in I}) \in LC_2$. Moreover, the quotient map $q : E \longrightarrow hE$ is a linear contraction. Suppose $(F, (e_j)_{j\in J}) \in LC_2$ and $f : (E, (d_i)_I) \longrightarrow (F, (e_j)_J)$ is a morphism in $LC_1$. For $(x, y) \in \ker(q) = \{(a, b) \in E\times E \mid q(a)=q(b)\}$, $d_i(x-y)=0$ for all $i\in I$ and hence $e_j(f(x-y))=0$ for all $j\in J$. Thus $\ker(q) \subseteq \ker(f)$. So there exists a unique map $\bar{f} : hE \longrightarrow F$ with $f=\bar{f}\circ q$. It is easy to show that $\bar{f}$ is a linear contraction and hence $q$ is the $LC_2$-reflection of $(E, (d_i)_I)$. This completes the proof.

COROLLARY 1. 6. *The category $LC_2$ is closed under the formation of products and extremal subobjects in $LC_1$.*

Let $((E_i, (d_\lambda)_{\lambda\in\Lambda_i}))_{i\in I}$ be a family in $LC_1$ and let $l_\infty(I, E_i)=\{(x_i)$

$\in \Pi E_i |$for any $f \in \Pi A_i$, $\sup_{i\in I} d_{f(i)}(x_i) < \infty\}$. For each $f \in \Pi A_i$, we define $d_f : l_\infty(I, E_i) \longrightarrow \boldsymbol{R}$ by $d_f((x_i)) = \sup_{i\in I} d_{f(i)}(x_i)$ for all $(x_i) \in l_\infty(I, E_i)$. Then $l_\infty(I, E_i)$ is a linear subspace of $\Pi E_i$, $d_f$ is a semi-norm on $l_\infty(I, E_i)$ and $(d_f)_{f\in \Pi A_i}$ is closed under finite joins. Hence $(l_\infty(I, E_i), (d_f)_{f\in \Pi A_i})$ is an object of $LC_1$. Under the above notation, one has immediately the following:

REMARK 1.7. (1) *For each* $\alpha \in I$, *the projection map* $p_\alpha : (l_\infty(I, E_i), (d_f)_{f\in \Pi A_i}) \longrightarrow (E_\alpha, (d_\lambda)_{\lambda\in A_\alpha})$ *is a linear contraction.*

(2) *If each* $(E_i, (d_\lambda)_{\lambda\in A_i})$ *belongs to* $LC_2$, *then sodoes* $(l_\infty(I, E_i), (d_f)_{f\in \Pi A_i})$.

## 2. Ultraproducts in the category $LC_2$

In this section, generalizing ultraproducts, we introduce a concept of ultracolimits.

For any ultrafilter $\mathcal{U}$ on a set $I$, $(\mathcal{U}, \subseteq)$ is a poset and hence it will be considered as a category which will be again denoted by $\mathcal{U}$.

DEFINITION 2.1. Let $\boldsymbol{A}$ be a category and $\mathcal{U}$ an ultrafilter on a set $I$.

(1) A colimit $((q_J)_{J\in\mathcal{U}}, L)$ of a diagram $D : \mathcal{U}^{op} \longrightarrow \boldsymbol{A}$ is said to de a *D-ultracolimit.*

(2) Let $(A_i)_{i\in I}$ be a family in $\boldsymbol{A}$. Suppose $D : \mathcal{U}^{op} \longrightarrow \boldsymbol{A}$ is given as follows: for $J \in \mathcal{U}$, $D(J) = \prod_{j\in J} A_j$ and $D(J \longrightarrow K) = \prod_{j\in J} A_j \xrightarrow{p_{J,K}} \prod_{k\in K} A_k$, where $\prod_{j\in J} A_j$ is the product of $(A_j)_{j\in J}$ in $\boldsymbol{A}$ and $p_{J,K}$ is the projection. Then a $D$-ultracolimit $((q_J)_{J\in\mathcal{U}}, L)$ is said to be an *ultraproduct* of the family $(A_i)_{i\in I}$ with respect to $\mathcal{U}$ and we write $L = \prod_I {}_{\mathcal{U}} A_i$ or $\Pi_{\mathcal{U}} A_i$.

The following is now immediate from the definition

THEOREM 2.2. *If* $\boldsymbol{A}$ *is a cocomplete category and has products, then* $\boldsymbol{A}$ *has ultraproducts and hence every topological category and algebraic category have ultraproducts.*

Let $((E_i, (d_\lambda)_{\lambda\in A_i}))_{i\in I}$ be a family in $LC_2$ and $\mathcal{U}$ an ultrafilter on

$I$. For any $(x_i)\in l_\infty(I, E_i)$ and $f\in \Pi\Lambda_i$, $\{d_{f(i)}(x_i)\mid i\in I\}$ is bounded in $\boldsymbol{R}$ and hence $\lim_{\mathcal{U}} d_{f(i)}(x_i)$ exists (see [8]). The set $\{(x_i)\in l_\infty(I, E_i) \mid \lim_{\mathcal{U}} d_{f(i)}(x_i)=0$ for all $f\in \Pi\Lambda_i\}$ will be denoted dy $N_{\mathcal{U}}$.

PROPOSITION 2. 3. *$N_{\mathcal{U}}$ is a closed subspace of $l_\infty(I, E_i)$.*

*Proof.* By the properties of limits, it is clear that $N_{\mathcal{U}}$ is a linear subspace of $l_\infty(I, E_i)$. Take any net $((x_i^\alpha))_{\alpha\in D}$ in $N_{\mathcal{U}}$ such that $((x_i^\alpha))$ converges to $(x_i)$ in $l_\infty(I, E_i)$. Suppose that there is an $f\in \Pi\Lambda_i$ such that $\lim_{\mathcal{U}} d_{f(i)}(x_i)=r>0$. Then there exists a $\beta\in D$ such that $d_{f(i)}(x_i^\beta - x_i)<r/4$ for all $i\in I$. Since $\{i\in I\mid d_{f(i)}(x_i)>3r/4\}\in\mathcal{U}$, $\{i\in I\mid d_{f(i)}(x_i^\beta)>r/2\}\in\mathcal{U}$. But $\{i\in I\mid d_{f(i)}(x_i^\beta)<r/2|$ also belongs to $\mathcal{U}$, which is a contradiction.

By Proposition 2. 3, we now have the quotient space $l_\infty(I, E_i)/N_{\mathcal{U}}$ and we write $(x_i)_{\mathcal{U}}=(x_i)+N_{\mathcal{U}}$ for $(x_i)\in l_\infty(I, E_i)$. Moreover, for any $f\in \Pi\Lambda_i$, we define $\bar{d}_f((x_i)_{\mathcal{U}})=\inf\{d_{f(i)}((x_i)+(y_i))\mid (y_i)\in N_{\mathcal{U}}\}$. Then $\bar{d}_f$ is clearly a semi-norm on $l_\infty(I, E_i)/N_{\mathcal{U}}$ and $(\bar{d}_f)_{f\in \Pi\Lambda_i}$ is closed under finite joins. Since $N_{\mathcal{U}}$ is a closed subspace of $l_\infty(I, E_i)$, $(l_\infty(I, E_i)/N_{\mathcal{U}}, (\bar{d}_f)_{f\in \Pi\Lambda_i})$ is an object of $LC_2$ which will be also denoted by $l_\infty(I, E_i)N_{\mathcal{U}}$. Using the above notation, we have the following:

THEOREM 2. 4. *Forany $(x_i)\in l_\infty(I, E_i)$, $\bar{d}_f((x_i)_{\mathcal{U}})=\lim_{\mathcal{U}} d_{f(i)}(x_i)$.*

*Proof.* Let $\lim_{\mathcal{U}} d_{f(i)}(x_i)=r$. Take any $\varepsilon>0$ and let $I_\varepsilon=\{i\in I\mid r-\varepsilon<d_{f(i)}(x_i)<r+\varepsilon\}$. Then $I_\varepsilon\in\mathcal{U}$ and for $(y_i)\in N_{\mathcal{U}}$, let $I'=\{i\in I\mid r-\varepsilon<d_{f(i)}(x_i+y_i)\}$. Then $I'\supseteq I_{\varepsilon/2}\cap\{i\in I\mid d_{f(i)}(y_i)<\varepsilon/2\}\in\mathcal{U}$ and hence $\sup_I d_{f(i)}(x_i+y_i)\geq \sup_{I'} d_{f(i)}(x_i+y_i)\geq r-\varepsilon$. So $\bar{d}_f((x_i)_{\mathcal{U}})\geq r$. Let $z_i=0$ if $i\in I_\varepsilon$ and $z_i=-x_i$ if not. Then $(z_i)\in N_{\mathcal{U}}$. Thus $\bar{d}_f((x_i)_{\mathcal{U}})=\inf_{(z_i)\in N_{\mathcal{U}}}(\sup d_{f(i)}(x_i+z_i))\leq r+\varepsilon$. Hence $\bar{d}_f((x_i)_{\mathcal{U}})=\lim_{\mathcal{U}} d_{f(i)}(x_i)$.

Clearly for $K\subseteq J$ in $\mathcal{U}$, the projection $p_{J,K}: l_\infty(J, E_j)\longrightarrow l_\infty(K, E_k)$ is a linear contraction.

NOTATION 2. 5. For a family $(E_i)_{i\in I}$ in $LC_2$ and an ultrafilter $\mathcal{U}$ on $I$, let $D:\mathcal{U}^{op}\longrightarrow LC_2$ be given by $D(J\longrightarrow K)=l_\infty(J, E_j)\xrightarrow{p_{J,K}} l_\infty(K, E_k)$. Then $D$ is a functor. In the following, the $D$-ultracolimit will be denoted by $\Pi_{\mathcal{U}} E_i$ in $LC_2$ if it exists.

THEOREM 2. 6. *In the category* $LC_2$, $\Pi_{\mathcal{U}}E_i = l_\infty(I, E_i)/N_{\mathcal{U}}$.

*Proof.* Let $q_I : l_\infty(I, E_i) \longrightarrow l_\infty(I, E_i)/N_{\mathcal{U}}$ be the quotient map, i.e., $q_I((x_i)) = (x_i)_{\mathcal{U}}$. Then by the definition of semi-norms on $l_\infty(I, E_i)/N_{\mathcal{U}}$, $q_I$ is an $LC_2$-morphism. Take any $J \in \mathcal{U}$ and $((x_i), (y_i)) \in \ker(p_{I,J})$, then for all $i \in J$, $x_i = y_i$ and hence for any $f \in \Pi A_i$, $\lim_{\mathcal{U}} d_{f(i)}(x_i - y_i) = 0$, so that $(x_i - y_i) \in N_{\mathcal{U}}$. Hence there is a unique linear map $q_J : l_\infty(J, E_i) \longrightarrow l_\infty(I, E_i)/N_{\mathcal{U}}$ with $q_J \circ p_{I,J} = q_I$ for all $J \in \mathcal{U}$. Take any $f \in \Pi A_i$ and any $(y_j)_J \in l_\infty(J, E_i)$. Let $x_i = y_i$ if $i \in J$ and $x_i = 0$ if not, then clearly $p_{I,J}((x_i)_I) = (y_j)_J$ and one has: $\bar{d}_f(q_J((y_j))) = \bar{d}_f(q_J(p_{I,J}((x_i)))) = \bar{d}_f(q_I((x_i))) \le d_f((x_i)) = \sup_I d_{f(i)}(x_i) = \sup_J d_{f(i)}(y_i) = d_{f|J}((y_i)_J)$. Hence $q_J$ is an $LC_2$-morphism. Moreover, we have $q_k \circ p_{J,K} = q_J$ for all $K \subseteq J$ in $\mathcal{U}$. Suppose $((h_J)_{J \in \mathcal{U}}, (Y, (e_\alpha)_{\alpha \in H}))$ is a natural sink for $D$. Take any $(x_i)$ in $N_{\mathcal{U}}$. Then for any $\alpha \in H$, there exists an $f \in \Pi A_i$ such that $e_\alpha(h_I((a_i))) \le d_f((a_i))$ for all $(a_i) \in l_\infty(I, E_i)$ Take any $\varepsilon > 0$. Since $(x_i) \in N_{\mathcal{U}}$, there is a $K \in \mathcal{U}$ such that $\sup_K d_{f(i)}(x_i) < \varepsilon$. Let $b_i = x_i$ if $i \in K$ and $b_i = 0$ if not. Then $e_\alpha(h_I((x_i))) = e_\alpha(h_K \circ p_{I,K}((x_i))) = e_\alpha(h_K \circ p_{I,K}((b_i))) = e_\alpha(h_I((b_i))) \le d_f((b_i)) = \sup_K d_{f(i)}(x_i) < \varepsilon$, so that $e_\alpha(h_I((x_i))) = 0$ for all $\alpha \in H$. Since $(Y, (e_\alpha)_H)$ belongs to $LC_2$, $h_I((x_i)) = 0$; hence $N_{\mathcal{U}} \subseteq h_I^{-1}(0)$, so that there exists a unique map $\bar{h} : l_\infty(I, E_i)/N_{\mathcal{U}} \longrightarrow Y$ such that $\bar{h} \circ q_I = f_I$. Since $q_I$ is onto, $\bar{h}$ is linear. Take any $e_\alpha$ in $(e_\alpha)_{\alpha \in H}$. Then there exists an $f \in \Pi A_i$ such that $e_\alpha(h_I((a_i))) \le d_f((a_i))$ for all $(a_i) \in l_\infty(I, E_i)$. Take any $\varepsilon > 0$ and any $(x_i)_{\mathcal{U}} \in l_\infty(I, E_i)/N_{\mathcal{U}}$. Then there exists a $(y_i) \in N_{\mathcal{U}}$ with $d_f((x_i) + (y_i)) < \bar{d}_f((x_i)_{\mathcal{U}}) + \varepsilon$. Hence $e_\alpha(\bar{h}((x_i)_{\mathcal{U}})) = e_\alpha(\bar{h}(((x_i) + (y_i))_{\mathcal{U}})) = e_\alpha(h_I((x_i) + (y_i))) \le d_f((x_i) + (y_i)) \le \bar{d}_f((x_i)_{\mathcal{U}}) + \varepsilon$. Thus $e_\alpha(\bar{h}((x_i)_{\mathcal{U}})) \le \bar{d}_f((x_i)_{\mathcal{U}})$. Hence $\bar{h}$ is an $LC_2$-morphism. This completes the proof.

Let $Norm_1$ denote the category of normed linear spaces and linear contractions between them.

PROPOSITION 2. 7. *The category* $Norm_1$ *is coreflective in* $LC_2$.

*Proof.* Take any $(E, (d_i)_{i \in I})$ in $LC_2$. Let $F = \{x \in E \mid \sup_I d_i(x) < \infty\}$ and $d : F \longrightarrow \boldsymbol{R}$ be defined by $d(x) = \sup d_i(x)$ for all $x \in F$. By a routine calculation, $F$ is a linear subspace of $E$ and since $(d_i)_{i \in I}$ is total, $d$ is a norm on $F$. Let $j : F \longrightarrow E$ be the inclusion map. Then clearly $j$ is a linear contraction. Suppose $(X, \bar{d})$ is in $Norm_1$ and $f : (X, \bar{d}) \longrightarrow (E, (d_i)_{i \in I})$ is a linear contraction. Since for each $i \in I$, $d_i(f(x)) \leq \bar{d}(x)$ for all $x \in X$, $\sup d_i(f(x)) < \infty$ for all $x \in X$ and hence $f(X) \subseteq F$. Let $g : (X, \bar{d}) \longrightarrow (F, d)$ be the corestriction of $f$ to $E$, then clearly $g$ is a linear contraction and $j \circ g = f$. Since $j$ is 1-1, such a $g$ with $j \circ g = f$ is unique. Thus $j : F \longrightarrow E$ is the $Norm_1$-coreflection of $E$.

Corollary 2. 8. *$Norm_1$ is closed under formation of colimits in $LC_2$.*

For any family $((E_i, d_i))_{i \in I}$ in $Norm_1$, $l_\infty(I, E_i)$ is precisely the product $\{(x_i) \in \Pi E_i \mid \sup_I d_i(x_i) < \infty\}$ of the family in $Norm_1$ and $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, E_i) \mid \lim_{\mathcal{U}} d_i(x_i) = 0\}$.

Corollary 2. 9. *The ultraproduct of a family $(E_i)_{i \in I}$ with respect to $\mathcal{U}$ in $Norm_1$ is $l_\infty(I, E_i) / N_{\mathcal{U}}$.*

*Proof.* It is immediate from Theorem 2. 6 and Proposition 2. 7.

We note that $Ban_1$ is epireflective in $Norm_1$ and closed under formation of coequalizers is $Norm_1$, and that for any family $((E_i, d_i))_{i \in I}$ in $Ban_1$, $l_\infty(I, E_i)$ in $Norm_1$ (or $LC_2$) is precisely the product of the family in $Ban_1$. Thus one has the following.

Corollary 2. 10. ([8]) *For any family $((E_i, d_i))_{i \in I}$ in $Ban_1$ and an ultrafilter $\mathcal{U}$ on $I$, the ultraproduct of the family in $Ban_1$ is given by $l_\infty(I, E_i) / N_{\mathcal{U}}$.*

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