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THE CONTINUITY OF DERIVATIONS AND MODULE HOMOMORPHISMS*

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1. Introduction

If T is a linear operator from a Banach space X into a Banach space Y, we let $S(T)=\{y\in Y \mid \text{there is a sequence } \{x_n\} \text{ in } X \text{ with } x_n\to 0 \text{ and } Tx_n\to y\}$ and call it the separating space of T. By Closed Graph Theorem, T is continuous if and only if $S(T)=\{0\}$. A derivation D from a Banach algebra A to a Banach algebra A-module X is the linear map from A to X which satisfies the identity

$$D(ab)=aDb+(Da)b\,(a, b\in A).$$

It is easily checked that $S(D)$ is a closed submodule of X. The continuity ideal for a derivation $D: A\longrightarrow X$ is

$$\mathfrak{I}(D)=\{a\in A \mid aS(D)=\{0\}\}.$$

Clearly $\mathfrak{I}(D)$ is a closed ideal in A. In Section 2, we show that if A is a Banach algebra satisfying some conditions, then every derivation from A to any Banach A-module is continuous. In Section 3, we define the k-differential subspace W_k of X and prove that if $D: C^n[0,1]\longrightarrow X$ is a discontinuous derivation, $F=\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the hull of $\mathfrak{I}(D)$ and if for some $k, 1\le k\le n$, $D(z)\in W_k$, then the following are equivalent

(1) $D(\mathfrak{I}(D))\subset W_k$,
(2) $\cap_{i=1}^m M_{n,n-k}(\lambda_i)\subset\mathfrak{I}(D)$,
(3) $D(C^n[0,1])\subset W_k$.

Also we show that if $D: C^n[0,1]\longrightarrow X$ is continuous and if $D(z)$ is an eigenvector for $\rho(z)$, then D is determined by $D(z)$ such that $D(f)=f'(\lambda)D(z)$, $\lambda\in[0,1]$ and $f\in C^n[0,1]$. In Section 4, we show that if $L(M)=l^1(w)$, then every module homomorphism from M into any $l^1(w)$-module is continuous, where w is a weight function.

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2. Derivations

We need the following lemmas which is found in [8, 10] to prove our main result.

LEMMA 2.1. *Let X and Y be Banach spaces, let U be a linear mapping from X into Y, and let T_n and R_n be sequences of bounded linear mappings on X and Y respectively, such that $UT_n - R_nU$ is continuous for all n. Then there is a natural number N such that*

$$(R_1 \cdots R_n S(U))^- = (R_1 \cdots R_N S(U))^-, \quad (n \geq N).$$

LEMMA 2.2. *Let A be a separable commutative Banach algebra, X a Banach A-module and $D: A \longrightarrow X$ a derivation with the continuity ideal $\mathfrak{I}(D)$. Then D is continuous on $\mathfrak{I}(D)^2$ if $\mathfrak{I}(D)^2$ is closed.*

LEMMA 2.3. *Let A be a commutative Banach algebra with identity and X a Banach A-module. Let $D: A \longrightarrow X$ be a discontinuous derivation with the separating space $S(D)$ and the continuity ideal $\mathfrak{I}(D)$. Then there is a discontinuous derivation $D_0: A \longrightarrow X$ with the separating space $S(D_0)$ and the continuity ideal $\mathfrak{I}(D_0)$ satisfying*

(1) *$D = a_0 D$ for some $a_0 \in A$.*

(2) *For each $a \in A$, either $(aS(D_0))^- = S(D_0)$ or $aS(D_0) = \{0\}$.*

(3) *$\mathfrak{I}(D_0) \supseteq \mathfrak{I}(D)$ and $\mathfrak{I}(D_0)$ is a prime ideal of A.*

Christensen [6] has shown that if A is a separable Banach algebra such that A^2 has finite codimension in A, then A^2 is closed.

THEOREM 2.4. *Let A be a separable commutative Banach algebra satisfying the following conditions;*

(1) *If I is a closed prime ideal of infinite codimension in A, then there is sequences $\{a_n\}$, $\{b_n\}$ in A satisfying $b_n a_1 \cdots a_{n-1} \notin I$ and $b_n a_1 \cdots a_n \in I$ for all $n \geq 2$.*

(2) *For every maximal ideal M, M^2 is of finite condimension in A.*

Then every derivation from A into a Banach A-module is continuous.

Proof. We may assume that A has an identity. Suppose that D is a discontinuous derivation from A into a Banach A-module X. By Lemma 3, there is a discontinuous derivation $D_0: A \longrightarrow X$ with the continuity ideal $\mathfrak{I}(D_0)$ which is a closed prime ideal. We claim that

$\mathfrak{J}(D_0)$ has a finite codimension in A. In fact, if $\mathfrak{J}(D_0)$ is of infinite codimension in A, then there is a subsequences $\{a_n\}$, $\{b_n\}$ in A such that $b_n a_1 \cdots a_{n-1} \notin \mathfrak{J}(D_0)$ and$b_n a_1 \cdots a_n \in \mathfrak{J}(D_0)$ for all $n \geq 2$. Let $T_n a = a_n a$ for all $a \in A$, $R_n x = x$ and $U_n x = b_n x$ for all $x \in X$. Then for each n,

$$(D_0 T_n - R_n D_0)(a) = D_0(a_n a) - a_n D_0(a) = (D_0 a_n) a.$$

Thus $D_0 T_n - R_n D_0$ is continuous for all n. On the other hand

$$U_n R_1 \cdots R_n S(D_0) = (b_n a_1 \cdots a_n) S(D_0) = \{0\}.$$

But

$$U_n R_1 \cdots R_{n-1} S(D_0) = (b_n a_1 \cdots a_{n-1}) S(D_0) \neq \{0\} \qquad \text{for all } n \geq 2.$$

This is a contradiction to the Lemma 2.1. Therefore $\mathfrak{J}(D_0)$ is a closed prime ideal having finite codimension in A, and so $\mathfrak{J}(D_0)$ is maximal. By the condition (3) $\mathfrak{J}(D_0)^2$ is of finite codimension in A. Christensen's Theorem implies that $\mathfrak{J}(D_0)^2$ is closed. By the Lemma 2.2, D_0 is continuous on $\mathfrak{J}(D_0)^2$. Since $\mathfrak{J}(D_0)^2$ is of finite codimension, D_0 is continuous in A. This is also a contradiction to the discontinuity of D_0. We complete the proof.

Remark. The condition (2) of Theorem 2.4 is necessary because if A has a maximal ideal M such that M^2 is not of finite codimension in A, then there is a discontinuous derivation from A into $\boldsymbol{C}$, the field of complex numbers [11].

3. Derivations on $C^n[0,1]$

Let $C^n[0,1]$ denote the algebra of all complex valued functions on $[0,1]$ which has n continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$\|f\|_n = \max_{t \in [0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is $[0,1]$ and also $C^n[0,1]$ is singly generated by $z(t) = t$.

We use the notation

$$M_{n,k}(\lambda) = \{f \in C^n[0,1] \mid f^{(j)}(\lambda) = 0, \ j = 0, 1, \ldots, k\}, \ \lambda \in [0,1].$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(\lambda)$ of functions vanishing at λ. A Banach $C^n[0,1]$-module is a Banach space X together with a continuous homomorphism

$$\rho : C^n[0,1] \longrightarrow B(X).$$

DEFINITIONS 3.1. *Let X be a Banach $C^n[0,1]$-module. The k-differential subspace is the set $W_k (0 \leq k \leq n)$ of all vectors x such that the map*

$$p \longrightarrow \rho(p')x$$

is continuous for the $C^{n-k+1}[0,1]$ norm on P, where P is the dense subalgebra of polynomials in $C^n[0,1]$

LEMMA 3.2. *Let X be a $C^n[0,1]$-module. A vector x lies in the k-differential subspace W_k if and only if the map*

$$p \longrightarrow \rho(p)x$$

is continuous for the $C^{n-k}[0,1]$ norm on P.

Proof. We use the elementary inequality

$$\|p'\|_{n-k} \leq (n-k+1)\|p\|_{n-k+1}, \quad p \in P,$$

$$\frac{1}{2}\|p\|_{n-k+1} \leq \|p'\|_{n-k}, \quad p \in P \cap M_{n,0}(0).$$

If $x \in W_k$, there exists a constant $L > 0$ such that

$$\|\rho(p')x\| \leq L\|p\|_{n-k+1}, \quad p \in P.$$

Let $q = p - p(0)$, for some $p \in P$. Then $q \in P \cap M_{n,0}(0)$ and

$$\|\rho(p')x\| \leq L\|q\|_{n-k+1} \leq 2L\|q'\|_{n-k} = 2L\|p'\|_{n-k}.$$

However, such p' exhaust P.

Conversely, suppose that $\|\rho(p)x\| \leq M\|p\|_{n-k}$, $p \in P$. Then

$$\|\rho(p')x\| \leq M\|p'\|_{n-k}$$
$$\leq (n-k+1)M\|p\|_{n-k+1},$$

so $x \in W_k$.

The k-differential subspace W_k occurs in the work of S. Kantorovitz [9]. Bade and Curtis proved the Lemma 3.2 in the case $k=1$ [2]. Note that $W_n \subset W_{n-1} \subset \cdots \subset W_0 = X$, and $\rho(p)x \in W_k$ if $x \in W_k, p \in P$. A nontrivial derivation $D : C^n[0,1] \longrightarrow X$ will be called singular if D vanishes on P (equivalently $D(z) = 0$). A singular derivation is, of course, discontinuous. A derivation D is decomposable if D can be expressed in the form $D = E + F$, where E is continuous and F is singular. We need the following lemmas which is found in [2].

LEMMA 3.3. *A derivation $D : C^n[0,1] \longrightarrow X$ is decomposable if and only if $D(z) \in W_1$. If D is decomposable and $D = E + F$, then its sing-*

ular part F vanishes also on $\mathfrak{J}(D)^2$.

LEMMA 3.4. *If D is a continuous derivation of $C^n[0,1] \longrightarrow X$, then $D(z) \in W_1$ and*

$$D(f) = \gamma(f')D(z) \text{ for all } f \in C^n[0,1],$$

where $r : C^{n-1}[0,1] \longrightarrow B(W_1)$ is a unique continuous homomorphism under $C^{n-1}[0,1]$ norm such that $\gamma(p)x = \rho(p)x$, for all $x \in W_1$, $p \in P$.

It is well known that if $D : C^n[0,1] \longrightarrow X$ is a discontinuous derivation, then the hull of $\mathfrak{J}(D)$ is finite [2].

THEOREM 3.5. *Let $D : C^n[0,1] \longrightarrow X$ be a discontinuous derivation and let $F = \{\lambda_1, \lambda_2, \dots \lambda_m\}$ be the hull of $\mathfrak{J}(D)$. If for some k, $1 \le k \le n$, $D(z) \in W_k$, then the following are equivalent;*

(1) $D(\mathfrak{J}(D)) \subset W_k$,

(2) $\bigcap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \mathfrak{J}(D)$,

(3) $D(C^n[0,1]) \subset W_k$.

Proof. (1)$\Rightarrow$(2). Since $D(z) \in W_k \subset W_1$, by Lemma 3.3, $D = E + F$ where E is continuous and F is singular. By Lemma 3.4, $D(f) = E(f) + F(f) = \gamma(f')E(z) + F(f)$ for all $f \in C^n[0,1]$. If $D(f) \in W_k$, for $f \in \mathfrak{J}(D)$, then $F(f) \in W_k$. By Theorem 3.2 in [2], $(z-\lambda_1)^n(z-\lambda_2)^n \cdots (z-\lambda_m)^n \in \mathfrak{J}(D)$. Since $\mathfrak{J}(D) = \mathfrak{J}(F)$,

$\rho((z-\lambda_1)^n \cdots (z-\lambda_m)^n)F(f) = \gamma_1(f')\rho((z-\lambda_1)^n \cdots (z-\lambda_m)^n)F(z) = 0$ for all $f \in \mathfrak{J}(D)$. Let $y = \rho((z-\lambda_2)^n \cdots (z-\lambda_m)^n)F(f)$, for $f \in \mathfrak{J}(D)$. If $\rho(z-\lambda_1)^l y \neq 0$, but $\rho(z-\lambda_1)^{l+1}y = 0$, then for any polynomial p,

$$\rho(p)y = \sum_{i=0}^{l} \frac{p^{(i)}(\lambda)}{i!}(\rho(z) - \lambda_1)^i y.$$

And the vectors

$$y,\ (\rho(z)-\lambda_1)y,\ \dots,\ (\rho(z)-\lambda_1)^l y$$

are linearly independent. By Lemma 3.2, $\|\rho(p)y\| \le M\|p\|_{n-k}$, $M>0$. Therefore $l \le n-k$. *Then* $\rho(z-\lambda_1)^{n-k+1}y = 0$. If we continue this process, for all $\lambda_2, \dots, \lambda_m$,

$$\rho((z-\lambda_1)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1})F(f) = 0.$$

Since $\mathfrak{J}(D)$ is of finite codimension, for all $f \in C^n[0,1]$,

$$\rho((z-\lambda_1)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1})F(f) = 0.$$

Thus $(z-\lambda_1)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1} \in \mathfrak{J}(F) = \mathfrak{J}(D)$.

(2) $\Rightarrow$ (3). Choose $e_k \in C^n[0, 1]$, $k=1, 2, \dots, m$ such that $e_k(\lambda)=1$ in a neighborhood of λ_k and $e_k(\lambda)=0$ in a neighborhood of $F-\{\lambda_k\}$. Let $e_0=1-\sum_{i=1}^m e_i$. Then

$$e_0 \in \cap_{i=1}^m M_{n,n}(\lambda_i) \subset \Im(D),$$

$$D(f)=\sum_{i=0}^m \rho(e_i) D(f), \quad f \in C^n[0, 1].$$

Let $D_i(\cdot)=\rho(e_i) D(\cdot)$. Then D_0 is continuous and D_i is discontinuous $(i=1, 2, \dots, m)$. We have

$$\text{hull}(\Im(D_i))=\{\lambda_i\} \ (i=1, 2, \dots, m)$$

and

$$\Im(D)=\cap_{i=1}^m \Im(D_i).$$

Suppose $D(z) \in W_k$ and $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$. Since $e_j \in M_{n,n-k}(\lambda_i)$, if $i \neq j$,

$$g e_j \in \cap_{i=1}^m M_{n,n-k}(\lambda_i) \text{ for } g \in M_{n,n-k}(\lambda_i).$$

Hence

$$\rho(g) D_j(\cdot)=\rho(g)\rho(e_j) D(\cdot)$$
$$=\rho(g e_i) D(\cdot)$$

is continuous. Note that

$$M_{n,n-k}(\lambda_i) \subset \Im(D_j) \ (j=1, 2, \dots, m).$$

Since $D(z) \in W_k$, $D_j(z) \in W_k$ $(j=1, 2, \dots, m)$. Thus it suffices to prove the theorem when the hull of $\Im(D)$ is a single point, which we may suppose to be zero. Since $D(z) \in W_k \subset W_1$, by Lemma 3. 3, we have $D=E+F$ where E is continuous and F is singular. Since $D(z)=E(z) \in W_k$, $E(C^n[0, 1]) \subset W_k$. From $\Im(D)=\Im(F)$,

$$M_{n,n-k}(0) \subset \Im(F).$$

So $z^{n-k+1} \in \Im(F)$. For all $f \in C^n[0, 1]$,

$$\rho(z^{n-k+1}) F(f)=\gamma_1(f')\rho(z^{n-k+1}) F(z)=0.$$

For $f \in C^n[0, 1]$, $p \in P$,

$$\|\rho(p) F(f)\|=\|\rho(p(0)+P'(0) z+\cdots+\frac{p^{(n-k)}(0)}{(n-k)!} z^{n-k}) F(f)\|$$
$$\leq L\|p\|_{n-k}.$$

Thus $F(f) \in W_k$ for all $f \in C^n[0, 1]$, Therefore we complete the proof.

THEOREM 3. 6. *Let $D: C^n[0, 1] \longrightarrow X$ be a continuous derivation. Then $D(z)$ is an eigenvector of $\rho(z)$ if and only if $D(f)=f'(\lambda) D(z)$ for some eigenvalue λ.*

Proof. If $D(z)$ is an eigenvector of $\rho(z)$, then $\rho(z-\lambda)D(z)=0$ for some eigenvalue λ of $\rho(z)$. Since for all $p\in P, D(p)=\rho(p')D(z)=p'(\lambda)D(z)$, we have

$$D(f)=\gamma_1(f')D(z)=f'(\lambda)D(z) \text{ for all } f\in C^n[0,1].$$

Conversely, suppose $D(f)=f'(\lambda)D(z)$ for all $f\in C^n[0,1]$. Let $p(z)=\alpha_1(z-\lambda)+\alpha_2(z-\lambda)^2$, $(\alpha_1,\alpha_2\neq 0)$. Then

$$D(p)=\rho(\alpha_1+2\alpha_2(z-\lambda))D(z)=p'(\lambda)D(z)$$

and so $\rho(z-\lambda)D(z)=0$.

4. Module homomorphisms

Let A be a Banach algebra, X a right A-module and T a module homorphism from A into X. If we make X into an A-module by defining $ax=0$, $(a\in A,\ x\in X)$, then T becomes a derivation. Thus if every module derivation from A into any A-module is continuous, then T is continuous. The following lemma is another version of Lemma 3. 2 in [**12**]

LEMMA 4. 1. *Let A be a Banach algebra with unit and X a Banach A-module. If every module homomorphism from X into A^* (regard as a A-module under the dual action) is continuous, then every module homomorphism from X into any Banach A-module is continuous.*

PROOF. Let Y be a Banach A-module and $T: X\to Y$ a module homomorphism. If $y\in S(T)$, we can choose $f\in Y^*$ such that $f(y)=\|y\|$ from Hahn-Banach Theorem. Define $R_f: Y\to B^*$ by $R_f(y)(a)=f(ay)$. Then R_f is a bounded linear operator and a module homomorphism because of a dual action. Thus $R_fT: Z\to A^*$ is a A-module homomorphism, and so continuous. Since $y\in S(T)$, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n=0$ and $\lim_{n\to\infty} T(x_n)=y$. Thus

$$\begin{aligned} 0&=\lim_{n\to\infty} R_fT(x_n)\\ &=\lim_{n\to\infty} f(Tx_n)\\ &=f(y)\\ &=\|y\| \end{aligned}$$

and so $y=0$. Therefore $S(T)=\{0\}$ and hence T is continuous.

A real valued function w defined on $Z^+=\{n\in Z\,|\,n\geq 0\}$ is a weight function if $w(n)>0$ $(n\in Z^+)$ and if

$$w(m+n)\leq w(m)w(n), \quad (m, m\in Z^+).$$

For convenience we let

$$l^1(w)=\{x=(x(0), x(1), \dots) \mid \sum_{n=0}^{\infty} |x(n)|w(n)<\infty\}$$

$$l^\infty(w^{-1})=\left\{y=(y(0), y(1), \dots) \mid \left\{\frac{|y(n)|}{w(n)}\right\} \text{ is bounded}\right\}$$

$$M=\{x\in l^1(w) \mid x(0)=0\}$$

$$Lx=(x(1), x(2), \dots) \text{ if } x=(x(0), x(1), \dots)\in l^1(w)$$

$$Rx=(0, x(0), x(1), \dots) \text{ if } x=(x(0), x(1), \dots)\in l^1(w)$$

Then it is well known that $(l^1(w), \|\cdot\|)$ and $(l^\infty(w^{-1}), \|\cdot\|_\infty)$ are Banach algebras of power series, where

$$\|x\|=\sum_{n=0}^{\infty}|x(n)|w(n)$$

$$\|y\|_\infty=\sup\left\{\frac{|y(n)|}{w(n)} \mid n\in Z^+\right\}$$

and M is a closed ideal of $l^1(w)$ [4, 5]. But M has neither an identity nor abounded approximate identity. Also $l^1(w)^*=l^\infty(w^{-1})$ the duality is implemented by

$$\langle x, y\rangle=\sum_{n=0}^{\infty}x(n)y(n).$$

By Lemma 4. 1, if every module homomorphismfrom $l^1(w)$-module X into $l^\infty(w^{-1})$ is continuous, then every module homomorphism from X into any $l^1(w)$-module is continuous.

THEOREM 4. 2. *Let w be a weight function. If $L(M)=l^1(w)$, then every module homomorphism from M into any Banach $l^1(w)$-module is continuous.*

Proof. Let X be a Banach $l^1(w)$-module and let $T: M\to X$ be a module homomorphism. Note that R is a module homomorphism. Since $L(M)=l^1(w)$, $R: l^1(w)\to M$ is a surjective module homomorphism and $\|R\|\leq w(1)$. By Open Mapping Theorem, $L: M\to l^1(w)$ is a continuous map. Then TR is a module homomorphism from $l^1(w)$ into X. Since $l^1(w)$ has an identity, TR is continuous. If $x_n\to 0$, then

$$T(x_n)=T(RL(x_n))=TR(L(x_n))\to 0.$$

By the Closed Graph Theorem, T is continuous.

REMARK. If $w(n)=1$, then $L(M)=l^1(w)$. If $w(n)=\exp(-n^2)$, then $L(M)\supsetneq l^1(w)$. Also if $L(M)=l^1(w)$, then $l^1(w)$ is semisimple [5]. The following corollary is well known.

COROLLARY 4. 3. *If $L(M)=l^1(w)$ and $T: M \longrightarrow M$ is a multiplier, then T is continuous.*

Proof. If T is a multiplier on M, then T is a module homomorphism. By Theorem 4. 2, T is continuous.

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