Evaluation of Probabilistic Finite Element Method in Comparison with Monte Carlo Simulation

Lee, Jae Young, Koh, Hong Seok
Department of Agricultural Engineering, Chonbuk National University

Abstract □ The formulation of the probabilistic finite element method was briefly reviewed. The method was implemented into a computer program for frame analysis which has the same analogy as finite element analysis. Another program for Monte Carlo simulation of finite element analysis was written. Two sample structures were assumed and analized. The characteristics of the second moment statistics obtained by the probabilistic finite element method was examined through numerical studies. The applicability and limitation of the method were also evaluated in comparison with the data generated by Monte Carlo simulation.

Keywords □ Structural Analysis, Structural Reliability, Finite Element Method, Probabilistic Finite Element Method, Stochastic Finite Element Method, Monte Carlo Simulation

I. INTRODUCTION

The technology of the finite element method has advanced rapidly since the 1950's. The method is now accepted as the most powerful technique for the numerical solution of a variety problems arising in engineering and science. In structural mechanics in particular, the method has been so diversified and generalized as to cover most of the problems encountered in a practical design situation. Many efforts are still going on to facilitate its application and also to achieve more realistic modelling of structural behavior. Most of the developments in finite element method have so far been within the category of deterministic analysis, which premises deterministic structural characteristics and deterministic structural behavior. In reality, however, there always exists a variability and uncertainty in material properties, load conditions and other factors affecting structural behavior. The variability and uncertainty will sometimes be significant enough to blur the accuracy of the rigorously sophisticated deterministic analysis. In such cases, it is more reasonable to present solutions in probabilistic terms so that it can take account of the stochastic nature of the structural characteristics. This fact has motivated the probability-based method of analysis proposed recently under the name of *'probabilistic finite element method'* or *'stochastic finite element method'*. § The probabilistic finite element method is intended to reflect the variability of input data to the analysis and to obtain information about the variability of the output from the analysis. For example, in case that the mean and the standard deviation of the material properties are known, the probabilistic finite element method produces, as output results, the standard deviation as well as the mean of displacements and stresses. This is an advantage of probabilistic analysis over the deterministic one which involves only the mean values, truncating the additional information.

It is also possible by probabilistic finite element method to extrapolate the variability observed in small and simple test specimens to a large and/or complex structural system. It suggests that the method can be used to estimate the *a priori* distribution of structural strength. The main target of the method is to estimate the variability of the structural response under given variability of structural characteristics, and thus to evaluate the risk of structural failure which is generally expressed in terms of safety index.

The current probabilistic finite element

§ The probabilistic finite element method and the stochastic finite element method are identical one. The term 'probabilistic' rather than 'stochastic' is used in this paper, because current 'stochastic' finite element method is based on second moment statistics rather than full probability distribution functions.

method has some negative features as well. The method is in a second moment format, and therefore, produces no specific information other than second moment statistics. The method is derived on the basis of first order approximation which is higher than the first order terms in Taylor's expansion are neglected. Consequently, it's applicability may be questionable under the situation of large variability.

On the other hand, more specific information on the probability distribution of structural behavior may be infered simply by Monte Carlo simulation. The simulation with finite element method is not practical, if not infeasible, chiefly due to prohibitive computing time. However, the simulation may be used as a proper tool to calibrate the probabilistic finite element method.

In this study, formulation of the probabilistic finite element method in a second moment format was briefly reviewed. A computer program was written to implement and to test the formulation for linear analysis. The second moment statistics of the computed results was examined through numerical studies. They were compared with the results obtained from Monte Carlo simulation. The applicability and limitation of the probabilistic finite element method was evaluated on the basis of this comparison.

II. PREVIOUS WORKS IN PROBABILISTIC FINITE ELEMENT METHOD

In order to take account of the stochastic nature in structural analysis, Monte Carlo simulation was frequently employed (Baratts, 1979, DeBonis, 1980). However, it required prohibitive computing time. This fact might have stimulated the efforts to embed the treatment of uncertainty within the finite element method. Hart and Collins (1970) suggested a statistical approach based on the linearization of variations about the mean values obtained from deterministic finite element analysis. The linearization is understood as a first order approximation by a Taylor's series expansion of given functions. They presented some of the basic principles in the treatment of randomness in finite element modelling. They verified the validity of their statistical model by a Monte Carlo simulation. Cambou (1975) applied a similar approach for treatment of uncertainty in soil or rock mechanics problems. Ingra and Baecher (1978) applied Hart and Collins' formulation to a two dimensional finite element model to study the settlement of foundation subject to a uniform vertical strip load. Handa and Anderson's formulation (1981) was also based on a similar approach like Hart and Collin's. But they expressed the covariance matrices of the output variables in explicit forms, and related the analysis results to risk analysis of structural failure. Hisada and Nakagiri (1981) applied perturbation technique for stochastic treatment of finite element method for two dimensional continuum structures with uncertainty in structural geometries.

Cambou (1975) suggested treatment of uncertainty from various sources : uncertainty in continuum characteristics, load conditions, boundary conditions and calculation method. However, he showed, in a case study, that the uncertainty in Young's modulus and load conditions had the greatest influence on the overall uncertainty which affects the analysis results. Some of the later works by others (Handa and Andersson, 1981, Hisada and Nakagiri, 1981, Ingra and Baecher, 1978) also focused mainly on the uncertainty in these two factors. Ingra and Baecher (1978) indicated that treating Poisson's ratio deterministically simplifies the computation significantly, and showed qualitatively the relative insensitivity of results to Poisson's ratio. Hisada and Nakagiri (1981) considered the uncertainty in boundary conditions. The uncertainty in calculation method was neglected in most of the reviewed literatures.

The random elastic characteristics of each finite element are represented by their local average over the element. Ingra and Baecher (1978) employed an exponential decay autocorrelation function for spatial correlation of soil properties. Vanmarcke and Grigoriu (1983) introduced the concept of variance function in a spatial random field, and exemplified its application to a stochastic finite element analysis of a simple beam. Vanmarcke (1981) suggested a simplified treatment of the correlation function of the random material property in terms of a single parameter called *'the scale of fluctuation'*. He applied the scale of fluctuation to obtain the covariances and the local averages over rectangular elements. He also mentioned the possibility of extending the methodology to non-rectangular elements.

The probabilistic finite element method is not so much extensively developed, at present, as the deterministic ones. This may be partly due to the difficulties inherent in probabilistic analysis, and partly due to failure of implementing the method as a useful and practical tool for probability-based structural design and analysis. Application of the method has been restricted to specific research purposes. Previous works in probabilistic finite element method have been inclined toward probabilistic treatment of uncertainty in properties of finite element and its responses. Not much attention has been paid to generalization or efficient implementation of the method in practice. The method has been restricted to linear elastic analysis, even though most of the structures, around failure region, are subject to large deformation or nonlinear stress-strain behavior.

III. FORMULATION OF THE METHOD

1. Brief Summary of Finite Element Method

The finite element method can be summarized as a procedure to assemble and solve a system of simultaneous equations of the form,

$$[K]\{U\} = \{F\} \tag{1}$$

where
$\{F\}$ = force vector
$[K]$ = stiffness matrix
$\{U\}$ = displacement vector

The displacement vector is the basic unknowns. The force vector and the stiffness matrix are obtained by assembling element nodal force vectors and element stiffness matrices, respectively. An element stiffness matrix is determined by integration over the element domain. That is

$$[k] = \int_\Omega [B]^T[D][B]\,d\Omega \tag{2}$$

where
$[k]$ = element stiffness matrix
$[B]$ = strain-displacement matrix
$[D]$ = constitutive matrix
Ω = element domain (length, area or volume of the element)

The matrices [B] and [D] are determined by structural geometries and material properties which are given as input data. Load conditions are also given as input data. Accordingly, [K] and $\{F\}$ in the system equation are obtained from given information. The system equation is solved to find the unknown nodal displacements $\{U\}$. Once the displacement vector is determined, stresses at an arbitrary point within an element are explicitly given by the stress vector

$$\{q\} = [D][B]\{u\}, \tag{3}$$

because $\{U\}$ consists of the nodal displacement vector $\{u\}$ for each element. The nodal stresses are determined after stress smoothing processes.

2. Probabilistic Finite Element Method

In probabilistic finite element method, all or part of the inputs are considered to have probabilistic nature. Those input variables which are probabilistic are defined as input random variables. In a second moment approach, not only their means but also their variances and covariances are given as input data. The output variables, namely displacements and stresses also inevitably have probabilistic nature and thus are defined as output random variables. Therefore, the variances and the covariances as well as the means of the output random variables should be obtained from the analysis.

Random variables, either input or output, can be decomposed into the deterministic part and the probabilistic part. Suppose that input random variables compose a random vector $\{X\} = \lfloor X_1 X_2 \cdots X_n \rfloor^T$, where n is the total number of input random variables. The ith input random variable X_i is decomposed into the deterministic part X_i° and the probabilistic part δX_i. That is

$$X_i = X_i^\circ + \delta X_i \quad (i=1,\cdots,n) \tag{4}$$

The value of any random variable evaluated at $X_i = X_i^\circ (\forall i)$ is also defined as the deterministic part of the random variable, and is denoted with superfix $^\circ$. For example, $[K^\circ]$ and $[\frac{\partial K^\circ}{\partial X}]$ represent the deterministic parts of the stiffness matrix and of its derivative, respectively. By such definition of $\{F^\circ\}$, $[K^\circ]$ and $\{U^\circ\}$, the relation

$$[K^{\circ}]\{U^{\circ}\}=\{F^{\circ}\} \tag{5}$$

is equivalent to the system equation of deterministic analysis.

Output random variables, namly $\{U\}$ and $\{q\}$ can be expanded about their deterministic values as follows :

$$\{U\}=\{U^{\circ}\}+\sum_{i,j}\{\frac{\partial U}{\partial X_i}\}\delta X_i + \frac{1}{2}\sum_{i,j}\{\frac{\partial^2 U}{\partial X_i \partial X_j}\}\delta X_i \delta X_j + \cdots$$

$$\{q\}=\{q^{\circ}\}+\sum_{i,j}\{\frac{\partial q}{\partial X_i}\}\delta X_i + \frac{1}{2}\sum_{i,j}\{\frac{\partial^2 q}{\partial X_i \partial X_j}\}\delta X_i \delta X_j + \cdots \tag{6}$$

Neglecting higher than first order terms, equations (6) can be simplified into the following matrix notations.

$$\{U\}=\{U^{\circ}\}+[\frac{\partial U}{\partial X}]\{\delta X\}$$

$$\{q\}=\{q^{\circ}\}+[\frac{\partial q}{\partial X}]\{\delta X\} \tag{7}$$

where $[\frac{\partial U}{\partial X}]$ and $[\frac{\partial q}{\partial X}]$ will be defined later. Equation (7) leads to first order approxim ation. The mean of displacements is approxim ated as

$$\{\mu_U\}=E[\{U\}]=\{U^{\circ}\} \tag{8}$$

which is equivalent to the deterministic part of the displacement vector. The covariance matrix of displacement is given by[5]

$$[\sigma_U]=E[(\{U\}-\{\mu_U\})(\{U\}-\{\mu_U\})^T] = [\frac{\partial U}{\partial X}][\sigma_X][\frac{\partial U}{\partial X}]^T \tag{9}$$

where $[\sigma_X]$ denotes covariance matrix of input random variables, which is also given as input data. Thus, the problem is reduced to that of evaluating $[\frac{\partial U}{\partial X}]$. Differentiation of equation (1) with respect of each of the input random variables leads to

$$[\frac{\partial K}{\partial X_i}]\{U\}+[K]\{\frac{\partial U}{\partial X_i}\}=\{\frac{\partial F}{\partial X_i}\} \tag{10}$$

where $[\frac{\partial U}{\partial X_i}]$ should be distinguished from $[\frac{\partial U}{\partial X}]$ in equation (9). Rearranging the equation, one obtains

$$[K]\{\frac{\partial U}{\partial X_i}\}=\{\frac{\partial F}{\partial X_i}\}-[\frac{\partial K}{\partial X_i}]\{U\}. \tag{11}$$

Let

$$\{F'_i\}=\{\frac{\partial F}{\partial X_i}\}-[\frac{\partial K}{\partial X_i}]\{U\}.$$

Then,

$$[K]\{\frac{\partial U}{\partial X_i}\}\ \{F'_i\} \tag{12}$$

Equation (12) is in the same form as the system equation (1). The right hand side of the equation can be determined explicitly after getting $\{U^{\circ}\}$ from equation (5). It is noted that all of $\{\partial U/\partial X_i\}$ can be obtained by solving equation only once, because $[K]$ does not change. The matrix $[\partial U/\partial X]$ consists of all set of $\{\partial U/\partial X_i\}$. That is,

$$[\frac{\partial U}{\partial X}]=[\{\frac{\partial U}{\partial X_1}\},\{\frac{\partial U}{\partial X_2}\},\cdots\{\frac{\partial U}{\partial X_n}\}] \tag{13}$$

Once the moments for displacement are obtained, the mean and the covariance matrix of stress is determined through similar procedure. They are respectively

$$\{\mu_q\}=\{q^{\circ}\}=[D^{\circ}]\ [B^{\circ}]\ \{U^{\circ}\} \tag{14}$$

and

$$[\sigma_q]=[\frac{\partial q}{\partial X}][\sigma_x][\frac{\partial q}{\partial X}]^T \tag{15}$$

Here, $[\partial q/\partial X]$ is again an assembly of $\{\frac{\partial q}{\partial X_i}\}$

$$[\frac{\partial q}{\partial X}]=[\{\frac{\partial q}{\partial X_1}\}\{\frac{\partial q}{\partial X_2}\}\cdots\{\frac{\partial q}{\partial X_n}\}] \tag{16}$$

where $\{\partial q/\partial X_i\}$ are obtained simply by differentiating equation (3).

$$\{\frac{\partial q}{\partial X_i}\}=[\frac{\partial D}{\partial X_i}][B]\{U\}+[D][\frac{\partial B}{\partial X_i}]\{U\} + [D][B]\{\frac{\partial U}{\partial X_i}\} \tag{17}$$

If the random variable X_i denotes material property, $[\partial B/\partial X_i]$ vanishes, and equation (17) becomes,

$$\{\frac{\partial q}{\partial X_i}\} = [\frac{\partial D}{\partial X_i}][B]\{U\} + [D]\{B\}\{\frac{\partial U}{\partial X}\} \tag{18}$$

On the other hand, if X_i is associated with load conditions, both $[\partial D/\partial X_i]$ and $\{\partial U/\partial X_i\}$ vanish, and the relation is reduced to

$$\{\frac{\partial q}{\partial X_i}\} = [D][B]\{\frac{\partial U}{\partial X_i}\} \tag{19}$$

where $\{\partial U/\partial X_i\}$ is already obtained from equation (12).

The above formulation is in a generalized form. The equations include a number of dummy operations, and $[\partial K/\partial X_i]$ and $\{\partial F/\partial X_i\}$ are sparsely populated. It is an important subject for probabilistic finite element method to develop a systematic approach to save computational efforts and storage requirements involved in evaluating $[\partial q/\partial X]$ and $[\partial U/\partial X]$. It is also crucial to obtain $[\partial K/\partial X_i]$ in pararell with $[K]$. As for $\{\partial F/\partial X_i\}$, the previous studies have been restricted to the case of concentrated nodal loads. In this study, distributed loads as well as loads between nodes were treated as random variabels. In actual finite element analysis of continuum structures, the distributed load may be evaluated by work equivalent formulation. In this case, the work equivalent nodal loads should be treated probabilistic in accordance with the formulation. In addition, probabilistic manipulation of stress smoothing should be devised.

3. Monte Carlo Simulation of Finite Element Analysis

Probability distribution of structural behaviors may be infered by Monte Carlo simulation. Data sets of input random variables such as material properties and load conditions are generated by random deviators of assumed probability distribution functions. The finite element solutions to each of the data sets produce samples of output random variables such as displacements and bending moments, from which their probability distribution or second moment statistics can be constructed.

In this study, the random deviates were primarily generated using IMSL library functions. However, the library functions do not include the random deviate generator for the extreme type I distribution applied to load conditions. Therefore, uniform deviates were generated and transformed into extreme type I deviates, as shown in Fig. 1, using the cumulative distribution function of extreme type I distribution.

IV. IMPLEMENTATION

The finite element method, either deterministic or probabilistic, is practical only by the use of a computer. Accordingly, computerization is an essential part of the method. Computing time and storage requirement for the method are inevitably significant, and therfore their minimization is very important. For example, the matrices $[\sigma_X]$ and $[\partial q/\partial X]$ in equation (9), and $[\partial U/\partial X]$ in equation (15) take up huge amount of storage space. Owing to the independence between certain variables, these matrices can be properly partitioned, and the non-zero entries can be stored economically. There is also a possibility of storing these matrices in out-of-core memory. Evaluation of the right hand side of equation (11) is especially time consuming. Most of the null operations can be avoided considering the relationship between variables.

The probabilistic finite element method for linear analysis executed in the following sequence :

1) Read input data : the structural geometries, material properties and load conditions are given as input data, or generated from a chosen correlation function.
2) Compute the element stiffness matrices and the element nodal force vectors (deterministic).
3) Assemble the global system equations (deterministic) into a compact one dimensional array.
4) Obtain deterministic nodal displacements by solving the system equations.
5) Compute the first derivative of the stiffness matrix and force vector with respect to random variables.
6) Obtain derivatives of the displacement vector by solving again the system equations with the same left hand side and modified right hand side.
7) Compute the expected values and the

covariance matrix of the nodal displacements.

8) Compute the first derivative of the strain-displacement matrix and the constitutive matrix with respect to random variables.
9) Compute the expected values and covariance matrix of stresses.
10) Print the computed results.

The Monte Carlo simulation of finite element analysis proceeds in the following sequence :

1) Generate random deviates for input random variables.
2) Convert the random deviates into physical values.
3) Assemble the global system equations, solve them, and obtain output random variables.
4) Generate a sequence of output random variables by repeating 1)-3) for a given number of samples.
5) Establish the probability distribution or the second moment statistics of output random variables from the generated values.

The computer programs for probabilistic finite element analysis and for Monte Carlo simulation were independently written, and their computing times were compared under an equivalent computing environment.

V. NUMERICAL STUDY

The validity and the applicability of the probabilistic finite element method was examined through numerical studies with two test cases as shown in Fig. 2. The subjects of investigation were the horizontal displacement at node 3 and the bending moment at node 5 of member 3 in smaple structure I, and the vertical displacement and the bending moment at node 5 of member 2 in smaple structure II. In the first part of this numerical study, the characteristics of the second-order statistics obtained from probabilistic finite element method were examined. The second part was intended to evaluate the limitation and the applicability of the probabilistic finite element method in comparison with the statistics from Monte Carlo simulation.

1. Characteristics of the Second-order Statistics from Probabilistic FEM

The second moment statistics were generated by the probabilistic finite element method for the following three cases :

- when Young's modulus of each member is random and all the other input variables are deterministic.
- when applied loads are random and all the other input variables are deterministic
- when both Young's moduli and applied loads are random and all the other input variables are deterministic

Sample structure I and II were analized respectively for all the above three cases. But the general tendency of the second-order statistics for both sample structures turned out to be almost identical. Thus, the characteristics of the second-order statistics are presented here in terms of sample structure I only.

a. Case of Random Young's Modulus

The Young's modulus of each member was assumed to be random with a uniform coefficient of variation. Equal correlation between Young's moduli is also assumed.

Fig. 3(a) shows the relationship between the variability of Young's modulus and that of displacement. The linearity in the relationship is due to linearization expressed by equation (7). The actual relationship may not be linear as will be shown by Monte Carlo simulation. The slope increases as the correlation increases, and reaches 45° when the correlation become unity. Fig. 3(b) shows the relationship between the variability of Young's modulus and that of bending moment. In contrast to the case of displacement, the slope decreases as the correlation increases. The variability in bending moment vanishes when the correlation becomes unity. It agrees with the expected actual behavior.

b. Case of Random Applied Loads

Applied loads are assumed to be random and equally correlated with uniform coefficient of variation. Fig. 4(a) and (b) shows respectively the relationship between the variability of applied loads and that of displacement and between the variability of applied loads and that of bending moment. In this case, the slope increases with increasing correlation for both displacement and bending moment. As shown in the figures, the correlation has less effect in comparison with the case of random Young's modulus.

c. Case of Random Young's Modulus and Random Applied Loads

Fig.5 shows the same relationship as previous ones for the case in which both Young's moduli and applied loads are random. The relationships are not linear in this case. In most cases, the variability of applied loads is much greater than that of material properties. The figure implies that the effect of the correlation between material properties may not be significant for those cases.

The above numerical study shows that the computed results of the probabilistic finite element analysis are reasonable in general and give some information on the overall tendency of output random variables. But the specific values of the second moment statistics may deviate, to some degree, from the actual probability distribution, as was investigated in the following numerical study.

2. Comparsion with the Results from Monte Carlo Simulation

First, the stability of Monte Carlo simulation was examined to estimate the minimum number of samples required for reliable comparison. Fig. 6 and Fig. 7 shows that at least 1000 simulations are required to obtain stable solutions for both displacements and bending moments. Thus 1000 samples were used for all the comparisons.

The relationship between the variability of Young's modulus and that of output variables were obtained and compared with the previously obtained results by the probabilistic finite element method. In the first case, the Young's moduli are assumed to be independent random variables while other input variables are deterministic. In Fig. 8, the results obtained from normally distributed and lognormally distributed Young's moduli are compared with the probabilistic finite element solutions. The difference between Monte Carlo simulation with normally distributed Young's modulus and the probabilitic finite element solution rapidly increases as the c.o.v. of the Young's modulus exceeds 0.3. Furthermore, when the c.o.v. is taken larger than 0.3, negative Young's moduli were generated within the range of 1000 samples, which disable the program. On the other hand, the probabilistic finite element analysis agrees well with the simulation with lognormally distributed Young's modulus, even for relatively large variation of young's modulus. Such a tendency is commonly observed in both displacement (Fig. 8 (a) and (c)) and bending moment (Fig. 8 (b) and (d)). This suggests partially the validity of the probabilistic finite element method for large range of variability in material properties, because the material properties are usually assumed to be lognormally distributed.

In the second case, the applied loads are assumed to be independent random variables, and all the other variables to be deterministic. Fig. 9 shows comparisons between probabilistic finite element method and the simulation with normally distributed and extreme type I distributed applied loads. The probabilistic finite element method agrees quite closely with the simulation with extreme type I distributed loads for all levels of variability.

Lastly, the means obtained by the simulation and the probabilistic finite element method were compared as shown in Fig. 10. Probabilistic finite element method produces only one single value of mean regardless of c.o.v., while the simulation indicates that the mean varies depending on the c.o.v.

The computing time for Monte Carlo simulation with 1000 samples was around 300 times greater than that of probabilistic finite element analysis.

VI. CONCLUSIONS

The results obtained from the probabilistic finite element method are reasonable in general, and give some information on the overall tendency of output random variables. In most cases, the second moment statistics generated by the method agrees well with the counterparts by Monte Carlo simulation for both mean and variance of the displacement and the bending moment. Particularly, the method gives a solution quite close to the simulation when the Young's moduli are assumed to have lognormal distribution, or when applied loads are assumed to have an extreme type I distribution. However, the deviation of the probabilistic finite element solution from Monte Carlo simulation is significant in case of normally distributed random Young's modulus with high c.o.v. Therefore, the method is not directly applicable in this case. For such circumstances, the method should be revised on the basis of second order approximation rather than the current first order approximation. Still the current method may be applicable for a large range of variability in material properties as well as in load conditions, because the material properties and the load conditions are usually assumed to be lognormally distributed and extreme type I distributed, respectively.

The probabilistic finite element method will serve as a good substitute for the expensive Monte Carlo simulation to obtain the second moment statistics of finite element analysis, although the method is not very informative about the full probability distribution of the output random variables. The drastic savings in computing time is the greatest advantage of the probabilistic finite element method over the Monte Carlo simulation, which may compensate for other shortcomings.

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