〈Research Paper〉

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Asymptotic Distribution of Sample Autocorrelation Function for the First-order Bilinear Time Series Model+

Won Kyung Kim*

ABSTRACT

For the first-order bilinear time series model $X_t = aX_{t-1} + e_t + be_{t-1}X_{t-1}$ where $\{e_t\}$ is a sequence of independent normal random variables with mean 0 and variance σ^2, the asymptotic distribution of sample autocorrelation function is obtained and shown to follow a normal distribution. The variance of the asymptotic distribution is of a complicated form and hence a bootstrap estimate of the variance is proposed for large sample inference. This result can be used to distinguish between different bilinear models.

1. Introduction

Time series analysis has been well developed within the framework of linear models. However, in recent times we have become more aware of the fact that there are many environments in which the resulting data sets cannot be modelled as linear models. Wegman, Schwartz and Thomas(1989) provide a rich source of examples emanating from the oceanographic and meteorological sciences which are clearly non-linear. Therefore, it is natural to ask if there exists non-linear models to provide a better fit to reality. One of the classes of non-linear models which attracted considerable attention is the class of bilinear models, initially discussed by Granger and Andersen(1978) and subsequently studied by Subba Rao(1981), Tuan Dinh Pham and Lanh Tat Tran(1981), Kumar(1986), and Gabr (1988).

The general form of a bilinear time series $\{X_t, t=0, \pm1, \pm2, \cdots\}$, denoted by BL(p, q, P, Q), is defined by

$$X_t - \sum_{i=1}^{p} a_i X_{t-i} = e_t + \sum_{j=1}^{q} C_j e_{t-j} + \sum_{r=1}^{P}\sum_{s=1}^{Q} b_{rs} X_{t-r} e_{t-s} \tag{1.1}$$

where $\{e_t\}$ is a sequence of independent identically distributed random variables with mean 0 and variance σ^2, and is also assumed to be independent of X_s for $s<t$. The model(1.1) is linear in X's and also in e's separately, but not in both. The linear model ARMA (p, q) is a special case of the model (1.1) when $b_{rs}=0$ for all r and s. Hence, bilinear models provide a natural generalization of linear models. Some properties of the model(1.1), for example, stationarity, invertibility, and ergodicity, have been well studied. For other statistical analysis, however, the general bilinear model is too complex to be analyzed successfully. In this paper, we focus attention on the first-order bilinear time series model BL(1, 0, 1, 1),

$$X_t = aX_{t-1} + e_t + be_{t-1}X_{t-1} \tag{1.2}$$

where $\{e_t\}$ is a sequence of independent normal random variables with mean 0 and variance $\sigma^2 < \infty$.

An usual tool for identifying time series model is the sample autocorrelation function. It is, however, known that the autocorrelation structure of the bilinear model BL(p, 0, p, 1) is analogous to that of the linear model ARMA(p, 1). Therefore, the sample autocorrelation function can not be used to distinguish between bilinear and linear models. Granger and Andersen(1978) suggested the autocorrelation function of the squared process $\{X_t^2\}$ to distinguish them, while Kumar (1986) considered the third-order moments. After a time series has been identified as a bilinear model, it is then necessary to distinguish between different bilinear models. This can be done by using the sample autocorrelation functions just as is done for linear models. The main purpose of this paper is to obtain the asymptotic distribution of the sample autocorrelation function of the bilinear model (1.2) so as to enable us to distinguish between bilinear models.

2. Higher-order Moments

To derive the asymptotic distribution of the sample autocorrelation function for the model (1.2), we need up to fourth-order moments. It is, however, known that not all higher-order moments of bilinear time series models exist.

Denote the k-th origin moment by μ'_k and

$$\mu(k) = E(X_t\, X_{t+k}), \tag{2.1}$$

$$\mu(k,l) = E(X_t\, X_{t+k}\, X_{t+l}), \tag{2.2}$$

and

$$\mu(k,l,m) = E(X_t\, X_{t+k}\, X_{t+l}\, X_{t+m}) \tag{2.3}$$

The sufficient conditions of existence of μ_3' and μ_4' are given by

$$|a| < 1, \quad a + 3b^2\sigma^2 < 1, \tag{2.4}$$

and

$$a^4 + 6a^2b^2\sigma^2 + 3b^4\sigma^4 < 1, \tag{2.5}$$

respectively. Under these existence conditions, Sesay and Subba Rao(1988) obtained the Yule-Walker type difference equations for $\mu(k,l)$ and $\mu(k,l,m)$ for almost all values of k, l, and m. These results are used to evaluate the variance-covariance matrix of the asymptotic distribution in Section 3.

3. Asymptotic Distribution

Given the observations $X_1, X_2, \cdots, X_n$, we define the sample mean, sample autocovariance function and sample autocorrelation function, respectively, as

$$\bar{X}_n = \sum_{t=1}^{n} X_t / n, \tag{3.1}$$

$$\hat{\gamma}(h) = \sum_{t=1}^{n-h} (X_t - X_n)\ (X_{t+h} - \bar{X}_n) / n, \tag{3.2}$$

and

$$\hat{\rho}(h) = \hat{\gamma}(h) / \hat{\gamma}(0) \tag{3.3}$$

The asymptotic distribution of the $\bar{X}_n$ and $\hat{\gamma}(h)$ are derived in the following theorems.

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sum_{k=-\infty}^{\infty} \gamma(k)) \tag{3.4}$$

where

$$\sum_{k=-\infty}^{\infty} \gamma(k) = [\gamma(0)(1+a) + 2b\sigma^2\mu]/(1-a) \tag{3.5}$$

Proof. Define

$$U_t = e_t + \sum_{j=1}^{m} ((\prod_{k=1}^{j} (a + be_{t-k}))e_{t-j} \tag{3.6}$$

and

$$W_t = \prod_{k=1}^{m+1} (a + be_{t-k}) X_{t-m-1} \tag{3.7}$$

Then, U_t is $(m+1)$-dependent stationary process and X_t can be expressed by

$$X_t = U_t + W_t \tag{3.8}$$

Tuan Dinh Pham and Lanh Tat Tran(1981) showed that W_t converges in probability to zero. Hence, the asymptotic distribution of $\sqrt{n}(\bar{X}_n - \mu)$ is the same as that of $\sum^{n} (U_t - \mu_u)/\sqrt{n}$ where μ_u is the mean of U_t. Since $E(U_t^2)$ is finite, we have that for fixed m,

$$n^{-1/2} \sum_{t=1}^{n} (U_t - \mu_u) \xrightarrow{d} N(0, \sum_{k=-m}^{m} Cov(U_t, U_{t+k})) \tag{3.9}$$

As m tend to infinity, U_t converges to X_t and hence the variance of the asymptotic distribution converges to $\sum_{k=-\infty}^{\infty} Cov(X_t, X_{t+k})$ which can be easily expressed by (3.5).

Theorem 3.2. Let $\{X_t\}$ be a sequence of random variables satisfying the model (1.2). Suppose that $a^4 + 6a^2b\sigma^2 + 3b^4\sigma^4 < 1$. Then,

$$\sqrt{n} \begin{bmatrix} \hat{\gamma}(0) - \gamma(0) \\ \hat{\gamma}(1) - \gamma(1) \\ \vdots \\ \hat{\gamma}(r) - \gamma(r) \end{bmatrix} \xrightarrow{d} N(0, \Sigma) \tag{3.10}$$

where Σ is the $(r+1)\times(r+1)$ matrix whose element σ_{ij} is given by

$$\sigma_{ij} = \sum_{k=-\infty}^{\infty} Cov(X_t X_{t+i-1}, X_{t+k} X_{t+k+j-1}) \tag{3.11}$$

Proof. From the definition of $\hat{\gamma}(h)$, $h=0, 1, \cdots, r$, given in (3.2), we have

$$\hat{\gamma}(h) = n^{-1}\{ \sum_{t=1}^{n-h} X_t X_{t+h} - \bar{X}_n \sum_{t=1}^{n-h} (X_t + X_{t+h}) + (n-h) \bar{X}_n^2\} \tag{3.12}$$

By the ergodicity of $\{X_t\}$, both the second term and the third term converge in probability to $-E(X_t^2)$. Therefore, the asymptotic distribution of $\sqrt{n}(\hat{\gamma}(h) - \gamma(h))$ is the same as that of $n^{-1/2} \sum_{t=1}^{n-h} (X_t X_{t+h} - E(X_t X_{t+h}))$. Let U_t and W_t be defined as in (3.6) and (3.7) respectively. Then,

$$n^{-1/2} \sum_{t=1}^{n-h} (X_t X_{t+h} - E(X_t X_{t+h}))$$
$$= n^{-1/2}\{ \sum_{t=1}^{n-h} (U_t U_{t+h} - E(U_t U_{t+h})) + \sum_{t=1}^{n-h} (U_t W_{t+h} - E(U_t W_{t+h})) \tag{3.13}$$

Since U_t converges to a stationary process and W_t converges in probability to zero as $m\to\infty$, the second, third, and fourth terms in (3.13) converges in probability to 0 as $m\to\infty$. Hence the asymptotic distribution of $n^{-1/2}\ \Sigma\ (X_tX_{t+h}-E(X_tX_{t+h}))$ is the same as that of $n^{-1/2}\ \Sigma\ (U_tU_{t+h}-E(U_tU_{t+h}))$ as $m\to\infty$.

Now, for any ξ_h, $h=0, 1, \cdots, r$, let

$$P_n=n^{-1/2}\sum_{h=0}^{r}\sum_{t=1}^{n-h}\xi_h(U_tU_{t+h}-E(U_tU_{t+h})). \tag{3.14}$$

We then have

$$\lim_{n\to\infty} P_n=\lim_{n\to\infty}\ n^{-1/2}\sum_{t=1}^{n-h} Y_{t,h} \tag{3.15}$$

where

$$Y_{t,h}=\sum_{h=0}^{r}\xi_h(U_tU_{t+h}-E(U_tU_{t+h})) \tag{3.16}$$

Clearly, $Y_{t,h}$ is a stationary and $(m+r+1)$-dependent process with

$$E(Y_{t,h}Y_{t+h,h})=\ \xi_h'V\ \xi_h<\infty \tag{3.17}$$

where $\xi_h'=(\xi_0, \xi_1, \cdots, \xi_r)$ and V is the $(r+1)\times(r+1)$ symmetric covariance matrix with the ij-th element being

$$v_{ij}=Cov(U_t\ U_{t+i-1},\ U_{t+k}\ U_{t+k+j-1}).$$

Therefore, we have

$$n^{-1/2}\sum_{t=1}^{n-h} Y_{t,h}\xrightarrow{d} N(0,\ \xi_h'V^*\xi_h) \tag{3.18}$$

where the ij-th element of V^* is

$$v^*_{ij}=\sum_{k=-(m+r)}^{m+r} v_{ij}$$

As $m\to\infty$, V^* converges to Σ whose ij-th element, is

$$\sum_{k=-\infty}^{\infty} Cob(X_t\ X_{t+i-1},\ X_{t+k}\ X_{t+k+j-1}).$$

Hence, the proof follows from (3.14).

Remark The variance-covariance matrix Σ in (3.10) can be explicitly obtained by the work of Sesay and Subba Rao(1988). For example, the first diagonal element σ_{11} of Σ is given by

$$\sigma_{11}=\mu_4'-(\mu_2')^2+2\{\mu(0,1,1)-(\mu_2')^2\}+2\sum_{s=2}^{\infty}\{\alpha_1\mu(0,s-1,s-1)$$
$$+\alpha_2\mu(0,s-1)+\alpha_3\mu_2'-(\mu_2')^2\} \tag{3.19}$$

where

$$\mu_4'=\sigma^2(1-a^4-6a^2b^2\sigma^2-3b^2\sigma^4)^{-1}\ [3\sigma^2(8b^2\sigma^4-1)-96ab^3\sigma^4\mu$$
$$+6\{1+12b^2\sigma^2(a^2+b^2\sigma^2)\}\mu_2'+16ab(a^2+3b^2\sigma^2)\mu_3'] \tag{3.20}$$

$$\mu(0,1,1)=(a^2+b^2\sigma^2)\mu_4'+8ab\sigma^2\mu_3'+\sigma^2(1+12b^2\sigma^2)\mu_2' \tag{3.21}$$

$$\mu(0,s,s)=\alpha_1\mu(0,s-1,s-1)+\alpha_2\mu(0,s-1)+\alpha_3\mu_2',\ \ s\geq 2 \tag{3.22}$$

$$\mu_3'=\sigma^2(1-a^3-3ab^2\sigma^2)\ \ [6b^4\sigma^4+3(1+6ab^2\sigma^2)\mu+9(a^2b^2\sigma^2)\mu_2'] \tag{3.23}$$

$$\mu(0,1)=a\mu_3'+3b\sigma^2\mu_2' \tag{3.24}$$

$$\mu(0,s)=a\mu(0,s-1)+b\sigma^2\mu_2',\ \ s\geq 2 \tag{3.25}$$

$$\mu=(1-a)^{-1}b\sigma^2 \tag{3.27}$$

$$\alpha_1=a^2+b^2\sigma^2 \tag{3.28}$$

$$\alpha_2=4ab\sigma^2 \tag{3.29}$$

and

$$\alpha_3=\sigma^2(1+2b^2\sigma^2) \tag{3.30}$$

But, this explicit form of Σ is very complicated and is not useful in practice. Hence, we propse an estimate of each element of Σ by the bootstrap method in Section 4.

Corollary 3.3 Suppose that the conditions of Theorem 3.2 hold. Then,

$$\sqrt{n}\begin{bmatrix}\hat\rho(1)-\rho(1)\\ \hat\rho(2)-\rho(2)\\ \vdots\\ \vdots\\ \hat\rho(r)-\rho(r)\end{bmatrix}\xrightarrow{d} N(0,\ \Sigma^*/\gamma(0)) \tag{3.31}$$

where Σ^* is $(r\times r)$ symmetric covariance matrix whose elements are $\sigma_{ij}, i,j\neq 1$ as given in (3.11).

Proof. For h=1, 2,⋯, r, we have

$$\sqrt{n}(\hat\rho(h)-\rho(h))=\ \sqrt{n}(\hat\gamma(h)-\gamma(h)\ \hat\gamma(0)/\gamma(0))/\hat\gamma(0). \tag{3.32}$$

By the ergodicity of the model(1.2), $\hat\gamma(0)$ converges in probability to $\gamma(0)$. Therefore, the asymptotic distribution of $\sqrt{n}(\hat\rho(h)-\rho(h))$ is the same as that of $\sqrt{n}(\hat\gamma(h)-\gamma(h))/\gamma(0)$. Hence the proof is completed.

4. Bootstrap Estimate

For large sample inference about the autocorrelation function $\rho(h)$, h=1, 2,⋯, r, one typically uses the asymptotic distribution of $\sqrt{n}(\hat\rho(h)-\rho(h))/\hat\sigma_{\hat\rho}$ where $\hat\sigma_{\hat\rho}$ is the estimated standard error of $\hat\rho(h)$. Unfortunately there is no simple expression for $\hat\sigma_{\hat\rho}$.

It is known that the bootstrap method provides an estimate for standard error of a complicated statistic. Efron and Tibshirani (1986) proposed that a bootstrap estimate of the standard error of a correlation coefficient can be obtained and showed that such an estimate achieved a higher level of accurracy. The bootstrap estimate of $\hat\sigma_\rho$ can be similarly obtained as follows.

Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables satisfying the model (1.2) with unknown distribution of X_n being denoted by F. Then the bootstrap estimate of $\hat\sigma_{\hat\rho}=\{var_F(\hat\rho(h))\}^{1/2}$ is $\hat\sigma_{\hat\rho}=\{var_{\hat F}(\hat\rho(h))\}^{1/2}$ where $\hat F$ is the empirical distribution. This bootstrap estimate σ_ρ can be numerically evaluated by a Monte-Carlo algorithm which proceeds in three steps.

(i) Define the residuals

$$\hat e_t=X_t-\hat aX_{t-1}-\hat be_{t-1}\ X_{t-1},\ t=1,\ 2,\cdots,\ n, \tag{4.1}$$

where we assume $e_1=0$ and both $\hat a$ and $\hat b$ are least squares estimates of a and b, respectively. To obtain bootstrap samples, we need the centered residual $\tilde e_t$'s, i.e.,

$$\tilde e_t=\hat e_t-n^{-1}\sum_{t=1}^{n}\ \hat e_t \tag{4.2}$$

Since samples from $\hat F$ is the same as the random samples of size n with replacement from the actual sample, we can independently draw a sample of size n of the $\hat e_t^{*}$'s from the centered residuals. The bootstrap sample $X^*=(X_1^*,\ X_2^*,\cdots,\ X_n^*)$ can be obtained from where we assume $X_0^* = X_0$ and $\tilde{e}_0^* = 0$.

(ii) Suppose a large number of bootstrap samples, $X^*(1)$, $X^*(2), \cdots X^*(B)$ is drawn. For each bootstrap sample $X^*(b)$, $b=1, 2, \cdots, B$, we can evaluate the autocorrelation function $\hat{\rho}^*(b)$.

(iii) Calculate the standard deviation of the $\hat{\rho}^*(b)$

$$\hat{\sigma}_{\hat{\rho}*} = \{(B-1)^{-1} \sum_{b=1}^{B} (\hat{\rho}^*(h) - \hat{\rho}^*)^2\}^{1/2} \tag{4.4}$$

where

$$\hat{\rho}^* = B^{-1} \sum_{b=1}^{B} \hat{\rho}^*(b)$$

It was shown by Efron and Tibshirani (1986) that as $B \rightarrow \infty$, $\hat{\sigma}_{\hat{\rho}}^*$ approaches $\tilde{\sigma}_\rho$, the bootstrap estimate of standard error. Therefore, $\hat{\sigma}_{\hat{\rho}}^*$ can be considered as a bootstrap estimate.

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