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A CUSUM Chart Based on Log Probability Ratio Statistic+

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ABSTRACT

A new approximation method is proposed for the ARL of CUSUM chart which is based on the log probability ratio statistic. This method uses the condition of before-stopping time to derive the expectation of excess over boundaries. The proposed method is compared to some other approximation methods in normal and exponential cases.

1. Introduction

Suppose that a series of independent observations are observed sequentially from a certain process and the distribution of the process may change at an unknown point in time. One of the main objectives of control chart is to detect a change in distribution as soon as possible after its occurrance. One standard procedure for detecting a change in distribution is Shewhart chart originnted by Shawhart (1931). Another standard procedure is the cumulative sum(CUSUM) chart proposed by Page(1954). The CUSUM chart accumulates information with time so that it may be more sensitive than Shewhart chart which regards observations separately.

Let $\{X_i, i=1, 2, \cdots\}$ be independent random variables with density $f(x ; \theta)$ where θ usually denotes the quality of the process. The process $\{X_i, i=1, 2, \cdots\}$ is said to be in-control if $\theta=\theta_0$ and out-of-control if $\theta=\theta_1(>\theta_0)$. For convenience, only positive shifts of the parameter θ are considered and the subset of parameter (θ_0, θ_1) is regarded as an indifference zone.

When θ denotes the mean of the process, the CUSUM procedure for detecting a positive shift in θ is defined as follows : set

$$W_n=\sum_{i=1}^{n} (X_i-k)-\min_{0\leq l\leq n} \sum_{i=1}^{l} (X_i-k) \qquad (1.1)$$

and define the stopping time as

$$N=\min \{n ; W_n\geq h\}, \qquad (1.2)$$

at which an out-of-control signal is called, where k and h are suitably chosen constants and $\sum_{i=1}^{0} (X_i-k)=0$.

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The efficiency of a control chart is measured in terms of the expected stopping time, which is usually referred as the average run length (ARL). If two different control charts have the same ARL at $\theta=\theta_0$, and different at $\theta=\theta_1$, the one with smaller ARL at $\theta=\theta_1$ is called more efficient than the other.

This paper is arranged as follows. Section 2 defines the CUSUM chart based on the log probability ratio statistic (LPRS) as an optimal stopping rule in some sense. In Section 3, a general approximation method is proposed for calculation of ARL of the CUSUM chart defined in section 2. In section 4 and 5, the method proposed in section 3 is compared to the other approximation methods in normal and exponential cases. Section 6 gives conclusions and remarks.

2. A CUSUM Chart Based on LPRS

Suppose that $X_1, X_2, \cdots$ are i.i.d. random variables which are observed sequentially. Let $X_1, \cdots, X_{m-1}$ have density $f(x;\ \theta_0)$ while $X_m, X_{m+1}, \cdots$ have density $f(x;\ \theta_1)$ $(\theta_1>\theta_0)$ where the time of change m is unknown. For $m=1, 2, \cdots$, let P_m denote the distribution of the sequence $X_1, X_2, \cdots$ under which X_m is the first term with density $f(x;\ \theta_1)$.

One popular method for detecting a change in distribution from $f(x;\ \theta_0)$ to $f(x;\ \theta_1)$ is Page's(1954) approach based on the probability ratio consideration.

Let the LPRS at i-th time be

$$\Sigma_i = log \frac{f(X_i;\ \theta_1)}{f(X_i;\ \theta_0)} \tag{2.1}$$

Then the CUSUM procedure base on Z_i is defined as follows: set

$$T_n = \sum_{i=1}^{n} Z_i - \min_{0\leq l\leq n} \sum_{i=1}^{l} Z_i \tag{2.2}$$

and define the stopping time as

$$N = min\ \{n;\ T_n \geq h\} \tag{2.3}$$

at which an out-of-control signal is called, where h is a suitably chosen constant and $\sum_{i=1}^{0} Z_i=0$.

Lorden(1971) formulated a problem of optimal stopping for detection of a change in distribution by defining

$$E_1N = \sup_{m\geq 1} [ess\ sup\ E_m\ \{(N-m+1)^+ \mid X_1, \cdots, X_{m-1}\}] \tag{2.4}$$

where E_m denotes expectation under P_m and $(x)^+=x$, 0 if $x\geq 0$, <0.

Moustakides(1986) showed that Page's stopping time (2.3) is optimal in the sense that it minimizes $\bar{E}_1N$ subject to $E_\infty N\geq\gamma>0$.

This work is a generalization of Lorden's(1971) result where it was shown that (2.3) is optimal asymptotically in the sense that $\gamma\to\infty$.

Page's stopping rule (2.3) is essentially based on a sequence of sequential probability ratio test(SPRT) in which one rejects the null hypothesis that the process has density $f(x;\ \theta_0)$; if a test in the sequence accepts the null hypothesis, one immediately repeats the test.

The value of h is determined according to the trade off between the desired degree of protection against false signals and the sensitivity requirements. Thus, in order to determine the constant h, we need to calculate ARL. Unfortunately, however, the exact evaluation of ARL for the case where the process has a continuous distribution is hopeless in general.

The two major methods used for approximating the ARL are the Wald and Wiener process approximation method. However, both method underestimates the actual ARL and thus they can not be used

3. Approximations to the ARL

Let the stopping time N of the CUSUM chart based on Z_i be defined as (2.3). Define

$$T=min\ \{n\ ;\ \ S_n \notin (0,\ h)\} \tag{3.1}$$

where $S_n=\sum_{i=1}^{n} Z_i$ and

$$Q(\theta)=P(S_T \leq 0). \tag{3.2}$$

Then T and $Q(\theta)$ denote the sample number and operating characteristic(OC) function of the SPRT with boundaries (0, h).

By the mathematical equivalence of CUSUM procedure and sequence of SPRT, the ARL of CUSUM procedure can be obtained by using the average sample number(ASN) and OC function of SPRT. That is, the ARL is expressed by Page (1954) as

$$E_\theta N=\frac{E_\theta T}{1-Q(\theta)} \tag{3.3}$$

The ASN and OC function of the SPRT are not known explicitly in general, and thus neither the ARL. There have been three major technique to approximate the ARL. The one is the use of numerical methods such as Van Dobben de Bruyn(1968) and Goel and Wu(1971). Another is the use of Wald approximations such as Siegmund(1979), Kahn(1978). The other is the Wiener process approximations by Reynolds(1975) and Park(1987).

According to the results, numerical methods using exact equations are more accurate than approximate method. Nevertheless, for analytical purposes there are advantages to having the formula of ARL.

A new approximation technique for calculation of ASN and OC function of the SPRT is presented by using the condition of before-stopping time(CBST).

Let C_r and C_a be the excess of S^T over boundaries h and 0, respectively. That is,

$$C_r=S_T-h\ |_{\,S_T\geq h}\ \ ,\ \ C_a=S_T\ |_{\,S_T\leq 0} \tag{3.4}$$

Also let the expectation of C_r and C_m be

$$u=E[C_r],\ \ l=E[C_a] \tag{3.5}$$

The expectation of S_T can be expressed as

$$\begin{aligned} E[S_T]&=E[S_T\mid S_T\geq h]\cdot[1-Q(\theta)]+E[S_T\mid S_T\geq 0]\cdot Q(\theta)\\ &=[h+u]\cdot[1-Q(\theta)]+l\cdot Q(\theta) \end{aligned} \tag{3.6}$$

and also, by Wald equation,

$$E[S_T]=E[T]\cdot E[Z] \tag{3.7}$$

where Z denotes LPRS. From (3.6) and (3.7), we have

$$E[T]=\frac{[h+u]\cdot[1-Q(\theta)]+l\cdot Q(\theta)}{E[Z]} \tag{3.8}$$

if $E[Z]\neq 0$. For the case $E[Z]=0$, we use $E[S^2_T]$ instead of $E[S_T]$.

$$E[S^2_T]=E[S^2_T\mid S_T\geq h]\cdot[1-Q(\theta)]+E[S^2_T\mid S_T\leq 0]\cdot Q(\theta) \tag{3.9}$$

From (3.9) and (3.10), we have

$$E[T]=\frac{E[S^T_T \mid S_T\geq h]\cdot[1-Q(\theta)]+E[S^2_T \mid S_T\leq 0]\cdot Q(\theta)}{E[Z^2]} \tag{3.11}$$

if E[Z]=0.

For approximation of u and l, we use the CBST technique. We give the condition of CUSUM of Z_i up to T-1 for approximating u and l. That is,

$$\begin{aligned} u&=E[E\ \{S_T-h \mid S_{T-1},\ S_T\geq h\}] \\ &\approx E[S_T-h \mid S_{T-1}=E(S_{T-1} \mid S_T\geq h),\ S_T\geq h] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} l&=E[E\ \{S_T \mid S_{T-1},\ S_T\geq 0\}] \\ &\approx E[S_T \mid S_{T-1}=E(S_{T-1} \mid S_T\geq 0),\ S_T\geq 0] \end{aligned} \tag{3.13}$$

For approximation of $E[S_{T-1} \mid S_T\geq h]]$ and $E[S_{T-1}\leq 0]$, we let

$$\begin{aligned} E[S_{T-1} \mid S_T\geq h]&\approx E[S_{T-1} \mid S_T=h+u] \\ &\overset{d}{=} x_r(u) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} E[S_{T-1} \mid S_T\geq 0]&\approx E[S_{T-1} \mid S_T=l] \\ &\overset{d}{=} x_a(l) \end{aligned} \tag{3.15}$$

Then we have the following two approximation equations by (3.12), (3.14), and (3.13), (3.15),

$$u\approx E\ \{S_T-h \mid S_{T-1}=x_r(u),\ S_T\geq h\} \tag{3.16}$$

$$l\approx E\ \{S_T \mid S_{T-1}=x_a(l),\ S_T\geq 0\}. \tag{3.17}$$

The values of u and l as well as $x_r(u)$ and $x_a(l)$ can be obtained by solving equation (3.16) and (3.17) numerically.

For approximation of $E[S^2_T \mid S_T\geq h]$ and $E[S_T^2 \mid S_T\leq 0]$, we let

$$E[S^2_T \mid S_T\geq h]\approx E[S^2_T \mid S_{T-1}=x_r(u),\ S_T\geq h] \tag{3.18}$$

and

$$E[S^2_T \mid S_T\leq 0]\approx E[S^2_T \mid S_{T-1}=x_a(l),\ S_T\leq 0] \tag{3.19}$$

From Wald's fundamental identity [see Wald(1947)]

$$E[e^{d(\theta)ST}]=1 \tag{3.20}$$

where $d(\theta)$ is the unique nonzero solution d of $Ee^{dz}=1$. Also,

$$\begin{aligned} E[e^{d(\theta)ST}]&=E[e^{d(\theta)ST} \mid S_T\geq h]\cdot[1-Q(\theta)] \\ &\quad+E[e^{d(\theta)ST} \mid S_T\leq 0]\cdot Q(\theta) \end{aligned} \tag{3.21}$$

Thus, we have from (3.20) and (3.21)

$$Q(\theta)=\frac{E[e^{d(\theta)ST} \mid S_T\geq h]-1}{E[e^{d(\theta ST} \mid S_T\geq h]-E[e^{d(\theta)ST} \mid S_T\leq 0]} \tag{3.22}$$

if $[Z]\neq 0$, equivalently $d(\theta)\neq 0$.

For approximation of the OC function $Q(\theta)$, we use the same technique used for u and l to

That is, we approximate

$$E[e^{d(\theta)ST} \mid S_T \geq h] \approx E[e^{d(\theta)ST} \mid S_{T-1}=x_r(u),\ S_T \geq h] \tag{3.23}$$

and

$$E[e^{d(\theta)ST} \mid S_T \leq h] \approx E[e^{d(\theta)ST} \mid S_{T-1}=x_a(l),\ S_T \leq h] \tag{3.24}$$

If E[Z]=0, we use L'Hospital's rule to (3.22) and obtain the OC function as

$$Q(\theta)= \frac{h+u}{h+u-l} \tag{3.25}$$

In order to evaluate the accuracy of the ARL obtained by CBST method in section 4 and 5, comparisons are made with the existing results in normal and exponential cases. In each case of the distribution, the analytical expression of the ARL is derived.

4. Normal Case

Suppose that $\{X_i,\ i=1,\ 2,\cdots\}$ are i.i.d. with density $f(x\ ;\ \theta)=\frac{1}{\sqrt{2\pi}}\exp[-\frac{(x-\theta)^2}{2}]$ and consider the detection problem for $\theta=0$ versus $\theta=\theta_1(>0)$. Then the stopping time is defined as

$$N=min\ \{n\ ;\ S_n - \min_{0\leq l\leq n} S_l \geq h\} \tag{4.1}$$

where $S_n=\sum_{i=1}^{n} Z_i$ and $Z_i=\theta_1\ (X_i-\frac{\theta_1}{2})$.

The expressions of $x_r(u)$ and $x_a(l)$ in (3.14) and (3.15) are obtained as follows.

$$\begin{aligned} x_r(u) &= E[S_{T-1} \mid 0<S_{T-1}<h,\ S_T=h+u] \\ &= E[h+u-Z_T \mid 0<h+u-Z_T<h] \\ &= \theta_l \cdot [\frac{\phi(g(h+u))-\phi(g(u))}{\Phi(g(h+u))-\Phi(g(u))}+g(h+u)] \end{aligned} \tag{4.2}$$

$$\begin{aligned} x_a(l) &= E[S_{T-1} \mid 0<S_{T-1}<h,\ S_T=l] \\ &= E[S_T-Z_T \mid 0<S_T-Z_T<h,\ S_T=l] \\ &= \theta_l \cdot [\frac{\phi(g(l))-\gamma(g(l-h))}{\Phi(g(l))-\Phi(g(l-h))}+g(l)] \end{aligned} \tag{4.3}$$

where ϕ and Φ denote the density and distribution function of standard normal distribution, respectively, and $g(a)=\frac{a}{\theta_1}+\frac{\theta_1}{2}-\theta$.

Then u and l in (3.16) and (3.17) are derived as

$$\begin{aligned} u &= E[S_T-h \mid S_{T-1}=x_r(u),\ S_T \geq h] \\ &= E[Z_T+x_r(u)-h \mid Z_T+x_r(u) \geq h] \\ &= \theta_l \cdot [\frac{\phi\{g(h-x_r(u))\}}{\Phi(-g(h-x_r(u))\}} - g(h-x_r(u))] \end{aligned} \tag{4.4}$$

$$\begin{aligned} l &= E[S_T \mid S_{T-1}=x_a(l),\ S_T \leq 0] \\ &= E[Z_T+x_a(l) \mid Z_T+x_a(l) \leq 0] \end{aligned}$$

It was shown by Wald(1947, pp. 168-169) that u and l are monotone increasing function of $x_r(u)$ and $x_a(l)$, respectively, and the ranges of u and l are

$$\theta_1\left[\frac{\phi\{g(h)\}}{\Phi\{-g(h)\}}-g(h)\right]<u<\theta_1\left[\frac{\phi\{g(0)\}}{\Phi\{-g(0)\}}-g(0)\right] \tag{4.6}$$

and

$$-\theta_1\left[\frac{\phi\{g(0)\}}{\Phi\{g(0)\}}-g(0)\right]<l<-\theta_1\left[\frac{\phi\{g(-h)\}}{\Phi(g(-h)\}}-g(-h)\right] \tag{4.7}$$

Thus the solutions u and l of nonlinear equations (4.4) and (4.5) can be easily obtained by a simple numerical method such as bisection method.

For the case E[Z]=0 (i.e. $\theta=\theta_1/2$), the expression in (3.18) and (3.19) are obtained as follows.

$$\begin{aligned} &E[S_T^2 \mid S_{T-1}=x_r(u),\ S_T\geq h] \\ &=\{x_r(u)\}^2+2\theta_1 x_r(u)\ E[X-\theta \mid X-\theta\geq a] \\ &\qquad +\theta_1^2 E[(X-\theta)^2 \mid X-\theta\geq a] \\ &=\{x_r(u)\}^2+2\theta_1 x_r(u)\ \frac{\phi(a)}{\Phi(-a)}+\theta_1^2\left[a\cdot\frac{\phi(a)}{\Phi(-a)}+1\right] \end{aligned} \tag{4.8}$$

where $a=\dfrac{h-x_r(u)}{\theta_1}$, and similarly,

$$\begin{aligned} &E[S_T^2 \mid S_{T-1}=x_a(l),\ S_T\leq 0] \\ &=\{x_a(l)\}^2+2\theta_1 x_a(l)\ \left[\frac{-\phi(b)}{\Phi(b)}\right]+\theta_1^2\left[-b\cdot\frac{\phi(b)}{\Phi(b)}+1\right] \end{aligned} \tag{4.9}$$

where $b=\dfrac{-x_a(l)}{\theta_1}$.

The expressions in (3.23) and (3.24) are obtained as

$$\begin{aligned} &E[e^{d(\theta)ST} \mid S_{T-1}=x_r(u),\ S_T\geq h] \\ &=\frac{\Phi[d(\theta)\ \theta_1-g\{h-x_r(u)\}]}{\Phi[-g\{h-x_r(u)\}]} \\ &\quad\cdot Exp\left[-d(\theta)\ \theta_1\{g\{-x_r(u)\}-\frac{d(\theta)\ \theta_1}{2}\}\right] \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &E[e^{d(\theta)ST} \mid S_{T-1}=x_a(l),\ S_T\leq 0] \\ &=\frac{\Phi[g\{-x_a(l)-d(\theta)\ \theta_1\}}{\Phi[g\{-x_a(l)\}]} \\ &\quad\cdot Exp\left[-d(\theta)\ \theta_1\{g\{-x_a(l)-\frac{d(\theta)\ \theta_1}{2}\}\right] \end{aligned} \tag{4.11}$$

where $d(\theta)=1-\dfrac{2\theta}{\theta_1}$.

The ASN is calculated by submitting (4.4), (4.5), Q(θ), and $E[Z]=\theta_1(\theta-\theta_1)$ to (3.8), where Q(θ) is calculated by submitting (4.10) and (4.11) to (3.22). Finally the ARL is calculated by (3.3).

For the case E[Z]=0, the ASN is calculated by submitting (4.8), (4.9), (3.25), and $EZ^2=\theta_1^2$ to (3.11).

Park(1987) used bounds of conditional expectation (BCE) method to calculate the continuity correction terms. The ARL values of Kahn, BCE are taken from Park(1987). The "EXACT" and "TRUE" values are obtained by Van Dobben de Bruyn(1968) and Kemp(1958), respectively. Goel and Wu(1971) used the systems of linear algebraic equation (SLAE) to approximate the ARL.

In Table 1, it is seen that CBST method is successful in estimating the ARL and is better than Kahn's and BCE methods for all cases. One serious defect of Kahn's method is that it overestimates the ARL when $|\theta-\theta_{1/2}|$ is near to 0 and does not provide an expression for $|\theta-\theta_{1/2}|=0$.

In Table 2 and 3, CBST method is better than BCE for almost all the cases, but is not as good as SLAE in general. This is because analytic methods by approximate equations can hardly beat numerical methods.

According to the results in Table 1, 2 and 3, CBST method is shown to be better than the other approximation methods.

5. Exponential Case

Suppose that $\{X_i, i=1, 2, \cdots\}$ are i.i.d. with density $f(x;\lambda)=\lambda e^{-\lambda x}$, $x>0$ and consider the detection problem of failure rate for $\lambda=1$ versus $\lambda=\lambda_1(>1)$.

The stopping time of the CUSUM procedure is defined as

$$N=min\,\{n\,;\ S_n-\min_{0\le l\le n} S_l\ge h\} \tag{5.1}$$

Table 1. Values of the ARL for h=3, 5, and 8($\theta_1=1$)

	$\theta-\frac{\theta_1}{2}$	"EXACT"	Kahn	BCE	CBST
h=3	-1.0	2000	1824.80	1602.30	1977.39
	-0.6	195	208.47	179.36	188.36
	-0.2	32.8	36.43	32.39	31.86
	0.0	17.3	-	17.45	16.99
	0.2	10.7	18.41	10.90	10.54
	0.6	5.62	6.05	5.83	5.55
	1.0	3.75	3.86	4.00	3.71
	1.2	3.22	3.36	3.50	3.19
	1.6	2.54	2.75	2.86	2.51
	2.0	2.12	2.40	2.47	2.10
h=5	-0.6	2200	2382.40	2058.80	2151.30
	-0.2	104	113.53	103.09	101.58
	0.0	38.1	-	38.24	37.48
	0.2	19.4	28.41	19.64	19.22
	0.6	8.94	9.39	9.16	8.87
	1.0	5.75	5.86	6.00	5.70
	1.2	4.89	5.02	5.17	4.85
	1.6	3.79	4.00	4.11	3.75
	2.0	3.11	3.40	3.47	3.08
h=8	-0.6	82000	87620	75728	79192.55
	-0.2	428	465.49	428.12	422.08
	0.0	84.6	-	84.42	83.21
	0.2	33.6	43.41	33.92	33.44
	0.6	13.9	14.39	14.16	13.87
	1.0	8.75	8.86	9.00	8.70
	1.2	7.39	7.52	7.67	7.35
	1.6	5.66	5.88	5.98	5.63

Table 2. Values of the ARL for $h=10(\theta_1=1)$

$\theta-\frac{\theta_1}{2}$	"TRUE"	SLAE	BCE	CBST
-0.250	2133	2071	2052	2026.77
-0.125	381.67	400.28	400.66	395.41
0.000	125.24	124.28	125.21	123.71
0.125	57.67	59.30	59.61	59.00
0.250	36.36	36.71	36.95	36.55
0.500	20.34	20.37	20.59	20.29
0.750	14.04	14.06	14.29	14.00
1.000	10.75	10.75	11.01	10.70

Table 3. Values of the ARL for $h=2(\theta_1=1)$

$\theta-\frac{\theta_1}{2}$	"EXACT"	SLAE	BCE	CBST
-1.6	3800	3768	2518	5551.18
-1.2	610	613.80	475.74	695.19
-1.0	259	258.67	213.59	271.75
-0.6	54	54.27	49.77	53.31
-0.2	15.9	15.94	15.68	15.57
0.0	10.0	10.00	10.06	9.82
0.2	6.86	6.86	7.02	6.76
0.6	3.96	3.96	4.18	3.94
1.0	2.74	2.74	3.01	2.74
1.2	2.38	2.67	2.40	
1.6	1.89	1.89	2.23	1.94
2.0	1.58	1.58	1.97	1.67

where $S_n=\sum_{i=1}^{n} Z_i$ and $Z_i=-(\lambda_1-1)X_i+\log\lambda_1$.

Regular(1975) found the exact expression of the ARL as follows under the restriction $\log\lambda_1\geq h$.

$$E_\lambda N=1+\frac{\exp[\lambda\cdot\frac{h}{\lambda_1-1}]}{\exp[\lambda\cdot\frac{\log\lambda_1}{\lambda_1-1}]-(1+\frac{\lambda h}{\lambda_1-1})} \tag{5.2}$$

When $\log\lambda_1\geq h$, however, the ARL in control becomes to small to be used in practice. Therefore we restrict our attention only to $\log\lambda_1<h$.

The expressions of $x_r(u)$ in (3.14) for $\log\lambda_1<h$ is obtained as follows.

$$\begin{aligned} x_r(u) &= E[S_{T-1}\mid h+u-\log\lambda_1<S_{T-1}<h.\ S_T=h+u] \\ &= h+u-\log\lambda_1+(\lambda_1-1)\ E[X\mid 0<X<\frac{\log\lambda_1-u}{\lambda_1-1}] \\ &= \frac{\lambda_1-1}{\lambda}+h-\frac{\log\lambda_1-u}{1-\exp[-\lambda\cdot\frac{\log\lambda_1-u}{\lambda_1-1}]} \end{aligned} \tag{5.3}$$

Then, by (3.16)

$$u=E[S_T-h \mid S_{T-1}x_r(u),\ S_T\geq h]$$
$$=E[Z_T+x_r(u)-h \mid Z_T+x_r(u)\geq h]$$
$$=-\frac{\lambda_1-1}{\lambda}+\frac{\log \lambda_1+x_r(u)-h}{1-\exp[-\lambda\cdot\frac{\log \lambda_1+x_r(u)-h}{\lambda_1-1}]} \tag{5.4}$$

Also, for $0<x<h$,

$$E[S_T \mid S_{T-1}=x,\ S_T\leq 0]$$
$$=E[x+Z_T \mid x+Z_T\leq 0]$$
$$=x+\log \lambda_1-(\lambda_1-1)\cdot E[X \mid S\geq\frac{\log \lambda_1+x}{\lambda_1-1}] \tag{5.5}$$
$$=-\frac{\lambda_1-1}{\lambda}$$

which is independent of x. Thus

$$l=E[E\{S_T \mid S_{T-1},\ S_T\leq 0\}]$$
$$=-\frac{\lambda_1-1}{\lambda} \tag{5.6}$$

It can be easily seen that u is a monotone increasing function of $x_r(u)$ and the range of u is

$$-\frac{\lambda_1-1}{\lambda}+\frac{\log \lambda_1-h}{1-\exp[-\lambda\cdot\frac{\log \lambda_1-h}{\lambda_1-1}]}<u<$$
$$-\frac{\lambda_1-1}{\lambda}+\frac{\log \lambda_1}{1-\exp[-\lambda\cdot\frac{\log \lambda_1}{\lambda_1-1}]} \tag{5.7}$$

Thus the solution u of nonlinear equation (5.4) can be easily obtained by the bisection method. The expression in (3.23) is obtained as

$$E[e^{d(\lambda)ST} \mid S_{T-1}=x_r(u),\ S_T\geq h]$$
$$=e^{d(\lambda)(\log\lambda_1+x_r(u))}\cdot E[e^{-d(\lambda)(\lambda_1-1)X} \mid X\leq\frac{\log \lambda_1+x_r(u)}{\lambda_1-1}]$$
$$=\frac{\lambda\cdot e^{d(\lambda)(\log \lambda_1+x_r(u))}\cdot[1-e^{-\{\lambda+d(\lambda)(\lambda_1-1)\}\frac{\log \lambda_1+x_r(u)}{\lambda_1-1}}]}{\{\lambda+d(\lambda)\ (\lambda_1-1)\}\cdot\{1-e^{-\lambda\cdot\frac{\log \lambda_1+x_r(u)}{\lambda_1-1}}\}} \tag{5.8}$$

$\lambda\cdot\lambda^{d}$

Also, for $0<x<h$,

$$
\begin{aligned}
&E[e^{d(\lambda)S_T} \mid S_{T-1}=x,\ S_T\leq 0] \\
&= e^{d(\lambda)(\log\lambda_1+x)} \cdot E[e^{-d(\lambda_1-1)X} \mid X \geq \frac{\log\lambda_1+x}{\lambda_1-1}] \\
&= \frac{\lambda}{\lambda+d(\lambda)\ (\lambda_1-1)}
\end{aligned}
\tag{5.9}
$$

which is independent of x. Thus

$$
\begin{aligned}
E[e^{d(\lambda)S_T} \mid S_T\leq 0] &= E[E\{e^{d(\lambda)S_T} \mid S_{T-1},\ S_T\leq 0\}] \\
&= \frac{\lambda}{\lambda+d(\lambda)\ (\lambda_1-1)}
\end{aligned}
\tag{5.10}
$$

Submitting (5.4), (5.6), $Q(\theta)$, and $E[Z]=-\frac{\lambda_1-1}{\lambda}+\log\lambda_1$ to (3.8), where $Q(\theta)$ is calculated by submitting (5.8) and (5.9) to (3.22), we obtain ASN and then the ARL by (3.3).

In the following Table 4, comparisons are made with some other results where the parameter values of λ are restricted only to 1 and λ_1. Note that $d(\lambda)=1$ and -1 for $\lambda=1$ and λ_1, respectively. In Table 4, Lorden and Eisenberger(1973) obtained "exact" values using the results of Kiefer and Wolfowitz (1950), and the values of "exact" and Kahn are taken from Kahn(1978).

Table 4. Values of the ARL in Exponential Case

	λ	EXACT	Kahn	CBST
$\lambda_1=1.4$ $h=7.48925$	1 1.4	422.1 47.9	420.8 (0.31) 44.26(7.60)	426.12(0.95) 47.95(0.10)
$\lambda_1=1.6$ $h=6.52$	1 1.6	676.0 36.4	675.2 (0.12) 33.84(7.03)	685.04(1.34) 36.49(0.25)
$\lambda_1=1.9$ $h=4.09867$	1 1.9	342.0 20.2	341.4 (0.18) 18.47(8.56)	348.75(1.97) 20.25(0.25)

* numbers in parentheses denote the percent difference

In Table 4, Kahn's results tend to be better than CBST for $\lambda=1$, but worse for $\lambda=\lambda_1$. If the percent difference of the estimated ARL to the exact one is considered, we can easily see that the maximum always occurs at $\lambda=\lambda_1$ of Kahn.

According to the results in Table 4, it may be stated that CBST method is successful in estimating the ARL of exponential case.

6. Conclusions and Remarks

It has been shown by Moustakides(1986) that the CUSUM procedure based on LPRS is optimal in detecting a change in distribution. The problem of calculating the ARL is not completely solved yet despite of its optimality.

The two main approaches presented in the literatures for the estimation of the ARL are numerical and approximate methods, which have their own advantages and disadvantages. Numerical methods give accurate results but require much computing time. Approximation methods can save time but tend to be less accurate than numerical ones. Nevertheless, approximation methods with analytic expres-

That is, it gives the condition of S_{T-1} and then replace it by $E[S_{T-1}]$ to approximate the ASN and OC function of the SPRT.

Applying CBST technique to normal and exponential cases, it was shown that CBST is better than or at least as good as the other approximation methods.

References

1. Dvoretzky, A., Kiefer, J., and Wolfowitz, J.(1950). Sequential Decision Problems for Processes with Continuous Time Parameter. Testing Hypotheses, *The Annals of Mathematical Statistics*, 24, 254-264.
2. Ghos, B.K.(1970). *Sequential Tests of Statistical Hypothesis*, Addison-Wesley Publishing Company, Reading, Massachusetts.
3. Goel, A.L. and Wu, S.M.(1971). Determination of A.R.L. and a Contour Nomogram for Cusum Charts to Control Normal Mean, *Technometrics*, 13, 2, 221-230.
4. Kahn, R.A.(1978). Wald's Approximations to the Average Run Length in CUSUM Procedures, *Journal of Statistical Planning and Inference*, 2, 63-77.
5. Kemp, K.W.(1958). Formulae for Calculating the Operating Characteristic and the Average Sample Number of Some Sequential Tests, *Journal of the Royal Statistical Society* B 20, 379-386.

10/ 6. Lorden, G.(1971). Procedures for Reacting to a Change in Distribution, *The Annals of Mathematical Statistics*, 42, 1897-1908.

7. Lorden, G. and Eisenberger, I.(1973). Detection of Failure Rate Increases. *Technometrics* 15, 167-176.
8. Moustakides, G.V.(1986). Optimal Stopping Times for Detecting Changes in Distributions, *The Annals of Statistics*, 14, 1379-1387.
9. Page, EMS.(1954). Continuous Inspection Schemes, *Biometrika*, 41, 100-114.
10. Park, C.(1987). A Corrected Wiener Process Approximation for CUSUM ARLS, *Sequential Analysis*, 6, 3, 257-265.
11. Pollak, M.(1985). Optimal Detection of a Change in Distribution, *The Annals of Statistics*, 13, 206-227.
12. Regula, G.(1975). Optimal Cusum Procedures to Detect a Change in Distribution for the Gamma Family, Ph.D. *Thesis, Case Western Reserve University, Cleveland*, Ohio.
13. Reynolds, M.R., Jr.(1975). Approximations to the Average Run Length in Cumulative Sum Control Charts, *Technometrics*, 17, 1, 65-71.
14. Shewhart, W.A.(1931). *The Economic Control of Quality of Manufacture Product*, Von Nostrand, New York.
15. Siegmund, D.(1979). Corrected Diffusion Approximations in Certain Random Walk Problems, *Advances in Applied Probabilities*, 11, 710-719.
16. Van Dobben de Bruyn, C.S.(1968). *Cumulative Sum Tests*, Hafner, New York.
17. Wald, A.(1947). *Sequential Analysis*, Wiley, New York.