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On The Derivation Of A Certain Noncentral t Distribution+

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ABSTRACT

Let a p-component vector **y** have a p-variate normal distribution $N(b\theta, \Sigma)$, Σ unknown, **b** specified, then for testing $\theta=0$ against general θ, Khatri and Rao(1987) derive a certain t test and obtain its power function. This paper presents a direct derivation of this power function in terms of the original variates unlike Khatri and Rao(1987) who resort to the canonical transformations of the original variates and the conditional distributions.

1. Introduction

The GMANOVA model of Potthoff and Roy(1964) is

$$X=B\xi A+E \tag{1}$$

where X is $p\times n$, specified B is $p\times(p\text{-}t)$, specified A is $q\times n$, ξ is $(p\text{-}t)\times q$, and E in the usual sense is pn-variate normal with mean vector zero and covariance matrix $\Sigma\otimes I$, Σ unknown. Assuming all matrices of full rank, Khatri(1966) shows that the pair $(\hat{\xi}, S)$ where

$$\hat{\xi}=(B'S^{-1}B)^{-1}B'S^{-1}XAG^{-1},\quad S=X[I-A'G^{-1}A]X', \tag{2}$$

is sufficient for the pair (ξ, Σ). He shows that the likelihood ratio test criterion for testing the linear double compound $C\xi V=0$, against $C\xi V\neq 0$, C and V specified appropriate dimensional full rank matrices, is

$$\Lambda=|Q|/|Q+P|, \tag{3}$$

where

$$Q=C(B'S^{-1}B)^{-1}C',\quad P=(C\hat{\xi}V)(V'RV)^{-1}(C\hat{\xi}V)', \tag{4}$$

$$R=G^{-1}+G^{-1}AX'[S^{-1}-S^{-1}B(B'S^{-1}B)^{-1}B'S^{-1}]XA'G^{-1},$$

$$G=AA'. \tag{5}$$

In case C is $m\times(p\text{-}t)$, V is $q\times g$, then Q has a Wishart density with (n-q-t) d.f., and independent P is Wishart with g d.f. and both have the population covariance matrix $C(B'\Sigma^{-1}B)^{-1}C'$. Although Khatri (1966) derives the density of Λ in the null case, the nonnull density of Λ still remains uninvestigated.

Now Khatri and Rao(1987) assume $t=(p\text{-}1)$, $q=1$, and write(1) as

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$$Y = \theta \mathbf{b}\mathbf{J}' + E, \tag{6}$$

where p-component $\mathbf{b}$ is specified and $\mathbf{J}$ has all n components unity, and investigate the noncentral density of (3).

From (2) it follows that the pair $(\hat{\theta}, S)$

$$\hat{\theta} = \mathbf{b}'S^{-1}\bar{\mathbf{y}} / \mathbf{b}'S^{-1}\mathbf{b}, \quad S = YY' - n\bar{\mathbf{y}}\bar{\mathbf{y}}', \quad n\bar{\mathbf{y}}' = \mathbf{J}'Y', \tag{7}$$

is sufficient for the pair (θ, Σ). For testing $H_0: \theta = 0$, against $H_1: \theta \neq 0$, they propose the statistic

$$t = \frac{\sqrt{n(n-p)}\ \mathbf{b}'S^{-1}\bar{\mathbf{y}}}{[(1+n\bar{\mathbf{y}}'S^{-1}\bar{\mathbf{y}})\mathbf{b}'S^{-1}\mathbf{b} - n(\mathbf{b}'S^{-1}\bar{\mathbf{y}})^2]^{\frac{1}{2}}} \tag{8}$$

where S is the sample dispersion matrix and $\bar{\mathbf{y}}$ is the sample mean vector of a sample of size n on $\mathbf{y}$. They show that in the null case(8) has a t-density with (n-p) degrees of freedom and also derive the noncentral density of (8).

Khatri and Rao(1987) resort to canonical transformations of the original variates to obtain the noncentral density of (8). This paper presents this noncentral density in terms of the original variates.

We find it easier to derive the noncentral density of

$$\lambda = \frac{n\hat{\theta}^2(\mathbf{b}'S^{-1}\mathbf{b})}{(1+n\bar{\mathbf{y}}'S^{-1}\bar{\mathbf{y}})} = \frac{t^2/(n-p)}{1+t^2/(n-p)'} \tag{9}$$

rather than that of (8). Note that in our context(3) yields

$$1-\lambda = \frac{nR}{(1+n\bar{\mathbf{y}}'S^{-1}\bar{\mathbf{y}})} \quad ; \quad nR = 1 + n\bar{\mathbf{y}}'S^{-1}\bar{\mathbf{y}} - n\hat{\theta}^2(\mathbf{b}'S^{-1}\mathbf{b}). \tag{10}$$

The next section records some useful results and Section 3 develops the distribution theory. Sometimes the same symbol denotes different quantities, however, its meaning is made explicit in the context.

2. Some Useful Results

Let $\mathbf{y}$ be a p-component vector, H a p×p positive definite symmetric matrix, D_1, $(q_1 \times p)$ and D_2, $(q_2 \times p)$, $q_1 + q_2 \leq p$ be two given full rank matrices. Then Sverdrup(1947) shows that

$$\begin{aligned}
&\int_{y'Hy=u,\ D_{1y}=v_1,\ D_{2y}=v_2} f(y'Hy)dy = \pi^{\frac{p-q_1-q_2}{2}} (\Gamma(\tfrac{p-q_1-q_2}{2}))^{-1} \ |H|^{-1/2} \\
&|D_1H^{-1}D'_1|^{-\frac{1}{2}} \ |D_2H^{-1}D'_2 - D_2H^{-1}D'_1(D_1H^{-1}D'_1)^{-1} D_1H^{-1}D'_2|^{-\frac{1}{2}} \\
&[u - \mathbf{v}'_1(D_1H^{-1}D'_1\ \mathbf{v}_1 - (\mathbf{v}_2 - D_2H^{-1}D'_1(D_1H^{-1}D'_1)^{-1}\mathbf{v}_1)' \\
&(D_2H^{-1}D'_2 - D_2H^{-1}D'_1(D_1H^{-1}D'_1)^{-1} D_1H^{-1}D'_2)^{-1} \\
&(\mathbf{v}_2 - D_2H^{-1}D'_1(D_1H^{-1}D'_1)^{-1}\mathbf{v}_1)]^{1/2(p-q_1-q_2-2)}
\end{aligned} \tag{11}$$

For a suitable density function f the right hand side of (11) is the joint density of u, $\mathbf{v}_1$, and $\mathbf{v}_2$.

$$\int_{-\sqrt{u}}^{\sqrt{u}} exp\{\alpha x\}(u-x^2)^{p-1}dx = \sum_{r=0}^{\infty} \frac{\sqrt{\pi}\Gamma(p)\alpha^{2r}u^{p+r-\frac{1}{2}}}{2^{2r}r!\ \Gamma(p+r+\frac{1}{2})} \tag{12}$$

The following system of matrix identites may be easily verified

$$(S + n\bar{\mathbf{y}}\bar{\mathbf{y}}')^{-1} = S^{-1} - nS^{-1}\bar{\mathbf{y}}\bar{\mathbf{y}}'S^{-1}(1+n\bar{\mathbf{y}}'\ S^{-1}\bar{\mathbf{y}})^{-1}, \tag{13}$$

$$\bar{\mathbf{y}}'S^{-1}\bar{\mathbf{y}} = h/(1-nh), \quad h = \bar{\mathbf{y}}'H^{-1}\bar{\mathbf{y}}, \quad H = S + n\bar{\mathbf{y}}\bar{\mathbf{y}}', \tag{14}$$

$$\mathbf{b}'S^{-1}\bar{\mathbf{y}} = \mathbf{b}'H^{-1}\bar{\mathbf{y}}k(1-nh)^{-1} \tag{15}$$

$$\mathbf{b}'S^{-1}\mathbf{b} = \mathbf{b}'H^{-1}\mathbf{b} + n(\mathbf{b}'H^{-1}\bar{\mathbf{y}})^2/(1-nh), \tag{16}$$

$$(\mathbf{b}'S^{-1}\mathbf{b})^{-1} = (\mathbf{b}'H^{-1}\mathbf{b})^{-1} - g^2n(1-nh+nm)^{-1}, \tag{17}$$

$$g=(\mathbf{b}'H^{-1}\mathbf{b})^{-1}\,\mathbf{b}'H^{-1}\bar{\mathbf{y}},\quad m=g^2(\mathbf{b}'H^{-1}\mathbf{b}), \tag{18}$$
$$(\mathbf{b}'S^{-1}\mathbf{b})^{-1}\,\mathbf{b}'S^{-1}\bar{\mathbf{y}}=g/(1-nh+nm), \tag{19}$$
$$(\mathbf{b}'S^{-1}\bar{\mathbf{y}})^2/(\mathbf{b}'S^{-1}\mathbf{b})=m/(1-nh)(1-nz),\quad h=m+z, \tag{20}$$
$$\lambda=nm/(1-nz). \tag{21}$$

We now proceed with the distribution theory.

3. Distribution Theory

From (6) the joint density of S and $\bar{y}$ is

$$g(S,\ \bar{\mathbf{y}})=K\,exp\{-\frac{1}{2}tr\ \Sigma^{-1}[S+n\bar{\mathbf{y}}\bar{\mathbf{y}}']+n\theta\mathbf{b}'\Sigma^{-1}\bar{\mathbf{y}}\}\ |\,S\,|^{\frac{1}{2}(n-p-2)}. \tag{22}$$

and setting $H=S+n\bar{y}\bar{y}'$, the density of H and $\bar{y}$ is found to be

$$g(H,\ \bar{\mathbf{y}})=K\phi(H)\ exp\{n\theta\mathbf{b}'\Sigma^{-1}\bar{\mathbf{y}}\}\ (1-n\bar{\mathbf{y}}'\ H^{-1}\bar{\mathbf{y}})^{\frac{1}{2}(n-p-2)}, \tag{23}$$

where

$$\phi(H)=exp\{-\frac{1}{2}tr\ \Sigma^{-1}H\}\ |\,H\,|^{\frac{1}{2}(n-p-2)}. \tag{24}$$

and K as a generic letter, denotes the normalizing constants of density functions in this paper.
Now using (11), we integrate (23) with respect to $\bar{y}$ over the region

$$\bar{\mathbf{y}}'H^{-1}\bar{\mathbf{y}}=h,\quad (\mathbf{b}'H^{-1}\mathbf{b})^{-1}\,\mathbf{b}'H^{-1}\bar{\mathbf{y}}=\mathbf{v}_1,\quad \mathbf{b}'\Sigma^{-1}\bar{\mathbf{y}}=\mathbf{v}_2, \tag{25}$$

and obtain the density of H, h, $\mathbf{v}_1$, and $\mathbf{v}_2$ to be

$$g(H,\ h,\ \mathbf{v}_1,\ \mathbf{v}_2)=K\phi(H)\ |\,H\,|^{\frac{1}{2}}(\mathbf{b}'H^{-1}\mathbf{b})^{\frac{1}{2}}w^{-\frac{1}{2}}exp\{n\theta\mathbf{v}_2\}(1-nh)^{\frac{1}{2}(n-p-2)}$$
$$[h-(\mathbf{b}'H^{-1}\mathbf{b})\mathbf{v}_1^2-(\mathbf{v}_2-\mathbf{b}'\Sigma^{-1}\mathbf{b}\mathbf{v}_1)^2/w]^{\frac{1}{2}(p-4)}, \tag{26}$$

where

$$\mathbf{w}=(\mathbf{b}'\Sigma^{-1}H\Sigma^{-1}\mathbf{b}-(\mathbf{b}'H^{-1}\mathbf{b})^{-1}\ (\mathbf{b}'\Sigma^{-1}\mathbf{b})^2). \tag{27}$$

Futher, setting

$$\mathbf{v}_1=(\mathbf{b}'\ H^{-1}\ \mathbf{b})^{\frac{-1}{2}}\,z_1,\quad (\mathbf{v}_2-\mathbf{b}'\Sigma^{-1}\mathbf{b}\mathbf{v}_1)w^{\frac{-1}{2}}=z_2, \tag{28}$$
$$z=h-m=h-z_1^2, \tag{29}$$

we derive the joint density of z_1, z_2, z and H. In this density we first set $H=\Sigma^{1/2}\,G_1\,\Sigma^{1/2}$, and then set $G^1=PGP'$, where the $p\times p$ orthogonal matrix P is so chosen such that $\mathbf{b}'\Sigma^{-1/2}P=((b'\Sigma^{-1}b)^{1/2},\ 0,\ 0,\cdots 0)$. Thus the joint density of G, z, z_1, and z_2, is

$$g(G,\ z_1,\ z_2,\ z)=K\ exp\ \{-\frac{1}{2}tr\ G\}\ |\,G\,|^{\frac{1}{2}(n-p-1)}$$
$$\cdot\ exp\{n\theta z_2(\mathbf{g}'_{(1)}\ G_{22}^{-1}\ \mathbf{g}_{(1)})^{\frac{1}{2}}(\mathbf{b}'\Sigma^{-1}\mathbf{b})^{\frac{1}{2}}+n\theta\ z_1(\mathbf{b}'\Sigma^{-1}\mathbf{b})^{\frac{1}{2}}\ (g_{11}-\mathbf{g}'_{(1)}\ G^{-1}_{22}\mathbf{g}_{(1)})^{\frac{1}{2}}\}$$
$$\cdot\quad (z-z_2^2)^{\frac{p-4}{2}}(1-nz-nz_1^2)^{\frac{1}{2}(n-p-2)}, \tag{30}$$

where

$$G=\begin{bmatrix} g_{11} & \mathbf{g}'_{(1)} \\ \mathbf{g}_{(1)} & G_{22}\end{bmatrix},\quad G_{22}(p-1)\times(p-1). \tag{31}$$

From (30), using (12), integrate out z_2 first to find that

$$g(\mathrm{G},z_1 z)=Kexp\{-\frac{1}{2}\,trG\}\ |\,G\,|^{\frac{1}{2}(n-p-1)}$$
$$\cdot\ exp\{n\theta z_1(b'\Sigma^{-1}b)^{\frac{1}{2}}\,(g_{11}-g'_{(1)}G^{-1}_{22}g_{(1)})^{1/2}\}$$

$$\cdot (1-nz-nz_1^2)^{\frac{1}{2}(n-p-2)} z^{\frac{1}{2}(p-3)}$$

$$\cdot \sum_{r=0}^{\infty} \frac{(n\theta)^{2r}(g'_{(1)}G^{-1}_{22}g_{(1)})^r(b'\Sigma^{-1}b)^r z^r}{2^{2r} r!\, \Gamma(\frac{1}{2}(p-1)+r)} \tag{32}$$

Next we transform from z_1 to λ using

$$nz_1^2=\lambda(1-nz), \tag{33}$$

and integrate z, and find the joint density of G and λ to be

$$g(G,\lambda)=K exp\{-\frac{1}{2} trG\} \mid G \mid^{\frac{1}{2}(n-p-1)}(1-\lambda)^{\frac{1}{2}(n-p-2)}\lambda^{-\frac{1}{2}}$$

$$\cdot \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{n^{r+s}\theta^{2r+2s}(b'\Sigma^{-1}b)^{r+s}\Gamma(\frac{n-p+1}{2}+s)\lambda^s}{2^{2r} r!\, \Gamma(\frac{n}{2}+s+r)(2s)!}$$

$$\cdot (g'_{(1)}G^{-1}_{22}g_{(1)})^r(g_{11}-g'_{(1)}G^{-1}_{22}g_{(1)})^s. \tag{34}$$

Finally we integrate out G from (34), and find that

$$g(\lambda)=K exp\{-\frac{1}{2}n\theta^2 b'\Sigma^{-1}b\}\ \lambda^{-\frac{1}{2}}\ (1-\lambda)^{\frac{1}{2}(n-p-2)}$$

$$\cdot \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{n^{r+s}\theta^{2r+2s}(b'\Sigma^{-1}b)^{r+s}(\Gamma(\frac{n-p+1}{2}+s))^2\lambda^s 2^s}{2^r r!\, \Gamma(\frac{n}{2}+s+r)(2s)!} \tag{35}$$

The expression(35) is the noncentral density of λ. Khatri and Rao(1987, p.181, equation(3.4)) set $\theta=1$ in (35), and contend that λ is also useful for testing the hypothesis that

$$H_0:\ E(y)=0,\ against\ H_1:\ E(y)=\mathbf{b}\ (specified), \tag{36}$$

where (6), with $\theta=1$, now represents a sample of size n on $\mathbf{y}$ of (36).

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