A NOTE ON IDEALS WHICH ARE MAXIMAL AMONG NONVALUATION IDEALS

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In this paper $R$ will be an integral domain. A Noetherian ring with unique maximal ideal is called a local ring. An ideal of $R$ is called a valuation ideal if it is the contraction of an ideal of some valuation overring of $R$. It is known that every primary ideal of a Noetherian domain $R$ is a valuation ideal if and only if $R$ is a Dedekind domain. From this fact we come to be interested in the ideals which are maximal among nonvaluation ideals. One might guess that such an ideal has to be a primary ideal, but this is false. We will show that such an ideal $I$ in a Noetherian domain $R$ is a primary ideal if and only if its radical $\sqrt{I}$ is a maximal ideal. In the case that $R$ is a two dimensional regular local ring, we will show that $I$ is a primary ideal. Note that $\sqrt{I}$ is not always a prime ideal. But it will turn out that $\sqrt{I}$ is a prime ideal if $R$ is a local domain. This will be used to prove that in a two dimensional regular local ring, $I$ is always a primary ideal. For undefined terms and general information, the reader is referred to [2].

**Lemma 1.** Let $R$ be a commutative ring such that the set $Z(R)$ of zero divisors is a union of finite number of prime ideals. Then any regular ideal of $R$ is generated by regular elements.

*Proof.* This follows from [1, Lemma B]

**Lemma 2.** Let $R$ be a local domain and $I$ an ideal of $R$. If $I$ is maximal among nonvaluation ideals of $R$, then $\sqrt{I}$ is a prime ideal.

*Proof.* Let $M$ be the maximal ideal of $R$. If $\sqrt{I} = M$, then there is nothing to prove. So let us assume that $\sqrt{I} \subseteq M$. Choose $a \in M \setminus \sqrt{I}$.
Then for each \( k \geq 1 \), \( I \subseteq I+(a^k) \). Now by passing to \( R/I \) and using Krull’s intersection theorem \([3, \text{ Theorem 142}]\), we deduce that \( I = \cap_{k=1}^{\infty} (I+(a^k)) \). Put \( I+(a^k) = I_k \). Now \( I = \cap_{k=1}^{\infty} I_k, I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k \supseteq I_{k+1} \ldots \), and each \( I_k \) is a valuation ideal. To prove that \( \sqrt{I} \) is a prime ideal, suppose that \( xy \in I^2 \) for \( x, y \in R \). Then \( xy \in (I_k)^2 \) for each \( k \). So either \( x \) or \( y \) is in \( I_k \) since \( I_k \) is a valuation ideal \([2, \text{ Lemma 24.4}]\). Hence at least one of \( x \) and \( y \) is contained in infinitely many \( I_k \)'s, which implies that either \( x \) or \( y \) is contained in \( \cap_{k=1}^{\infty} I_k = I \). Now suppose \( xy \in \sqrt{I} \) for \( x, y \in R \). Then \((xy)^n \in I \) for some \( n > 0 \). So \( x^{2n}y^{2n} = (xy)^{2n} \in I^2 \). From the previous argument, either \( x^{2n} \) or \( y^{2n} \) is contained in \( I \). From this, we conclude that \( x \) or \( y \in \sqrt{I} \) and hence \( \sqrt{I} \) is a prime ideal.

**Lemma 3.** Let \( R \) be an integral domain and \( I \) an ideal which is maximal among nonvaluation ideals. If \( P \) is a prime ideal containing \( I \), then \( R/P \) is a valuation ring.

**Proof.** Let \( \bar{x}, \bar{y} \) be two nonzero elements of \( \bar{R} = R/P \), so that \( x \notin P \), \( y \notin P \). Since \( I \subseteq P \subseteq P+(xy) \), we have that \( P+(xy) \) is a valuation ideal of \( R \). For some valuation overring \( V \) of \( R \), \((xy)+P = ((xy)+P) \ V \cap R \). Then either

\[
[x^2V \subseteq ((xy)+P) \ V \text{ or } y^2V \subseteq ((xy)+P) \ V]
\]

or

\[
[x^2V \supseteq ((xy)+P) \ V \text{ and } y^2V \supseteq ((xy)+P) \ V]
\]

**Case I.** \( x^2V \subseteq ((xy)+P) \ V \Rightarrow x^2 \in ((xy)+P) \ V \cap R = (xy)+P \Rightarrow x^2 = rxy+p \) for \( r \in R \) and \( p \in P \Rightarrow x(x-ry) \in P \Rightarrow x-ry \in P \) since \( x \notin P \Rightarrow x \in (y)+P \Rightarrow (x)+P \subseteq (y)+P \Rightarrow (\bar{x}) \subseteq (\bar{y}) \).

**Case II.** \( ((xy)+P) \ V \subseteq x^2V \text{ and } ((xy)+P) \ V \subseteq y^2V \Rightarrow xy = x^2v, xy = y^2v' \) for some \( v, v' \in V \Rightarrow x^2y^2 = x^2y^2v' \Rightarrow vv' = 1 \Rightarrow x^2 = xyv' \) from \( xy = x^2v \Rightarrow x^2 \in ((xy)+P) \ V \cap R \). This reduces to case I. Thus either \( (\bar{x}) \subseteq (\bar{y}) \) or \( (\bar{y}) \subseteq (\bar{x}) \). Hence \( R/P \) is a valuation ring.

**Corollary 4.** Let \( R \) be a local domain and \( I \) an ideal maximal among nonvaluation ideals. Then \( R/\sqrt{I} \) is a principal ideal domain.
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**Proof.** This follows from Lemma 2 and Lemma 3.

Let $R$ be a Noetherian domain and $I$ an ideal maximal among nonvaluation ideals. Let $D$ be a Dedekind domain which is not a DVR. The set of nonvaluation ideals of $D$ is not empty. Let us choose an ideal $I$ which is maximal among nonvaluation ideals. If $I$ is a primary, then $I = I_P \cap R$, where $P = \sqrt{I}$, and hence $I$ is a valuation ideal since $D_P$ is a DVR. This contradicts our choice of $I$. So $I$ need not be a primary ideal. In the next theorem, we give a necessary and sufficient condition for $I$ to be a primary ideal.

**Theorem 5.** Let $D$ be a Noetherian domain and $I$ an ideal maximal among nonvaluation ideals. Then $I$ is a primary ideal if and only if $\sqrt{I}$ is a maximal ideal.

**Proof.** ($\Rightarrow$) Suppose that $I$ is a primary ideal. Let $\sqrt{I} = P$. We want to show that $P$ is a maximal ideal. If not, there exists a maximal ideal $M$ such that $P \subset M$. In $R = D/I, Z(R) = P/I$. Let $x$ and $y$ be regular elements of $R$, so $x, y, xy \notin P$. Then $I + (xy) \supseteq I$ so $I + (xy)$ is a valuation ideal of $D$. Then for some valuation overring $V$ of $R$, $I + (xy) = (I + (xy))V \cap D$. As in the proof of Lemma 3, we deduce that either $x^2V \subseteq ((xy) + I)V$ or $y^2V \subseteq ((xy) + I)V$. We may assume that $x^2V \subseteq ((xy) + I)V$. We can find $r \in R$ such that $x(x-ry) \in I$ as we did in the case I of the proof of Lemma 3. Since $I$ is a primary ideal and $x \notin P = \sqrt{I}$, so $(x) \subseteq (y)$. Thus in $D/I$, the regular principal ideals are totally ordered. It is easy to see that every element of $(M \setminus P)/I$ is a regular element of $D/I$. So $M/I$ is a regular ideal and it is generated by regular elements by Lemma 1. Since $D/I$ is Noetherian, $M/I$ is finitely generated and hence $M/I$ is a principal ideal. By the Krull's principal ideal theorem, $M/I$ is a minimal prime ideal of $D/I$. So $M/I = P/I$ and $M = P$, which contradicts our assumption that $P \subset M$. Therefore we conclude that $P$ is a maximal ideal.

($\Leftarrow$) is obvious.

Let $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$ be a chain of prime ideals of an integral domain $R$. Then there always exists a valuation overring $V$ of $R$ such that $P_iV \cap D = P_i$ for each $i = 1, \cdots, n$. This fact is crucial in proving
Theorem 6. Let \((R, M)\) be a two dimensional regular local domain and \(I\) an ideal maximal among nonvaluation ideals of \(R\). Then \(\sqrt{I}\) is the maximal ideal of \(R\).

Proof. If \(P = \sqrt{I}\) is not the maximal ideal, then \(P\) is a minimal prime ideal of \(R\). Since \(R\) is a UFD, there exists an \(a \in R\) such that \(P = (a)\). By corollary 4, \(R/P\) is a PID. So \(M = P + (b)\) for some \(b \in R\) and \(M = (a, b)\). It is easy to see that \(A \equiv \{J \mid J\) is an ideal of \(R\) \((a^2) \subseteq J \subseteq (a)\} = \{(a^2, ab^k)\}_{k=0}^{\infty}\). We claim that \(I \subseteq A\). We have to show that \(P^2 \subseteq I\). For otherwise, \(P^2 \not\subseteq I\) and \(P^2 \not\subseteq I + (b^n)\) for some \(n\) by Krull's intersection theorem. Let \(J = (I + (b^n)) \cap P\). Then \(Pb^n \subseteq J\). Since \(I \subseteq J \subseteq P\), we have that \(\sqrt{I} \subseteq \sqrt{J} \subseteq P\). Thus \(\sqrt{J} \subseteq M\). Choose \(z \in M \setminus \sqrt{J}\). Then following the same argument as in the proof of Lemma 2, we can show that \(J = \bigcap_{k=1}^{\infty} (J + (z^k))\). Put \(J + (z^k) = J_k\). Then each \(J_k\) is a valuation ideal since \(J_k\) properly contains \(I\). Now let \(x = a, y = b^n\). Now \(xy \in J\), which implies that \(xy \in J_k\) for each \(k\). For each \(k\), either \(x^2\) or \(y^2\) belongs to \(J_k\) since \(J_k\) is a valuation ideal \([2,\text{ Lemma }24,4]\), and hence either \(x^2\) or \(y^2\) belongs to infinitely many \(J_k\). So \(x^2 \in J = \bigcap_{k=1}^{\infty} J_k\) or \(y^2 \in J\), i.e., \(a^2 \in J\) or \(b^{2n} \in J\). This contradicts that \(0 \not\subseteq I\) and \(b \not\in P\). Thus \(a^2 \not\in I\). Now \(I \subseteq A\), so that \(I = (a^2, ab^n)\) for some \(n \geq 0\). We can choose a valuation domain \(V\) such that \(PV \cap R = P\) and \(MV \cap R = M\). Obviously \(IV \cap R \subseteq A\), so \(IV \cap R = (a^2, ab^n)\) for some \(k\). But \(k \leq n\) since \(I \subseteq IV \cap R\). We will show that \(k = n\), so that \(I = IV \cap R\). Suppose \(k < n\). Then \(ab^k \in IV = (a^2, ab^n) V \Rightarrow b^k \in (a, b^n) V \Rightarrow b^k (1 - b^{n-k} v) \in a V = V \Rightarrow b^k \in a V \) since \(1 - b^{n-k} v (n-k > 0)\) is a unit of \(V\) (note that \(b\) is a nonunit of \(V\) since \(b \in M\) and \(MV \neq V\)) \Rightarrow b^k \in a V \cap D = P \Rightarrow b \in P,\) which contradicts that \(P \neq M\). Thus \(k = n\), so \(I = IV \cap R\) is a valuation ideal. But this contradicts that \(I\) is not a valuation ideal. Therefore \(\sqrt{I}\) is the maximal ideal of \(R\).
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References


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