INVARIANT SUBMEANS AND SEMIGROUPS OF NONEXPANSIVE MAPPINGS ON UNIFORMLY
CONVEX BANACH SPACES

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1. Introduction

Let $S$ be a semitopological semigroup i.e., $S$ is a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ from $S$ into $S$ are continuous. Let $E$ be a uniformly convex Banach space and $\mathcal{V} = \{T_t \; ; \; t \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of $E$ into $C$, i.e., $T_{ts}x = T_tT_sx$, $t, s \in S$, $x \in C$, and the mapping $(t, x) \mapsto T tx$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology. Let $\text{AP}(S)$ be all continuous almost periodic functions on $S$. i.e., $f \in C(S)$ such that $\{rsf ; s \in S\}$ is relatively compact in the norm topology.

Lau[1], in 1985, proved that if the space of almost periodic functions on $S$ has a left invariant mean, $C$ is a closed convex subset of a Hilbert space $H$, and there exist $x \in C$ with relatively compact orbit, then $C$ contains a common fixed point for $\mathcal{V} = \{T_t \; ; \; t \in S\}$.

In this paper we prove that if $\text{AP}(S)$ has an invariant submean, $\mathcal{V} = \{T_t \; ; \; s \in S\}$ is a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of an uniformly convex, uniformly smooth Banach space and $C$ contains an element of relatively compact orbit, then $C$ contains a common fixed point for $S$.

2. Preliminaries

Let $S$ be a semitopological semigroup and $B(S)$ be the Banach space of all bounded real valued functions on $S$ with supremum norm. Let $D$ be a subspace of $B(S)$ containing constants. A real valued function
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\( \mu \) on \( D \) is called **submean** on \( D \) if the following conditions are satisfied:

1) \( \mu(f+g) \leq \mu(f) + \mu(g) \) for every \( f, g \in D \).
2) \( \mu(\alpha f) = \alpha \mu(f) \) for every \( f \in D \) and \( \alpha \geq 0 \).
3) For \( f, g \in X \), \( f \leq g \) implies \( \mu(f) \leq \mu(g) \).
4) \( \mu(c) = c \) for every constant \( c \).

Let \( \mu \) be a submean on \( D \) and \( f \in D \). Then, according to times and circumstances, we use \( \mu_t(f(t)) \) instead of \( \mu(f) \).

For \( s \in S \) and \( f \in B(S) \), we define \( l_s f(t) = f(st) \) and \( r_s f(t) = f(ts) \) for all \( t \in S \). Let \( D \) be a subspace of \( B(S) \) containing constants which is \( l_s \)-invariant, i.e., \( l_s(D) \subset D \) for each \( s \in S \). Then a submean \( \mu \) on \( D \) is said to be left invariant if \( \mu(f) = \mu(l_s f) \) for all \( s \in S \) and \( f \in D \). Similarly, we can define a right invariant submean on a \( r_s \)-invariant subspace of \( B(S) \) containing constants. A left and right invariant submean is called an invariant submean.

Let \( E \) be a Banach space, and let \( E^* \) be its dual. The value of \( f \in E^* \) at \( x \in E \) will be denoted by \( \langle x, f \rangle \). With each \( x \in E \), we associate the set

\[
J(x) = \{ f \in E^*, \langle x, f \rangle = \| x \|^2 = \| f \|^2 \}.
\]

Using the Hahn–Banach theorem, it is immediately clear that \( J(x) \neq \emptyset \) for each \( x \in E \). The multi-valued operator \( J : E \to E^* \) is called the **duality mapping**. As well known ([2, p.130]), if \( E^* \) is uniformly convex (or equivalently, \( E \) is uniformly smooth), \( J \) is single-valued, and \( J \) is uniformly continuous on each bounded subset of \( E \) when \( E \) has the strong topology while \( E^* \) has the weak* topology.

Let \( E \) be a uniformly smooth Banach space with duality mapping \( J : E \to E^* \). A map \( T \) with domain \( D(T) \) is said to be **accretive** if, for any \( x, y \in E \) and all \( \lambda > 0 \),

\[
\| \lambda x + Tx - (\lambda y + Ty) \| \geq \lambda \| x - y \|.
\]

Equivalently, \( T \) is accretive if and only if

\[
(Tx - Ty, J(x-y)) \geq 0
\]

for all \( x, y \in D(T) \) (see [5, p.245]). The range of \( \lambda I + T \), \( R(\lambda I + T) \), is known to be all of either for all \( \lambda > 0 \), or no \( \lambda > 0 \) (see [4]); in the first case, \( T \) is called **\( m \)-accretive**. In this case, the resolvent \( J_{\lambda} = (I + \lambda T)^{-1} \) is a nonexpansive mapping defined on \( E \) for each positive \( \lambda \).

Let \( \mathcal{F} \) be a family of \( m \)-accretive operators with common domain
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Let $S(\mathcal{F})$ be the semigroup of nonexpansive mappings on $E$ generated by $\{J_f^T; T \in \mathcal{F}\}$. Equip $S(\mathcal{F})$ with the strong operator topology. Then $S(\mathcal{F})$ is a topological semigroup i.e., the multiplicative on $S(\mathcal{F})$ is jointly continuous.

3. Lemmas

**Lemma 3.1.** Let $S$ be a semitopological semigroup, let $D$ be a subspace of $B(S)$ containing constants and let $\mu$ be a submean on $D$. Let $\{x_t; t \in S\}$ be a bounded subset of a Banach space $E$ and let $C$ be a closed convex subset of $E$. Suppose that for each $x \in C$, the real-valued function $G$ on $C$ by

$$G(x) = \mu_t||x_t - x||^2.$$  

Then the real-valued function $G$ on $C$ is continuous and convex.

**Proof.** Let $x_n \to x$, and $M = \sup \{||x_t - x_n|| + ||x_t - x||; n=1,2,\ldots, \text{ and } t \in S\}$. Then, since

$$||x_t - x_n||^2 - ||x_t - x||^2 = (||x_t - x_n|| + ||x_t - x||)$$  

$$- (||x_t - x_n|| - ||x_t - x||) \leq M||x_t - x_n|| - ||x_t - x|| \leq M||x_n - x||$$  

for every $n=1,2,\ldots$ and $t \in S$, we have

$$\mu_t||x_t - x_n||^2 \leq \mu_t||x_t - x||^2 + M||x_n - x||.$$  

Similarly, we have

$$\mu_t||x_t - x||^2 \leq \mu_t||x_t - x_n||^2 + M||x_n - x||.$$  

So, we have $|G(x_n) - G(x)| \leq M||x_n - x||$. This implies that $G$ is continuous on $C$.

Let $\alpha$ and $\beta$ be nonnegative numbers with $\alpha + \beta = 1$ and $x, y \in C$. Then, since

$$||x_t - (\alpha x + \beta y)||^2 \leq \alpha||x_t - x||^2 + \beta||x_t - y||^2$$  

we have $G(\alpha x + \beta y) \leq \alpha G(x) + \beta G(y)$. This implies that $G$ is convex on $C$.

**Lemma 3.2.** Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $E$, let $S$ be a semitopological semigroup, and let $\{x_t; t \in S\}$ be a bounded subset of $E$. Let $D$ be a subspace of $B(S)$ such that $D$ contains constants and for any $x \in C$ and $u \in E$, functions $h$
and $g$ defined by $h(t) = \|x_t - z\|^2$, $g(t) = \langle u, J(x_t - z) \rangle$ for all $t \in S$ are in $D$ such that $\lim_{t \to \infty} \|x_t - z\|^2$ exists for all $z \in C$. Let $\mu$ be a submean on $D$ satisfying the following condition: if $\lim_{t \to \infty} x_t = \alpha$ and $\lim_{t \to \infty} y_t = \beta$ then $\mu_t(x_t \pm y_t) = \mu_t(x_t) \pm \mu_t(y_t)$. Let $z_0 \in C$ and $\mu_t \|x_t - z_0\|^2 = \min_{y \in C} \mu_t \|x_t - y\|^2$.

Then $\mu_t(z - z_0, J(x_t - z_0)) \leq 0$ for all $z \in C$.

**Proof.** For $z$ in $C$ and $0 \leq \lambda \leq 1$, we have

$$\|x_t - z_0\|^2 = \|x_t - \lambda z_0 - (1 - \lambda) z + (1 - \lambda) (z_0 - z)\|^2$$

$$\geq \|x_t - \lambda z - (1 - \lambda) x\|^2$$

$$+ 2(1 - \lambda) \langle z - z_0, J(x_t - \lambda z_0 - (1 - \lambda) z) \rangle$$

since $J(x)$ is the subdifferential of the convex function $\frac{1}{2} \|x\|^2$ ([2, p. 97]). Since $E$ is uniformly smooth, the duality map is uniformly continuous on bounded subset of $E$ from the strong topology of $E$ to the weak* topology of $E^*$. Therefore

$$\langle z - z_0, J(x_t - \lambda z_0 - (1 - \lambda) z) - J(x_t - z_0) \rangle \leq \epsilon$$

if $\lambda$ is closed enough to 1. Consequently, we have

$$\langle z - z_0, J(x_t - z_0) \rangle$$

$$= \langle \epsilon + \langle z - z_0, J(x_t - \lambda z_0 - (1 - \lambda) z) \rangle \rangle$$

$$\leq \epsilon + \frac{1}{2(1 - \lambda)} \left\{ \|x_t - z_0\|^2 - \|x_t - \lambda z_0 - (1 - \lambda) z\|^2 \right\}.$$ 

Hence, by hypothesis,

$$\mu_t(z - z_0, J(x_t - z_0)) \leq \epsilon + \frac{1}{2(1 - \lambda)} \left\{ \mu_t \|x_t - z_0\|^2 - \mu_t \|x_t - \lambda z_0 - (1 - \lambda) z\|^2 \right\}$$

$$\leq \epsilon$$

since $\lim_{t \to \infty} \|x_t - z_0\|^2$ and $\lim_{t \to \infty} \|x_t - \lambda z_0 - (1 - \lambda) z\|^2$ exists.

**Lemma 3.3.** Let $C$ be a closed convex subset of a uniformly convex and uniformly smooth Banach space $E$, let $S$ be a semitopological semigroup, and let $\{x_t : t \in S\}$ be a bounded set of $E$. Let $D$ be a subspace of $B(S)$ such that $D$ contains constants and for any $z \in C$ and $u \in E$, functions $h$ and $g$ defined by $h(t) = \|x_t - z\|^2$ and $f(z) = \langle u, J(x_t - z) \rangle$ for all $t \in S$ are in $D$ such that $\lim_{t \to \infty} \|x_t - z\|^2$ exists for all $z \in C$. Let $\mu$ be a submean on $D$ satisfying the following condition: if $\lim_{t \to \infty} x_t = \alpha$
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and \( \lim_{t \to \infty} y_t = \beta \) then \( \mu_t(x_t \pm y_t) = \mu_t(x_t) \pm \mu_t(y_t) \). Then, the set

\[
M = \{ u \in C ; \mu_t \| x_t - u \|^2 = \min_{z \in C} \mu_t \| x_t - z \|^2 \}
\]

consists of one point.

Proof. Let \( g(z) = \mu_t \| x_t - z \|^2 \) for every \( z \in C \) and \( \gamma = \inf \{ g(z) ; z \in C \} \). Then, since the function \( g \) on \( C \) is convex, continuous and \( g(z) \to \infty \) as \( \| z \| \to \infty \) from \([6, p.79]\), there exists \( u \in C \) with \( g(u) = \gamma \). Therefore \( M \) is nonempty. From lemma 3.2 and \( u \in M \),

\[
\mu_t \langle z - u, J(x_t - u) \rangle \leq 0
\]

for all \( z \in C \). We show that \( M \) consists of one point. Let \( u, v \in M \) and suppose \( u \neq v \). Then by \([3, Theorem 1]\), there exists a positive number \( k \) such that

\[
\langle x_t - u - (x_t - v), J(x_t - u) - J(x_t - v) \rangle \geq k
\]

for all \( t \in S \). Therefore

\[
\mu_t \langle v - u, J(x_t - u) - J(x_t - v) \rangle \geq k > 0.
\]

On the other hand, since \( u, v \in M \), we have \( \mu_t \langle v - u, J(x_t - u) \rangle < 0 \) and \( \mu_t \langle u - v, J(x_t - v) \rangle < 0 \). Since

\[
\begin{align*}
\langle v - u, J(x_t - u) - J(x_t - v) \rangle \\
= \langle v - u, J(x_t - u) \rangle + \langle u - v, J(x_t - v) \rangle,
\end{align*}
\]

\[
\begin{align*}
\mu_t \langle v - u, J(x_t - u) - J(x_t - v) \rangle \\
\leq \mu_t \langle v - u, J(x_t - u) \rangle + \mu_t \langle u - v, J(x_t - v) \rangle \\
< 0.
\end{align*}
\]

This is a contradiction. Therefore \( u = v \).

4. Semigroup of nonexpansive mappings with bounded orbit

Let \( S \) be a semitopological semigroup. Let \( C(S) \) be the Banach space of bounded continuous real-valued functions on \( S \). Let \( AP(S) \) denote the space of all continuous almost periodic functions on \( S \). i.e., \( f \in C(S) \) such that \( \{ rsf ; s \in S \} \) is relatively compact in the norm topology where \( (rsf)(t) = f(is) \).

**Theorem 4.1.** Let \( S \) be a semitopological semigroup. Let \( \mathcal{G} = \{ T_s ; s \in S \} \) be a continuous representation of \( S \) as nonexpansive mappings on a closed convex subset \( C \) of a uniformly convex uniformly smooth Banach space \( E \) into \( C \). If \( AP(S) \) has an invariant submean, and \( x \in C \) with
relatively compact orbit, then there exists \( u \in C \) such that \( T_s u = u \) for all \( s \in S \).

**Proof.** We first prove that for any \( z \in C \) and \( y \in E \) the function \( h \) and \( g \) defined by

\[
 h(t) = \| T_t x - z \|^2 \quad \text{and} \quad g(t) = \langle y, J(T_t x - z) \rangle
\]

for all \( t \in S \) are in \( \text{AP}(S) \). It is clear that \( h \in \text{C}(S) \). Let \( h_x(t) = \| T_t x - z \|^2 \). Then \( r_z h_x(t) = h_{w}(t) \) where \( w = T_s x \). Let \( \tau : x \rightarrow h_x(t) \). If we can show that \( \tau \) is continuous when \( C(S) \) has the supnorm topology, the \( \tau(O(x)) \) is a compact subset of \( C(S) \) containing \( \{ r_z h ; s \in S \} \) where \( O(x) = \{ T_s x ; s \in S \} \). In particular, \( h \in \text{AP}(S) \). To see that \( \tau \) is continuous, let \( \{ x_n \} \) be a sequence in \( C \), \( x_n \rightarrow x \) and \( M = \sup_{t \in S} \| T_t x - z \| \) for all \( x \in C \) with relatively compact orbit, then

\[
 \| \tau(x_n) - \tau(x) \| = \sup_{t \in S} \| T_t x_n - z \|^2 - \| T_t x - z \|^2 \\
 = \sup_{t \in S} (\| T_t x_n - z \| - \| T_t x - z \|) \| T_t x_n - z \| + \| T_t x - z \|) \\
 \leq 2M \sup_{t \in S} \| T_t x_n - T_t x \| \\
 \leq 2M \| x_n - x \|
\]

by nonexpansive of \( T_t \), \( t \in S \). Hence \( \| \tau(x_n) - \tau(x) \| \rightarrow 0 \) as \( x_n \rightarrow x \). Thus we have \( h \in \text{AP}(S) \).

Similarly, let \( g_x(t) = \langle y, J(T_t x - z) \rangle \). Then \( r_z g_x(t) = g_{w}(t) \) where \( w = T_s x \). Let \( \eta : x \rightarrow g_x(t) \) and \( x_n \rightarrow x \). Then we have

\[
 \| \eta(x_n) - \eta(x) \| = \sup_{t \in S} | \langle y, J(T_t x_n - z) \rangle - \langle y, J(T_t x - z) \rangle | \\
 = \sup_{t \in S} | \langle y, J(T_t x_n - z) - J(T_t x - z) \rangle |.
\]

Since \( J \) is uniformly continuous on bounded sets when \( E \) has its strong topology while \( E^* \) has its weak* topology and

\[
 \| (T_t x_n - z) - (T_t x - z) \| = \| T_t x_n - T_t x \| \\
 \leq \| x_n - x \|.
\]

Hence \( \| \eta(x_n) - \eta(x) \| \rightarrow 0 \) as \( x_n \rightarrow x \). Thus we have \( g \in \text{AP}(S) \).

Let \( \mu \) be an invariant submean on \( \text{AP}(S) \). Then, the set

\[
 M = \{ u \in C ; \mu || T_s x - u ||^2 = \min_{s \in S} \mu || T_s x - z ||^2 \}
\]

is invariant under every \( T_s \), \( s \in S \). In fact, if \( u \in M \) then for each \( s \in S \) we have

\[
 \mu || T_s x - T_s u ||^2 = \mu || T_s x - T_s u ||^2
\]

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\[ \mu_1 \| T_i T_j x - T_i u \| ^2 \leq \mu_1 \| T_i x - u \| ^2 \]

and hence \( T_i u \in M \). On the other hand, by Lemma 3.3, we know that \( M \) consists of one point. Therefore this point is a common fixed point of \( T_i, i \in S \).

**Theorem 4.2.** Let \( E \) be a uniformly convex uniformly smooth Banach space. Let \( \mathcal{F} \) be a family of \( m \)-accretive mappings with common domain \( D \) in \( E \). Suppose that \( AP(\mathcal{S}) \) has an invariant submean, and there exists a sequence \( \{x_n\} \) in \( D \) such that \( T(x_n) \to 0 \) for each \( T \in \mathcal{F} \), then there exists \( v \in E \) such that \( T(v) = 0 \) for all \( T \in \mathcal{F} \).

**Proof.** Define a function \( g : E \to \mathbb{R} \) by

\[ g(z) = \mu_1 \| x_n - z \| \]

for each \( z \in E \) and \( \gamma = \inf \{ g(z) \mid z \in E \} \), where \( \mu_1 \| x_n - z \| \) denotes the value of \( \mu \) at the bounded sequence \( \{ \| x_n - z \| \} \). Then, since the function \( g \) on \( E \) is continuous, convex and \( g(z) \to \infty \) as \( \| z \| \to \infty \), it follows from [6, p. 79], there exists \( v \in E \) with \( g(v) = \gamma \). So, putting \( M = \{ v \in E \mid g(v) = \gamma \} \), \( M \) is nonempty, bounded, closed, and convex. Let \( v \in M \), \( T \in \mathcal{F} \), and \( J = J_1 T = (I + T)^{-1} \). Then

\[
\mu_1 \| x_n - Jv \| = \mu_1 \| x_n - Jx_n + Jx_n - Jv \| \\
\leq \mu_1 \| x_n - Jx_n \| + \mu_1 \| x_n - v \| \\
\leq \mu_1 \| Tx_n \| + \mu_1 \| x_n - v \| \\
\leq \mu_1 \| x_n - v \|
\]

since \( Tx_n \to 0 \) by assumption. Therefore, \( M \) is invariant under \( J_1 T \) for each \( T \in \mathcal{F} \). In particular, \( M \) is invariant under the semigroup \( S \) generated by \( \{ J_1 T \mid T \in \mathcal{F} \} \). If \( AP(\mathcal{S}) \) has an invariant submean, then by Theorem 4.1, there exists \( v \in M \) with \( J_1 T(v) = v \) for all \( T \in \mathcal{F} \). i.e., \( T(v) = 0 \) for all \( T \in \mathcal{F} \).

**References**

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