ON DUALITIES FOR STRONGLY DECOMPOSABLE OPERATORS

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1. Notations and definitions

Throughout this note, $X$ denotes a complex Banach space, $B(X)$ the Banach algebra of all bounded linear operators of $X$ and $X^*$ the dual of $X$. For an operator $T \in B(X)$, $T^*$ denotes the dual operator of $T$. If $M$ is a closed $T$-invariant subspace of $X$, we write $T|M$ for the restriction and $T/M$ for the operator induced by $T$ on the quotient space $X/M$. For $N \subseteq X$, let $N^\perp$ be its annihilator in $X^*$, $\overline{N}$ the closure of $N$. The symbol $\sigma(T)$ stands for the spectrum of $T$. We denote $\mathcal{U}$ and $\mathcal{F}$ the class of all open subsets and the closed subsets in the finite complex plane $\mathbb{C}$ respectively. If $T$ has the single valued extension property, we denote $X_T(F) = \{ x \in X : \sigma(x, T) \subseteq F \}$. This is a linear subspace of $X$ but not necessarily closed even if $F$ is closed in $\mathbb{C}$. The set theoretic difference between two sets $A$ and $B$ is denoted by $A - B$.

**Definition 1.1 ([3]).** Let $T \in B(X)$. A $T$-invariant subspace $Z$ is said to be spectral maximal for $T$ if for any $T$-invariant subspace $Y$ such that $\sigma(T|Y) \subseteq \sigma(T|Z)$ we have that $Y \subseteq Z$.

We denote the set of all spectral maximal subspaces for $T$ by $\text{SM}(T)$.

**Definition 1.2 ([3]).** An operator $T \in B(X)$ is said to be decomposable if for any finite system $\{ G_1, G_2, \ldots, G_n \}$ of open subsets of $\mathbb{C}$ that cover $\sigma(T)$, there exist spectral maximal subspaces $\{ Y_1, Y_2, \ldots, Y_n \}$ such that $X = \sum_{i=1}^{n} Y_i$ and $\sigma(T|Y_i) \subseteq G_i$ for $i = 1, 2, \ldots, n$.

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It is known that if $T$ is decomposable, then $\text{SM}(T) = \{X_T(F) : F \in \mathcal{F}\}$.

**Definition 1.3 ([9]).** The $T$-invariant subspace $Y$ is called analytically invariant if for each $X$-valued analytic function $f$ defined on a region $V_f$ in $\mathbb{C}$ such that $(\lambda - T)f(\lambda) \in Y$ for $\lambda \in V_f$, then it follows that $f(\lambda) \in Y$ for $\lambda \in V_f$.

It is known that "Spectral Maximal" implies "Analytically invariant" but the converse is false. We denote the class of all analytically invariant subspaces for $T$ by $\text{AI}(T)$. Thus $\text{SM}(T) \subseteq \text{AI}(T)$.

**Definition 1.4 ([6]).** A decomposable operator is strongly decomposable if the operator $T|_Y$ is decomposable for every $T$-spectral maximal subspace $Y$.

2. **Analytical spectral resolvent (ASR)**

**Definition 2.1.** A map $E : \mathcal{U} \to \text{AI}(T)$ is said to be an analytic spectral resolvent of $T$ if

(i) $E(\phi) = \{0\}$,

(ii) For any finite open cover $\{G_1, G_2, \ldots, G_n\}$ of $\sigma(T)$,

$$X = \sum_{i=1}^{n} E(G_i),$$

(iii) $\sigma(T|E(G)) \subseteq \overline{G}$ for each $G \in \mathcal{U}$.

Thus an ASR is a spectral resolvent, which is defined in [5], whose range is analytically invariant subspaces.

The ASR for $T$ is not unique as well as the spectral resolvent, there are typical types of ASR for $T$.

**Remark 2.2.** Let $T$ be strongly decomposable, then the map $E$ defined by $E(G) = X_T(G)$ ($G \in \mathcal{U}$) is an ASR for $T$.

For, it is known that if $T$ is decomposable then $X_T(G)$ is analytically invariant for each $G \in \mathcal{U}$. Obviously $E(\phi) = \overline{X_T(\phi)} = \{0\}$, and $\sigma(T|E(G)) = \sigma(T|X_T(G)) \subseteq \overline{\sigma(T|X_T(G))}$ ($G \in \mathcal{U}$) hold since $\overline{X_T(G)} \subseteq X_T(G)$, and both $\overline{X_T(G)}$, $X_T(G)$ are analytically invariant under $T$; in fact, $X_T(G)$ is spectral maximal so it is analytically invariant. For any finite open cover $\{G_1, G_2, \ldots, G_n\}$ of $\sigma(T)$,
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\[ \sum_{i=1}^{n} E(G_i) = \sum_{i=1}^{n} X_T(G_i) \supset \sum_{i=1}^{n} X_T(G_i) = X_T(\bigcup_{i=1}^{n} G_i) = X_T(\sigma(T)) = X, \]
the second equality holds since \( T \) is strongly decomposable (see [6], p. 86, Lemma 12.7).

Remark 2.3. Let \( T \) be decomposable. The map \( E \) defined by \( E(G) = X_T(G) \) \((G \in \mathcal{U})\) is an ASR for \( T \).

Proof. Let \( \{G_i\}_{i=1}^{n} \) be any open covering of \( \sigma(T) \), it is known that
\[ X = X_T(\sigma(T)) \supset \sum_{i=1}^{n} X_T(G_i), \text{ thus } X = \sum_{i=1}^{n} X_T(G_i). \]
Obviously, \( \sigma(T | X_T(G)) \subset G \) and \( X_T(\{0\}) = \emptyset \).

Theorem 2.4. If \( T \) has an ASR \( E : \mathcal{U} \rightarrow A_I(T) \), then \( T \) is decomposable.

There are three different methods of proof on this theorem. Among those we give a proof using the following theorem.

Theorem 2.5 ([10]). For an operator \( T \), the following are equivalent.

(a) \( T \) is decomposable.
(b) For every open set \( G \) in \( C \), there is a \( T \)-invariant subspace \( M \) such that \( \sigma(T | M) \subset G \) and \( \sigma(T/M) \subset C - G \).

Proof of Theorem 2.4. Since \( \sigma(T | E(G)) \subset \overline{G} \) by definition, and \( \sigma(T/E(G)) \subset C - G \) holds if \( E(G) \) is analytically invariant under \( T \) (see [5], p. 60, Theorem 10). (In fact, \( \sigma(T/E(G)) \subset \sigma(T) - G \) since \( \sigma(T/E(G)) \subset \sigma(T) \)). Hence the conclusion follows by Theorem 2.5.

Further properties for an operator \( T \) having ASR were studied in [11].

3. A duality theorem for a strongly decomposable operator

In this section, we prove the main result, that is, if \( T \) is strongly decomposable with the spectrum \( \sigma(T) \) of \( T \), under some conditions, the dual operator \( T^* \) of \( T \) is strongly decomposable.

To begin with we list here some basic results.
PROPOSITION 3.1 ([1], p.1; [9], p.231). Let \( Y \) and \( Z \) be \( T \)-invariant subspaces such that \( Y \subset Z \). Then

1. \( Y \in AI(T) \) implies \( Y \in AI(T|Z) \)
2. \( Y \in AI(T|Z), \ Z \in AI(T) \) implies \( Y \in AI(T) \)
3. \( Z \in AI(T) \) if and only if \( Z/Y \in AI(T/Y) \)
4. \( (T|Z)|Y = T|Y \)
5. \( (T|Z)/Y = (T/Y)|Z/Y \)

We prove the following lemma using the above proposition.

LEMMA 3.2. Let \( T \) be decomposable. For an open set \( G \) in \( \mathcal{C} \), we put \( Y(G) = X_T(G), \ Z(G) = X_T(G), \ Y(G) = Z(G)/Y(G), \ \tilde{T} = T/Y(G) \)
\( \tilde{X}(G) = X/Y(G) \) and \( \tilde{T} \) is the dual operator of \( \tilde{T} \). Then
\( \tilde{\sigma}(\tilde{T}|Y(G)) \subseteq \tilde{G}, \ \tilde{\sigma}(\tilde{T}/Y) \subseteq \mathcal{C} - \tilde{G} \)
and \( Y \) is analytically invariant under \( \tilde{T} \).

Proof. Let \( G \) be arbitrary open in \( \mathcal{C} \) but fixed, let \( Y = Y(G), \ Z = Z(G) \) and \( \tilde{Y} = \tilde{Y}(G) \). By proposition 3.1, (3), \( \tilde{Y} = Z/Y \) is analytically invariant under \( \tilde{T} = T/Y \) since both \( Y \) and \( Z \) are analytically invariant under \( T \). Since \( T \) is decomposable, \( Y = X_T(G) \) is analytically invariant under \( T \), it is also analytically invariant under \( T|Z \). Thus we have
\( \tilde{\sigma}[(T|Z)/Y] \subseteq \tilde{\sigma}(T|Z) = \tilde{\sigma}(T|X_T(G)) \subseteq \tilde{G} \);
the first inclusion follows from the fact that, in general, if \( Y \) is analytically invariant (or spectral maximal) under \( T \), then \( \tilde{\sigma}(T) = \tilde{\sigma}(T|Y) \cup \tilde{\sigma}(T/Y) \) (see [9], p.227, Proposition 1.5).

Moreover, from the equality \( (T|Z)/Y = (T/Y)|(Z/Y) \), we have
\( \tilde{\sigma}(\tilde{T}|\tilde{Y}) = \tilde{\sigma}[(T|Y)|(Z/Y)] = \tilde{\sigma}(T|Z)/Y) \subseteq \tilde{G} \).
Since \( G \in \mathcal{U} \) is arbitrary, we have \( \tilde{\sigma}(\tilde{T}|\tilde{Y}(G)) \subseteq \tilde{G} \) for any \( G \in \mathcal{U} \).

Again fix \( G \). By the identification \( (T/Z)|X_T(G)^\perp \cong \sigma(T|Z) \), we get
\( \tilde{\sigma}(T^*|X_T(G)^\perp) = \sigma[(T/Z)^*] = \sigma(T/Z) \).
Furthermore, since \( Y \subset Z \), we have the following unitarily equivalence relation
\( (T/Y)^*|Z/Y)^\perp \cong T^*|Z^\perp \) (see [7], p.292, Lemma 5).
Therefore
\( \tilde{\sigma}((\tilde{T})^*|\tilde{Y}^\perp) = \tilde{\sigma}(T^*|Z^\perp) = \sigma[(T/Z)^*] = \sigma(T/Z) \)
\( = \sigma(T/X_T(G)) \subseteq \mathcal{C} - \tilde{G} \).
the last inclusion holds since \( Z(G) = X_T(G) = E(G) \) defines an ASR for \( T \) as we noted in Remark 2.3. In fact \( \sigma(T/X_T(G)) \subset \sigma(T) - G \) since \( \sigma(T/X_T(G)) \subset \sigma(T) \).

The arbitrariness of \( G \) implies \( \sigma[(\hat{T})^*|\hat{Y}(G)^\perp] \subset C - G \) for every \( G \in \mathcal{U} \). It follows that \( \sigma(\hat{T}/\hat{Y}) = \sigma[(\hat{T}/\hat{Y})^*] = \sigma[(\hat{T})^*|\hat{Y}(G)^\perp] \subset C - G \).

We have proved the lemma.

Now, we consider again the identification \( (T/X_T(G))^* = T^*|X_T(G)^\perp \). Since \( SM(T^*) = \{X_T(C - F)^\perp : F \in \mathcal{F}\} = \{X_T(G)^\perp : G \in \mathcal{U}\} \) (see [8], p. 1057, Remark), \( T^* \) is strongly decomposable if and only if \( T^*|X_T(G)^\perp \) is decomposable for every \( G \in \mathcal{U} \). Therefore, \( T^* \) is strongly decomposable if and only if \( T/X_T(G) \) is decomposable for every \( G \in \mathcal{U} \) since, in general, \( A \in B(X) \) is decomposable if and only if \( A^* \) is.

It is known that if \( T \) is strongly decomposable then \( T/M \) is decomposable for any spectral maximal space \( M \) for \( T \). Since \( X_T(G) = X_T(G \cap \sigma(T)) \), if \( \sigma(T) \) is finite then \( T/X_T(G) \) is decomposable for any \( G \in \mathcal{U} \), whence \( T^* \) is strongly decomposable.

Thus we have the following

**Proposition 3.3.** Let \( T \) be strongly decomposable. If the spectrum of \( T \) is finite, then \( T^* \) is strongly decomposable.

**Theorem 3.4.** Let \( T \) be strongly decomposable. If the spectrum \( \sigma(T) \) of \( T \) does not contain any isolated point, the interior of \( \sigma(T) = G_0 \) is nonempty and \( X_T(G_0) = X \) then \( T^* \) is strongly decomposable.

**Proof.** For those open sets such that \( G \cap \sigma(G) = \emptyset \), \( X_T[G \cap \sigma(T)] = \{0\} \), whence \( T/X_T(G) = T \) is decomposable. So we may assume without loss of generality that \( G \cap \sigma(T) \neq \emptyset \). Let \( G \in \mathcal{U} \) be arbitrary but fixed, and let \( H \) be any open set in \( C \). We put \( Y = X_T(G), Z = X_T(G \cup H), \bar{Y} = Z/Y, \bar{T} = T/Y \) and let \( (\bar{T})^* \) be the dual of \( \bar{T} \).

By the similar proof as that of Lemma 3.2, \( \bar{Y} \) is analytic invariant under \( \bar{T} \). Now, we prove that \( (\sigma(\bar{T}|\bar{Y}) \subset \bar{H}, \sigma(\bar{T}/\bar{Y}) \subset C - H \) for any \( H \in \mathcal{U} \).

Then, by Theorem 2.5., \( \bar{T} = T/X_T(G) \) is decomposable. Arbitrariness
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of $G$ implies that $T^*$ is strongly decomposable.

For an open set $H$ such that $\sigma(T) \cap H = \emptyset$,
$$\sigma(T \cap \{0\}) = \emptyset \subset H, \quad \sigma(T \cap \{0\}) = \sigma(T) \subset C - H;$$
where $0$ is the zero vector in $X/\overline{X_T(G)}$, that is, $\overline{X_T(G)} = 0$. Therefore, without loss of generality, we may assume that $\sigma(T) \cap H \neq \emptyset$. Since $\sigma(T) = \sigma(T/\overline{X_T(G)}) \subset C - G$, $\sigma(T) \cap H \neq \emptyset$, so $H - G \neq \emptyset$ and $\sigma(T) \cap H \neq \emptyset$.

**Case (a).** $\sigma(T \cap \overline{X_T(G \cup H)}) \neq \sigma(T)$.

Since $\overline{T|Z} = T|X_T(\overline{G \cup H})$ is decomposable, we have
\[ (**) \quad \sigma(\overline{T|Z}) - \sigma(T) \supset \sigma(T|Z) - \sigma(T|Y) \]
the last inclusion holds since $Y \in A(T|Z)$, so $\sigma(T|Z) \subset \sigma(T|Z)$; and since $\overline{Y} \in A(T|\overline{Y})$, $\sigma(T) \subset \sigma(T) = \sigma(T/\overline{X_T(G)}) \subset C - G$.

As we stated in Remark 2.2, $E(G) = \overline{X_T(G)}$ defines an ASR for $T$, $\sigma(T) \cap G \subset \sigma(T|X_T(\overline{G \cup H}) \subset \overline{G} \cap \sigma(T) \subset G \cup H \cap \sigma(T)$. Moreover since $(G \cup H) \cap \sigma(T) \neq \emptyset$, $\sigma(T)$ contains no isolated point, so $(G \cup H) \cap \sigma(T) = G \cup H \cap \sigma(T)$. Also $G \cap \sigma(T) = \overline{G} \cap \sigma(T)$.

Thus we have
$$\sigma(T|Z) - \overline{G} = \sigma(T|Z) - \overline{G} \cap \sigma(T|Z) \supset (G \cup H) \cap \sigma(T) - \overline{G} \cap \sigma(T|Z)$$
$$= [\overline{G} \cap \sigma(T)] \cup [\overline{H \cap \sigma(T)}] - \overline{G} \cap \sigma(T|Z) \neq \emptyset.$$

We claim that
$$[\sigma(T|Z) - \overline{G}]^- = \sigma(T|Z) - G.$$

Suppose $[\sigma(T|Z) - G]^- \subset \sigma(T|Z) - G$. Choose $\lambda$ belong to the right but not the left, then dist. $(\lambda, [\sigma(T|Z) - \overline{G}]^-) > 0$.

While $\lambda \in [\sigma(T|Z) - G]^- - [\sigma(T|Z) - \overline{G}] = \sigma(T|Z) \cap \partial G$, where $\partial G = \overline{G} - G$, the boundary of $G$. But this implies $\lambda \in [\sigma(T|Z) - \overline{G}]^-$, which is a contradiction. Therefore, we get, by (**), that
$$\sigma(T|\overline{Y}) = [\sigma(T|Z) - \sigma(T|Y)]^- \subset [G \cup H \cap \sigma(T) - \overline{G} \cap \sigma(T)]^-$$
$$= [(\overline{G} \cap \sigma(T)) \cup (\overline{H \cap \sigma(T)} - \overline{G} \cap \sigma(T))]^- \subset \overline{H \cap \sigma(T)} \subset \overline{H}.$$

i.e. $\sigma(T|\overline{Y}) \subset \overline{H}$.

**Case (b).** $\sigma(T|X_T(G \cup H)) = \sigma(T)$.

In this case, (**) can be written by

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\[ \sigma(T) - \bar{G} \subseteq \sigma(T) - \sigma(T|Y) \subseteq \sigma(T/Y) \subseteq \sigma(T) - G. \]

(i) If \( \sigma(T) - G = \emptyset \) then \( \sigma(\tilde{T}|\tilde{Y}) \subseteq \sigma(\tilde{T}) = \sigma(T/Y) = \emptyset. \)
Thus \( \sigma(\tilde{T}|\tilde{Y}) \subseteq \bar{H} \) for any \( H \in \mathcal{U}. \)

(ii) If \( \sigma(T) - \bar{G} \neq \emptyset \), then, by the same calculation as in (a), we have
\[ \sigma(\tilde{T}|\tilde{Y}) \subseteq \sigma(T/Y) \subseteq [\sigma(T) - \sigma(T|Y)]^{-1} \subseteq [\bar{G} \cup \bar{H}] \cap \sigma(T) - \bar{G} \cap \sigma(T)^{-1} \subseteq \bar{H}. \]

(iii) Finally, if \( \sigma(T) - \bar{G} = \emptyset \) but \( \sigma(T) - G \neq \emptyset \), then
\[ X = \overline{X_T(G_o) \subseteq X_T(G \cap \sigma(T))}. \]
Thus \( X/X_T(G) \) is the zero vector. Therefore, we have
\[ \sigma(\tilde{T}|\tilde{Y}) = \sigma[(T/Y) | X/X_T(G)] = \emptyset \subseteq \bar{H}. \]

For the second inclusion, the proof is the same as that of Lemma 3.2; by the identification \( (T/Y)^*(Z/Y)^\perp = T^*|Z^\perp \), we have
\[ \sigma(\tilde{T}/\tilde{Y}) = \sigma[(\tilde{T}/\tilde{Y})^\perp] = \sigma[(\tilde{T})^*|\tilde{Y}^\perp] = \sigma(T^*|Z^\perp) = \sigma(T/Z)^\perp = \sigma(T/Z) \subseteq \mathcal{C} - (G \cup H) \subseteq \mathcal{C} - H. \]
We completes the proof.

EXAMPLE 3.5. Let \( T \) be strongly decomposable with the spectrum \( \sigma(T) = [a, b], \ a < b. \) We prove that \( T^* \) is strongly decomposable. According to the Theorem 3.4, it is enough to show that \( \overline{X_T([a, b])} \)
\[ = X \text{ since } G_o = (a, b). \]

We choose a system of open sets \( G_n = \left( a - \frac{1}{n}, \ a + \frac{1}{n} \right) \cup \left( b - \frac{1}{n}, \ b + \frac{1}{n} \right) \) in \( \mathbb{R} \), \( n = 1, 2, \ldots \). Then \( \overline{G_{n+1}} \subseteq \overline{G_n} \) for \( n = 1, 2, \ldots \), whence
\[ X_T(\overline{G_{n+1}}) \subseteq X_T(\overline{G_n}), \overline{G_n} \cap \sigma(T) = [a, a + \frac{1}{n}] \cup (b - \frac{1}{n}, b], \text{ and} \]
\[ \bigcap_{n=1}^{\infty} [\overline{G_n} \cap \sigma(T)] = \lim_{n \to \infty} [\overline{G_n} \cap \sigma(T)] = (a, b) = \partial_{\mathbb{R}} \sigma(T). \]

In general, for any system of open sets \( \{H_i\}_{i=1}^s \) in \( \mathcal{C} \),
\[ X_T(\bigcup_{i=1}^s H_i) = \bigcup_{i=1}^s X_T(H_i) \text{ holds if } T \text{ is strongly decomposable (see [6], p. 86, Lemma 12.7). Therefore, we have} \]
\[ X = X_T([a, b]) \subseteq X_T([a, b] \cup G_n] = X_T([a, b]) + X_T(G_n). \]
Thus
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\[ X \subset X_T[(a, b)] + \bigcap_{n=1}^{\infty} X_T[\mathbb{G}_n \cap \sigma(T)] = X_T[(a, b)] + X_T[\bigcap_{n=1}^{\infty} \mathbb{G}_n \cap \sigma(T)] = X_T[(a, b)] + X_T[(\partial_R \sigma(T)) = X_T[(a, b)] + X_T[(a, b)] + X_T\{a, b\} \]

Moreover, since two closed sets \(\{a\}, \{b\}\) are disjoint

\[ X_T\{a, b\} = X_T\{a\} \oplus X_T\{b\} \]

Both \(X_T\{a\}\) and \(X_T\{b\}\) are contained in \(X_T[(a, b)]\); for, let \(\{\lambda_n\}\) be a sequence in \((a, b)\) such that \(\lambda_n \to b\) as \(n \to \infty\).

Since

\[ X_T(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}) \subset X_T(\{\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{n+1}\}) \subset X_T(a, b) \]

hold for any \(n \in N\), whence

\[ X_T\{b\} \subset X_T(\{\lambda_1, \lambda_2, \ldots, \lambda_m, \ldots, b\}) = \lim_{n \to \infty} X_T(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}) \subset X_T[(a, b)] \]

Similarly,

\[ X_T\{a\} \subset X_T[(a, b)] \]

Hence

\[ X_T\{a, b\} = X_T\{a\} \oplus X_T\{b\} \subset X_T[(a, b)] \]

and

\[ X = X_T[(a, b)] \]

**Theorem 3.6.** Let \(A = C[a, b]\) be the commutative Banach \(*\)-algebra of complex-valued continuous functions on \([a, b]\) endowed with the norm \(\|x\| = \sup_{t \in [a, b]} |x(t)| (x \in A)\) and the natural involution. The operator \(T\) of multiplication by independent variables in \(C[a, b]\) defined by \((Tx)(t) = tx(t) (t \in [a, b])\) is strongly decomposable and \(\sigma(T) = [a, b]\).

**Proof.** Let \(m \in C[a, b]\) be \(m(t) = t\), \(t \in [a, b]\). The multiplication operator \(T_m\) defined by \(T_m x = mx\). Since

\[ (T_m x)(t) = m(t)x(t) = tx(t), \text{ so } T = T_m \]

We prove that \(T_m\) is strongly decomposable: Since \([a, b]\) is compact Hausdorff for the usual topology, the maximal ideal space of \(A = C[a, b]\) is \([a, b]\) (See [13], p. 271, Example (a)). For every closed subset \(F\) of \([a, b]\) and \(t_0 \in F\), there exists a \(x \in C[a, b]\) such that \(x = 0\) on \(F\) and \(x(t_0) = 0\) thus \(C[a, b]\) is regular. By the Gelfand–Naimark theorem, \(A\) is also semisimple, whence every multiplication operator in \(A\) is super–decomposable (See [11], p. 42, Corollary 2.4), so it is strongly decomposable (See [11], p. 36, Theorem 1.3).

The fact \(\sigma(T) = [a, b]\) is well known.

**Corollary 3.7.** The operator of multiplication by independent variables
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in \( A = C[a, b] \) and its dual are strongly decomposable.

This follows from Example 3.5 and Theorem 3.6.

For the representation of \( T_m^* \), we consider \( A = C[a, b] \) as a Banach space, let \( A^* \) be its dual. By the Riesz's representation theorem \( T_m^* \) can be represented by Riemann–Stieltjes integral

\[
(T_m^*f)(x) = f(T_mx) = \int_a^b x(t) dw(t) \quad (x \in A, \ f \in A^*),
\]

where \( w \) is a bounded variation function on \( [a, b] \) and has the total variation \( \text{Var}(w) = \|f\| \).

References

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