GLOBAL HOLOMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A PARAMETER IN A STEIN MANIFOLDS

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1. Introduction

Let $S$ be a pure finite dimensional Stein space, $\mathbb{C}^n$ be the space of $n$ complex variables $z_1, z_2, \ldots, z_n$, $\Omega$ be a Stein domain of the product space $\mathbb{C}^n \times S$ of $\mathbb{C}^n$ and $S$. Let $\mathcal{O}_{\mathbb{C}^n}^{s}$ be the sheaf over $\Omega$ of germs of holomorphic functions of variables $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $s \in S$, $m$ be a positive integer, $a_{i_k}^j(z, s)$ be holomorphic functions on $\Omega$ for $i = 1, 2, \ldots, n$ and $p, q = 1, 2, \ldots, m$. We define sheaf homomorphisms $T_i, P_j$ and $P$ of $O^m : = \mathcal{O}_{\mathbb{C}^n}^{s}$ in $O^m$ for $i, j = 1, 2, \ldots, n$ by putting

$$T_i f = \left( \frac{\partial f_1}{\partial z_i} + \sum_{k=1}^{m} a_{1k}^i(z, s) f_k, \frac{\partial f_2}{\partial z_i} + \sum_{k=1}^{m} a_{2k}^i(z, s) f_k, \ldots, \frac{\partial f_m}{\partial z_i} + \sum_{k=1}^{m} a_{mk}^i(z, s) f_k \right),$$

$$P_j f = T_j f, \quad P_j f = T_{n-j+1}(P_{j-1} f) \quad (j = 2, 3, \ldots, n-1),$$

$$P f = P_n f$$

for $f = (f_1, f_2, \ldots, f_m) \in \mathbb{C}^m$, where we denote the column vector $f$ by $f = (f_1, f_2, \ldots, f_m)$ in order to conserve natural resources.

L. Ehrenpreis [2] considered an application of the sheaf theory to differential equations and gave a criterion for the existence of global solutions of differential equations where the existence of local solutions are assured. J. Kajiwara [5] applied the method of Ehrenpreis to ordinary differential equations in the analytic category. I. Wakabayashi [14] gave examples of domain of holomorphy in which an equation is not globally solvable. The equation $\frac{\partial f}{\partial z_1} = g$ is such an example, because there exists a simply connected domain in $\mathbb{C}^2$ on which $\frac{\partial f}{\partial z_1} = g$.

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has no global solution for some holomorphic functions $g$. H. Suzuki [12] stated a necessary and sufficient condition for the global existence of holomorphic solutions. S. I. Pinčuk [10] found sufficient conditions to solve the formulated problem and he found necessary and sufficient conditions for the solution of the problem. J. Kajiwara–T. Mori [6] obtained the necessary and sufficient condition that for any function $g \in H^0(Q, O^m)$ there is a function $f \in H^0(Q, O^m)$ satisfying the inhomogeneous equation $P_n f = g$ in case that $n=1$. M. Harita [4] obtained the condition in case that $n>1$. And J. Kajiwara–K.H. Shon [8] have obtained the equivalent relations for $H^1(Q, \text{Ker } P_n) = 0$ in case that $n=1$.

At first, we generalize the result of Kajiwara–Shon [8]. The method are based on the above [4, 8]. Nextly, we obtain the vanishing theorem of cohomology groups for domains, those are not Stein.

2. Preliminaries

Let $D$ be a Stein domain of the product space $C \times S$, $\mathcal{D}$ be the sheaf of germs of holomorphic functions on $D$ and $a_{pq}(z, s)$ be holomorphic functions on $D$. In case that $n=1$ of Section 1, we put $T_n f = Tf$ and let $\text{Ker } T$ be the kernel of $T$. Let $\phi : D \to S$ be the canonical projection. For $(z, s) \in D$, let $D(z, s)$ be the connected component of $\phi^{-1}(s)$ in $C \times \{s\}$ containing $(z, s)$, $\tilde{D}$ be the set of all cuts $D(z, s)$ for all $(z, s) \in D$, $\check{D}$, be the set of all simply connected $D(z, s)$ for $(z, s) \in D$, and define the mapping $\phi : \tilde{D} \to S$ by $\phi(D(z, s)) = s \in S$.

**Theorem 2.1 ([8]).** If $\tilde{D} = \check{D}$, if there exists a domain $E$ in $C \times S$ containing $D$ such that all coefficients $a_{pq}(z, s)$ are holomorphic in $E$ and that the space $\check{E}$ of cuts $E(z, s), (z, s) \in E$, is a Hausdorff space and if the parameter space $S$ is a Stein manifold, then the following properties (1), (2) and (3) are equivalent:

1. $H^1(D, \text{Ker } T) = 0$.
2. The dimension of $H^1(D, \text{Ker } T)$ is finite or countably infinite.
3. The set $\tilde{D}$ is a Stein manifold.

**Theorem 2.2 ([8]).** Let $D$ be a Stein domain of the product space $C \times S$ of $C$ and a pure finite dimensional Stein space $S$. If $H^1(D, \text{Ker } T)$
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\[ \frac{H^0(D, \mathcal{F}_m)}{TH^0(D, \mathcal{F}_m)} = 0, \text{ then either } D(z, s) \text{ is simultaneously simply} \]

connected for any \((z, s) \in D\) or \(D(z, s)\) is simultaneously doubly connected and satisfies \(H^0(D(z, s), \text{ Ker } T) = 0\) for any \((z, s) \in D\).

In case that \(D(z, s)\) is a doubly connected domain with \(H^0(Dz, s), \text{ Ker } T) = 0\) for any \((z, s) \in D\), then \(H^1(D, \text{ Ker } T) = 0\) holds if and only if \(\breve{D}\) is a Hausdorff space.

In case that \(D(z, s)\) is a simply connected domain for any \((z, s) \in D\), if the dimension of \(H^1(D, \text{ Ker } T)\) is finite or countably infinite and if all coefficients \(a_{pq}(z, s)\) are holomorphic in a domain \(E\) of \(\mathbb{C} \times S\) containing \(D\) such that \(\breve{E}\) is a Hausdorff space, then \(\breve{D}\) is a Hausdorff space and the domain \((D, \phi)\) over \(S\) is a domain of meromorphy. Moreover, if \(S\) is a Stein manifold then \(\breve{D}\) is also a Stein manifold. Conversely, if \(\breve{D} = \breve{D}\) is a Stein space, then we have \(H^1(D, \text{ Ker } T) = 0\).

3. Global holomorphic solutions

Let \(\phi_i : Q \longrightarrow \{(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, s)\}\) be the canonical projection. For \((z, s) = (z_1, z_2, \ldots, z_n, s) \in Q\), let \(\Omega_i(z, s)\) be the connected component of \(\phi_i^{-1}(x_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, s) \cap Q\) in \(\mathbb{C}^n \times \{s\}\) containing \((z, s) \in Q\) and \(\breve{Q}_i\) be the set of all cuts \(\Omega_i(z, s)\) for any \((z, s) \in Q\) \((i = 1, 2, \ldots, n)\). We define the mapping \(\phi_i : \breve{Q}_i \longrightarrow \{(x_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, s)\}\) by \(\phi_i(\Omega_i(z, s)) = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, s)\) for any \((z, s) \in Q\) and the canonical mapping \(\eta_i : Q \longrightarrow \breve{Q}_i\) by \(\eta_i(x_1, z_2, \ldots, z_n, s) = \Omega_i(z, s)\) for any \((z, s) \in Q\) \((i = 1, 2, \ldots, n)\). We define in the space \(\breve{Q}_i\) the strongest topology so that the mapping \(\eta_i\) is continuous. Then the mapping \(\phi_i\) is a local homeomorphism. We have short exact sequences of sheaves

\[ 0 \longrightarrow \text{ Ker } T_i \longrightarrow O^m \longrightarrow O^m \longrightarrow 0, \]

\[ 0 \longrightarrow \text{ Ker } P_j \longrightarrow O^m \longrightarrow O^m \longrightarrow 0 \]

and long exact sequences of cohomology groups

\[ \ldots \longrightarrow H^0(Q, O^m) \longrightarrow H^0(Q, O^m) \longrightarrow H^1(Q, \text{ Ker } T_i) \]

\[ \longrightarrow H^1(Q, O^m) \longrightarrow H^1(Q, O^m) \longrightarrow \ldots, \]

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\[ P_j \quad \cdots \quad H^0(Q, \mathcal{O}_m) \quad \longrightarrow \quad H^1(Q, \mathcal{O}_m) \quad \longrightarrow \quad H^1(Q, \ker P_j) \quad \longrightarrow \quad H^1(Q, \mathcal{O}_m) \quad \longrightarrow \quad \cdots \]

for \( i, j = 1, 2, \ldots, n \). Since \( Q \) is a Stein domain, we have \( H^i(Q, \mathcal{O}_m) = 0 \) for \( i \geq 1 \) and

\[
\begin{align*}
H^1(Q, \ker T_i) &= H^0(Q, \mathcal{O}_m) / T_i H^0(Q, \mathcal{O}_m), \\
H^1(Q, \ker P_j) &= H^0(Q, \mathcal{O}_m) / P_j H^0(Q, \mathcal{O}_m).
\end{align*}
\]

A necessary and sufficient condition for \( H^1(Q, \ker P) = 0 \) is that every function which is locally of a form \( Pf = g \) is also globally of the form (see [2]), and a necessary and sufficient condition that for any function \( g \in H^0(Q, \mathcal{O}_m) \) there exists a function \( f \in H^0(Q, \mathcal{O}_m) \) satisfying the form \( Pf = g \) is that there holds \( H^1(Q, \ker P) = 0 \).

Let \( E_i(z, s) \) be the connected component of \( \phi_i^{-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, s) \cap E_i \) in \( C^n \times \{s\} \) containing \( (z, s) \in E_i \) for any domain \( E_i \) in \( C^n \times S \). Hereafter, we consider the case that \( \Omega_i(z, s) \) is simply connected for each \( (z, s) \in \Omega \) and \( i = 1, 2, \ldots, n \).

**Lemma 3.1.** Let \( \Omega_i(z, s) \) be a simply connected domain for \( (z, s) \in \Omega \). If the dimension of \( H^1(Q, \ker P) \) is finite or countably infinite, then the dimensions of \( H^1(Q, \ker T_1) \) and \( H^1(Q, \ker P_{n-1}) \) are finite or countably infinite, respectively.

**Proof.** Since the dimension of \( H^1(Q, \ker P) \) is finite or countably infinite, we have \( \dim H^1(Q, \ker P) < +\infty \) by Y. T. Siu [11, Theorem 4] and then \( H^1(Q, \ker P) = 0 \) by Theorem 2.1. Therefore, we have \( H^0(Q, \mathcal{O}_m) = PH^0(Q, \mathcal{O}_m) \). Then there exists a function \( f \in H^0(Q, \mathcal{O}_m) \) such that \( Pf = g \) for any function \( g \in H^0(Q, \mathcal{O}_m) \). Letting \( f^1 = P_{n-1} f \), then we have \( T_1 f^1 = P_n f = g \). Hence we have \( H^1(Q, \ker T_1) = 0 \), that is, the dimension of \( H^1(Q, \ker T_1) \) is finite or countably infinite. For any function \( g \in H^0(Q, \mathcal{O}_m) \), we have \( T_1 g \in H^0(Q, \mathcal{O}_m) \). Hence there exists a function \( h \in H^0(Q, \mathcal{O}_m) \) such that \( T_1 g = Ph = T_1 (P_{n-1} h) \) for any \( g \in H^0(Q, \mathcal{O}_m) \). Thus we have \( T_1 (P_{n-1} f - g) = 0 \), and then \( P_{n-1} f = g \) for any function \( g \in H^0(Q, \mathcal{O}_m) \). So we have proved that \( H^1(Q, \ker P_{n-1}) = 0 \).

**Lemma 3.2.** Let \( \Omega_i(z, s) \) be simply connected domains for all \( 1 \leq i \leq n \) and \( (z, s) \in \Omega \). If the dimension of \( H^1(Q, \ker P) \) is finite or countably infinite, if there exists a domain \( E_i \) in \( C^n \times S \) containing \( \Omega \) for each \( i = 1, 2, \ldots, n \) and \( \Omega_i(z, s) \) is simply connected, then

\[
\begin{align*}
H^1(Q, \ker P) &= 0.
\end{align*}
\]
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1, 2, ⋯, n such that all coefficients \( a_{im}(z, s) \) are holomorphic in \( E_i \) and if the space \( \mathcal{E}_i \) of cuts \( E_i(z, s), (z, s) \in E_i \), is a Hausdorff space for each \( i = 1, 2, \cdots, n \), then the dimension of \( H^1(\Omega, \text{Ker } T_i) \) is finite or countably infinite, \( \mathcal{Q}_i \) is a Hausdorff space and the domain \((\mathcal{Q}_i, \phi_i)\) over the Stein space \( S \) is a domain of meromorphy for each \( i = 1, 2, \cdots, n \).

Proof. For \( i = 1 \), we have the result by Lemma 3.1 and Theorem 2.2. Suppose that the dimension of \( H^1(\Omega, \text{Ker } P_k) \) is finite or countably infinite for \( k < n \). By Lemma 3.1, we have the dimension of \( H^1(\Omega, \text{Ker } P_{n-k}) \) is finite or countably infinite and then \( H^0(\Omega, O^m) = P_{n-k}H^0(\Omega, O^m) \). Since \( T_{k+1}(P_{n-k-1}f) = P_{n-k}f \) for any \( f \in H^0(\Omega, O^m) \), we have \( H^1(\Omega, \text{Ker } T_{k+1}) = 0 \). That is, the dimension of \( H^1(\Omega, \text{Ker } T_{k+1}) \) is finite or countably infinite. And the remainder statements are desired by Theorem 2.2.

Theorem 3.3. Let \( \Omega \) be a Stein domain of \( C^n \times S \) and \( \Omega_i(z, s) \) be simply connected domains for \( (z, s) \in \Omega \) and \( 1 \leq i \leq n \). If the dimension of \( H^1(\Omega, \text{Ker } P) \) is finite or countably infinite, if there exists a domain \( E_i \) in \( C^n \times S \) containing \( \Omega \) for each \( i = 1, 2, \cdots, n \) such that all coefficients \( a_{im}(z, s) \) are holomorphic in \( E_i \) and if the space \( \mathcal{E}_i \) of cuts \( E_i(z, s), (z, s) \in E_i \), is a Hausdorff space for each \( i = 1, 2, \cdots, n \), then the dimension of \( H^1(\Omega, \text{Ker } T_i) \) is finite or countably infinite, \( \mathcal{Q}_i \) is a Hausdorff space and the domain \((\mathcal{Q}_i, \phi_i)\) over the Stein space \( S \) is a domain of meromorphy for each \( i = 1, 2, \cdots, n \). Conversely, if the simply connected domain \( \Omega_i(z, s) \) is a Stein space for each \( i = 1, 2, \cdots, n \), then \( H^1(\Omega, \text{Ker } P) = 0 \).

Proof. By Theorem 2.2 and Lemma 3.1 and 3.2, we have the theorem.

K. Oka [9] proved that every domain over \( C^n \) analytically convex in the sense of Hartogs is a domain of holomorphy. Therefore a domain of meromorphy over \( C^n \) coincides with a domain of holomorphy over \( C^n \). J. Kajiwara–E. Sakai [7] proved that the envelope of meromorphy of a domain over a Stein manifold \( S \) with respect to a family of meromorphic function on the domain is \( p \)-convex in the sense of F. Docquier–H. Grauert [1] and, therefore, is a Stein manifold. Especially, a domain of meromorphy over \( S \) coincides with a domain
of holomorphy over $S$.

**Lemma 3.4.** Under the assumption of Theorem 3.3, if $S$ is a Stein manifold, then $\mathcal{O}_i$ are Stein manifolds for all $i=1, 2, \cdots, n$.

**Proof.** For each $i=1, 2, \cdots, n$, since the unramified domain $(\mathcal{O}_i, \phi_i)$ over the Stein manifold $S$ is a domain of meromorphy by Theorem 3.3, it is pseudoconvex by Kajiwara–Sakai [7]. Thus $\mathcal{O}_i$ are Stein manifolds for all $i=1, 2, \cdots, n$ by Docquier–Grauert [1].

In case that $U_1 \subset U_2 \subset \cdots$ be a sequence of open Stein subsets in $X$ and $U=\bigcup_{j=1}^{\infty} U_j$, if $X$ is a Stein manifold, it is known that $U$ is Stein. And if $X$ is a Stein space, it is not known whether $U$ should be Stein. J.E. Fornaess [3] has given an example of a sequence of increasing Stein subsets in a manifold whose union is not Stein. If $U_1$ and $U_2$ are open Stein subsets of Stein space $X$, if $U=U_1 \cup U_2 \subset \subset X$ and if $\dim H^1(U, 0)<\infty$, then $U$ is Stein by L.M. Tovar [13].

Let $\mathcal{O}_{ij}=\mathcal{O}_i \cup \mathcal{O}_j$ for the above Stein manifold $\mathcal{O}_i$, each $i$ and $j$. Then the union $\mathcal{O}_{ij}$ is not necessarily Stein. We consider exclusively the case that $\mathcal{O}_i \cap \mathcal{O}_j \neq \phi$.

**Theorem 3.5.** Under the assumption of Lemma 3.4, if in addition $Q \subset E_i \subset \subset \mathbb{C}^n \times S$ for each $i=1, 2, \cdots, n$, then

$$H^1(\mathcal{O}_{jk}, \zeta) = H^0(\mathcal{O}_j \cap \mathcal{O}_k, \zeta) / R(H^0(\mathcal{O}_j, \zeta) \oplus H^0(\mathcal{O}_k, \zeta))$$

for some mapping $R$ and

$$H^q(\mathcal{O}_{jk}, \zeta) = 0 \quad (q \geq 2)$$

for any coherent analytic sheaf $\zeta$ on $\mathbb{C}^n \times S$ $(j, k=1, 2, \cdots, n)$.

**Proof.** The intersection $\mathcal{O}_j \cap \mathcal{O}_k$ is a Stein manifold for each $j, k=1, 2, \cdots, n$. For the coherent analytic sheaf $\zeta$, we have the Mayer–Vietoris exact sequence

$$
\begin{array}{c}
0 \rightarrow H^0(\mathcal{O}_{jk}, \zeta) \rightarrow H^0(\mathcal{O}_j, \zeta) \oplus H^0(\mathcal{O}_k, \zeta) \\
\rightarrow H^0(\mathcal{O}_j \cap \mathcal{O}_k, \zeta) \rightarrow H^1(\mathcal{O}_{jk}, \zeta) \rightarrow \cdots \\
\rightarrow H^{q-1}(\mathcal{O}_j \cap \mathcal{O}_k, \zeta) \rightarrow H^q(\mathcal{O}_{jk}, \zeta) \\
\rightarrow H^q(\mathcal{O}_j, \zeta) \oplus H^q(\mathcal{O}_k, \zeta) \rightarrow \cdots
\end{array}
$$
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for each \( j, k = 1, 2, \ldots, n \). By Lemma 3.4 and the theorem B of Cartan, we have

\[
H^q(\bar{\Omega}_j, \zeta) = H^q(\bar{\Omega}_k, \zeta) = 0, \\
H^q(\bar{\Omega}_j \cap \bar{\Omega}_k, \zeta) = 0 \quad (q \geq 1)
\]

for all \( j, k = 1, 2, \ldots, n \). Therefore, we have

\[
H^1(\bar{\Omega}_{jk}, \zeta) = H^0(\bar{\Omega}_j \cap \bar{\Omega}_k, \zeta) / R(H^0(\bar{\Omega}_j, \zeta) \oplus H^0(\bar{\Omega}_k, \zeta))
\]

and

\[
H^q(\bar{\Omega}_{jk}, \zeta) = 0 \quad (q \geq 2)
\]

for all \( j, k = 1, 2, \ldots, n \).

**Corollary 3.6.** A necessary and sufficient condition that for any function \( g \in H^0(\bar{\Omega}_j \cap \bar{\Omega}_k, \zeta) \) there exists a function \( f \in (H^0(\bar{\Omega}_j, \zeta) \oplus H^0(\bar{\Omega}_k, \zeta)) \) satisfying a form \( Rf = g \) is that there holds \( H^1(\bar{\Omega}_{jk}, \zeta) = 0 \). And a necessary and sufficient condition for \( H^1(\bar{\Omega}_{jk}, \zeta) = 0 \) is that every function which is locally of the form \( Rf = g \) is also globally of the form.

**References**


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