HOMOGENEOUS POLYNOMIALS SATISFYING CAUCHY INTEGRAL EQUALITIES

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1. Introduction

Let $\mathcal{P}_n$ be the class of all holomorphic homogeneous polynomials $\pi$ on $\mathbb{C}^n$ normalized so that

$$\max\{|\pi(z)|: |z_1|^2 + \cdots + |z_n|^2 = 1\} = 1.$$  

(1.1)

For $\pi \in \mathcal{P}_n$, if the sequence $C[\pi^{m+1}]$ of Cauchy integrals satisfies

$$C[\pi^{m+1}] = \gamma_m \pi^m, \quad m = 0, 1, 2, \ldots$$

(1.2)

for a sequence of positive numbers $\gamma_m$, then $\pi$ is said to satisfy the Cauchy integral equalities, CIE for short (See [1, 2]). Ahern and Rudin [1] noticed that if $\pi \in \mathcal{P}_n$ is a monomial or the sum-of-squares ($= z_1^2 + \cdots + z_n^2$) then it satisfies CIE and utilized this fact in their new proof of the BMOA-pullback theorem for such $\pi$. Choe [2] made more extensive study on CIE and asked whether there is a concrete characterization of $\pi \in \mathcal{P}_n$ satisfying CIE.

We observe that if $n = 2$, the sum-of-squares

$$z_1^2 + z_2^2 = 2\left(\frac{1}{\sqrt{2}}z_1 - \frac{i}{\sqrt{2}}z_2\right)\left(\frac{1}{\sqrt{2}}z_1 + \frac{i}{\sqrt{2}}z_2\right)$$

is obtained from the monomial $2w_1w_2$ in $\mathcal{P}_2$ by the unitary change of variables:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -i/\sqrt{2} \\ \frac{1}{\sqrt{2}} & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$  

This observation leads us to conjecture that if $\pi \in \mathcal{P}_2$ satisfies CIE then it can be transformed to a monomial by a unitary change of variables. We show in this paper that the conjecture is true for $\pi \in \mathcal{P}_2$ of degree $\leq 4$. More precisely we prove

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**Theorem 1.1.** If $\pi \in D_2$ of degree $\leq 4$ satisfies CIE then it can be transformed to a monomial by a unitary change of variables.

The proof is very technical. The CIE condition (1.2) gives an infinite number of nonlinear algebraic equations on the coefficients of the homogeneous polynomial $\pi$. If the degree of $\pi$ gets higher, the equations become too complicated to handle.

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Any unexplained notations are as in [3].

### 2. Known results on CIE

We summarize some known results on CIE for $D_n$ which will be used in the proof of Theorem 1.1.

**Proposition 2.1.** [1, Lemma 2.2] If $\pi(z) = b a z^2 \in D_n$ or $\pi(z) = z_1^2 + \cdots + z_n^2$ then $\pi$ satisfies CIE.

The following two propositions show that CIE holds only for very special polynomials in $D_n$.

**Proposition 2.2.** [2, 1, Remark 2.4] $\pi(z) = a_1 z_1^2 + \cdots + a_n z_n^2 (a_i \neq 0$ for every $i$) satisfies CIE if and only if $|a_1| = \cdots = |a_n| = 1$.

**Proposition 2.3.** [2, Example 3.8] If $d \geq 2$ and $\pi(z) = a_1 z_1^d + \cdots + a_n z_n^d (|a_1| = 1$ for every $i$) satisfies CIE then $d = 2$.

The following proposition gives a way of getting new polynomials satisfying CIE from an old one.

**Proposition 2.4.** [2, Lemma 3.6] If $\pi \in D_n$ satisfies CIE and $U$ is a unitary transformation of $C^n$ then $\pi \circ U$ also satisfies CIE.

### 3. Monomials and their unitary transforms

#### 3.1. The unitary group $U(2)$

We observe that any unitary matrix $U \in U(2)$ of $C^2$ is of the form

$$U = \begin{pmatrix} \mu \sqrt{1-r^2} & \lambda r \\ \nu r & -\bar{\mu}\lambda \sqrt{1-r^2} \end{pmatrix},$$

where $|\lambda| = |\mu| = |\nu| = 1, \ 0 \leq r \leq 1,$
which can be factored as $U = U_{\mu,\nu} V_r U_{1,\omega} (\omega = -\bar{\mu} \lambda)$, where

$$U_{\mu,\nu} = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \quad \text{and} \quad V_r = \begin{pmatrix} \sqrt{1-r^2} & -r \\ r & \sqrt{1-r^2} \end{pmatrix}.$$

### 3.2. Monomials in $\mathcal{D}_2$ and their transforms

We note that if $\pi(z) = b_{lm} z_1^l z_2^m \in \mathcal{D}_2$ then

$$|b_{lm}| = \sqrt{\frac{d^d}{l!m!}} (d = l + m)$$

by the normalization condition (1.1). By the unitary transformation corresponding to a suitable $U_{\mu,\nu}$, the monomial $\pi(z) = b_{lm} z_1^l z_2^m$ can be transformed to

$$(3.2) \quad \pi_{l,m}(z) = \sqrt{\frac{d^d}{l!m!}} z_1^l z_2^m \quad (d = l + m).$$

The unitary matrix

$$V = U_{1,\omega} = \begin{pmatrix} \sqrt{\frac{d}{l!}} & -\sqrt{\frac{m}{d}} \\ \sqrt{\frac{m}{d}} & \sqrt{\frac{l}{d}} \end{pmatrix}$$

transforms $\pi_{l,m}$ again to

$$(3.3) \quad \tilde{\pi}_{l,m} = \sqrt{\frac{d^d}{l!m!}} \left( \sqrt{\frac{l}{d}} z_1 - \sqrt{\frac{m}{d}} \omega z_2 \right)^l \left( \sqrt{\frac{m}{d}} z_1 + \sqrt{\frac{l}{d}} \omega z_2 \right)^m,$$

which has value 1 at $(1,0)$ if $l \geq 1$. We list this correspondence in the following table (3.4) for later references.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\pi_{l,m}(l \geq m)$</th>
<th>$\tilde{\pi}_{l,m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z_1$</td>
<td>$z_1$</td>
</tr>
<tr>
<td>2</td>
<td>$z_1^2$</td>
<td>$z_1^2$</td>
</tr>
<tr>
<td></td>
<td>$2z_1z_2$</td>
<td>$z_1^2 - (\omega z_2)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$z_1^3$</td>
<td>$z_1^3$</td>
</tr>
<tr>
<td></td>
<td>$3\sqrt{3}/2 z_1^2 z_2$</td>
<td>$z_1^3 - \frac{3}{2} z_1 (\omega z_2)^2 + \frac{1}{\sqrt{2}} (\omega z_2)^3$</td>
</tr>
<tr>
<td>4</td>
<td>$z_1^4$</td>
<td>$z_1^4$</td>
</tr>
<tr>
<td></td>
<td>$\frac{16}{3\sqrt{3}} z_1^3 z_2$</td>
<td>$z_1^4 - 2z_1^2 (\omega z_2)^2 + \frac{8}{3\sqrt{3}} z_1 (\omega z_2)^3 - \frac{1}{3} (\omega z_2)^4$</td>
</tr>
<tr>
<td></td>
<td>$4z_1^2 z_2^2$</td>
<td>$z_1^4 - 2z_1^2 (\omega z_2)^2 + (\omega z_2)^4$</td>
</tr>
</tbody>
</table>
PROPOSITION 4.1. Any homogeneous polynomial \( \pi \in \mathcal{P}_2 \) of degree \( d \geq 1 \) can be transformed to \( \pi' \) of the form

\[
\pi'(z_1, z_2) = z_1^d + a_2 z_1^{d-2} z_2^2 + \cdots + a_d z_2^d
\]

by a suitable unitary change of variables. (Note that the term \( z_1^{d-1} z_2 \) is missing in \( \pi' \)).

Proof. Suppose \( |\pi| \) attains its maximum at \((\zeta_1, \zeta_2)\) on the sphere \( S_2 \) of \( C^2 \). Choose a unitary transform \( U \) which maps \((1,0)\) to \((\zeta_1, \zeta_2)\) and set

\[
\pi'(z_1, z_2) = \pi \circ U(z_1, z_2) = z_1^d + a_2 z_1^{d-2} z_2^2 + \cdots + a_d z_2^d.
\]

Since \( |\pi'| \) attains its maximum 1 at \((1,0)\) and the vector field \( \frac{\partial}{\partial z_2} \) is tangential to \( S_2 \) at \((1,0)\), we have

\[
0 = \frac{\partial |\pi'|^2}{\partial z_2} (1, 0) = \pi'(1, 0) a_2 = a_1.
\]

This completes the proof.

We now proceed on the proof of Theorem 1.1.

4.2. The case \( d=1 \) or \( d=2 \). By Proposition 4.1, any \( \pi \in \mathcal{P}_2 \), of degree 1 can be transformed, by a unitary transformation, to \( \pi'(z) = z_1 \), a monomial. By Proposition 4.1 again any \( \pi \in \mathcal{P}_2 \) of degree 2 can be transformed to \( \pi'(z) = z_1^2 + a_2 z_2^2 \) by a unitary transformation. If \( \pi \) satisfies CIE, then either \( a_2 = 0 \) or \( |a_2| = 1 \) by Proposition 2.2. In either case, \( \pi' \) reduces to a monomial by a unitary transformation as we see in the table (3.4).

4.3. The case \( d=3 \). Suppose \( \pi \in \mathcal{P}_2 \) of degree 3 satisfies CIE. By Proposition 4.1, we may assume \( \pi \) is of the form

\[
\pi(z) = z_1^3 + a_2 z_1 z_2^2 + a_3 z_2^3.
\]

By another transformation corresponding to a suitable unitary matrix of the form \( U_{1,\omega} \), we may assume \( a_3 \geq 0 \). We compute \( C[\pi^2 \pi] \) as follows.
We note that
\begin{equation}
\pi(\zeta_1, \zeta_2)^2 = \zeta_1^6 + 2a_2\zeta_1^4\zeta_2^2 + 2a_3\zeta_1^3\zeta_2^3 + a_2^2\zeta_2^4 + 2a_2a_3\zeta_1\zeta_2^2 + a_3^2,$n=1
\end{equation}
and
\begin{equation}
\pi(\zeta_1, \zeta_2) \langle \zeta, \zeta \rangle^3 = \zeta_1^3(\zeta_1^6 + \bar{a}_2\zeta_1^4\zeta_2^2 + \bar{a}_3\zeta_1^3\zeta_2^3) + 3\zeta_1^2\zeta_2(\zeta_1^4\zeta_2^2 + \bar{a}_2\zeta_1^3\zeta_2^3 + \bar{a}_3\zeta_1^2\zeta_2^4) + 3\zeta_1\zeta_2^2(\zeta_1^4\zeta_2^2 + \bar{a}_2\zeta_1^3\zeta_2^3 + \bar{a}_3\zeta_1^2\zeta_2^4) + \zeta_2^3(\zeta_1^3\zeta_2^3 + \bar{a}_2\zeta_1^2\zeta_2^5 + \bar{a}_3\zeta_2^6).
\end{equation}
We use the orthogonality relations for monomials [3, Propositions 1.4.8 and 1.4.9] in the following computation of the Cauchy integral.

From (4.2) and (4.3), we have
\begin{equation}
C[\pi^2\pi](z) = \int_{S_2} \frac{\pi^2(\zeta)\pi(\zeta)}{(1 - \langle \zeta, \zeta \rangle^2)^2} d\sigma(\zeta)
\end{equation}
\begin{align*}
&= \sum_{j=0}^{\infty} \frac{(-2)^j}{3} \int_{S_2} \pi^2(\zeta)\pi(\zeta) \langle \zeta, \zeta \rangle^j d\sigma(\zeta) \\
&= 4 \left\{ \left( \frac{6!}{7!} + 2|a_2|^2 \frac{412!}{7!} + 2|a_3|^2 \frac{313!}{7!} \right) \zeta_1^3 \\
&\quad + 3 \left( 2a_2a_3 \frac{313!}{7!} + a_2^2a_3 \frac{214!}{7!} \right) \zeta_1^2 \zeta_2 \\
&\quad + 3 \left( 2a_2 \frac{412!}{7!} + a_2^2 \frac{214!}{7!} + 2a_2 |a_3|^2 \frac{51!}{7!} \right) \zeta_1 \zeta_2^2 \\
&\quad + \left( 2a_3^2 \frac{313!}{7!} + 2a_2 |a_3|^2 \frac{61!}{7!} + a_3 |a_3|^2 \frac{51!}{7!} \right) \zeta_2^3 \right\} \\
&= \frac{4 \cdot 4!}{7!} \left\{ (30 + 4|a_2|^2 + 3|a_3|^2) \zeta_1^3 \\
&\quad + 3(3a_2^2a_3 + 2a_2^2a_3^2) \zeta_1^2 \zeta_2 \\
&\quad + 3(4a_2 + 2a_2 |a_2|^2 + 10a_2 |a_3|^2) \zeta_1 \zeta_2^2 \\
&\quad + (3a_3 + 10 |a_2|^2a_3 + 30 |a_2|^3a_3^2) \zeta_2^3 \right\}.
\end{align*}
Comparing the coefficients in the CIE condition $C[\pi^2\pi] = \gamma_1\pi$, we have the following equations from (4.1) and (4.4). Recall that $a_3 \geq 0$ is assumed.

(a. 0) $30 + 4|a_2|^2 + 3a_3^2 = \tilde{\gamma}_1$,

(a. 1) $3a_2a_3 + 2a_2^2a_3 = 0$,

(a. 2) $3a_2(4 + 2|a_2|^2 + 10a_3^2) = a_2\tilde{\gamma}_1$,

(a. 3) $a_3(3 + 10 |a_2|^2 + 30a_3^2) = a_3\tilde{\gamma}_1$.

($\tilde{\gamma}_1 > 0$ is another constant).
Case 1. \(a_2 \neq 0\): Suppose \(a_2 \neq 0\). We solve (a.0) and (a.2) for \(|a_2|\) and get \(|a_2|=3\). If we set \(a_2=3\omega\) with \(|\omega|=1\), then
\[
\pi \left( \frac{1}{\sqrt{2}}, \frac{\omega}{\sqrt{2}} \right) = \left( \frac{1}{\sqrt{2}} \right)^3 + 3\omega \left( \frac{1}{\sqrt{2}} \right)^2 \frac{\omega}{\sqrt{2}} = \frac{4}{2\sqrt{2}} > 1,
\]
which is impossible for \(\pi \in \mathcal{D}_2\). Therefore \(a_2=0\) and \(\pi(z) = z_1^3\), a monomial.

Case 2. \(a_3 \neq 0\): Suppose \(a_3=0\). We solve (a.0) and (a.3) for \(a_3\) and get \(a_3=1\). But \(\pi(z) = z_1^3 + z_2^3\) cannot satisfy CIE by Proposition 2.3. We should have \(a_3 \neq 0\). From (a.1), we have \(a_2 = \frac{3}{2} \omega\) with \(\omega^3 = -1\).

Now, we solve (a.0) and (a.2) to get \(a_3 = \frac{1}{\sqrt{2}}\).

Therefore, we have
\[
\pi(z) = z_1^3 + \frac{3}{2} \omega z_1 z_2^2 + \frac{1}{\sqrt{2}} z_2^3 = z_1^3 - \frac{3}{2} z_1 (\omega z_2^2)^2 + \frac{1}{\sqrt{2}} (\omega^3 z_2)^3,
\]
which can be transformed to the monomial \(\frac{3\sqrt{3}}{2} z_1 z_2^2\) by a suitable unitary transformation as we see in the table (3.4).

4.4. The case \(d=4\). Suppose \(\pi \in \mathcal{D}_2\) of degree 4 satisfies CIE. By Proposition 4.1 we may assume \(\pi\) is of the form
\[
\pi(z) = z_1^4 + b_2 z_1^2 z_2^2 + b_3 z_1 z_2 + b_4 z_2^4.
\]
From the CIE condition \(C[\pi^2 \pi] = T_1 \pi\), we have the following equations on the coefficients as before.

(b.0) \(140 + 10 |b_2|^2 + 5 |b_3|^2 + 2 (b_2^2 + 2b_4) b_4 = T_1\),

(b.1) \(5b_3 b_2 + 2 (b_2^2 + 2b_4) b_5 + 5b_3 b_4 = 0\),

(b.2) \(60b_2 + 12 (b_2^2 + 2b_4) b_2 + 30b_2 |b_3|^2 + 30 (b_2^2 + 2b_2 b_4) b_4 = b_2 T_1\),

(b.3) \(20b_3 + 20 |b_2|^2 b_3 + 20 (b_2^2 + 2b_2 b_4) b_5 + 140b_3 |b_4|^2 = b_3 T_1\),

(b.4) \(2 (b_2^2 + 2b_4) + 5 (b_3^2 + 2b_2 b_4) b_2 + 35 |b_3|^2 b_4 + 140 |b_4|^2 b_4 = b_4 T_1\).

(\(T_1 > 0\) a constant)

Case 1. \(b_2=0, b_4=0\): From (b.4), \(b_2=0\); so \(\pi(z) = z_1^4\), a monomial.

Case 2. \(b_3=0, b_4 \neq 0\): Suppose \(b_2=0\). From (b.0) and (b.4), we have \(|b_4|=1\), which is impossible by Proposition 2.3. We should have \(b_2 \neq 0\). We may assume that \(b_2=\text{real}\) by a suitable unitary transformation. We solve (b.0), (b.2) and (b.4) for \(b_4\) and get \(b_4=1\); so \(\pi(z) = z_1^4 + b_2 z_1 z_2^2 + b_4 z_2^4\). Now we have to consider CIE condition for \(m=2\). Comparing the coefficients of \(z_1^3\) and \(z_1^5 z_2^2\) in \(C[\pi^2 \pi] = T_2 \pi^2\).
we have

\[ 332 + 17b_2^2 = \tilde{T}_2, \]
\[ 268 + 33b_2^2 = \tilde{T}_2, \quad \tilde{T}_2 > 0, \]
from which \( b_2 = \pm 2 \). Therefore we have \( \pi(z) = z_1^4 \pm 2z_1^2z_2^2 + z_2^4 \), which can be transformed to the monomial \( 4z_1^2z_2^2 \) by a unitary transformation as seen in the table (3.4).

Case 3. \( b_3 \neq 0, b_2 = 0 \): From (b.1), \( b_4 = 0 \). We show that this case cannot happen. We note that

\[ \pi^{m+1}(\zeta) = (\zeta_1^4 + b_3\zeta_2^2 \zeta_3^2)^{m+1} \]
\[ = b_3^{m+1} + (m+1) b_3^{m+3} \zeta_3^2 + \ldots \]
and

\[ \pi(\zeta) \zeta_1^m \zeta_2^2 \zeta_3^2 = b_3^m + b_3^{m+1} \zeta_3^2 \zeta_4^2 + \ldots. \]

From the orthogonality relations for monomials [3, Propositions 1.4.8 and 1.4.9], we have

\[ \int_{s_2} \pi^{m+1}(\zeta) \overline{\pi}(\zeta) \zeta_1^m \zeta_2^2 \zeta_3^2 d\sigma(\zeta) = b_3^m |b_3|^2 \frac{(m+1)! (3m+3)!}{(4m+5)!} + (m+1) b_3^m \frac{(m+4)! (3m)!}{(4m+5)!}, \]

and

\[ \int_{s_2} \pi^m(\zeta) \overline{\pi}(\zeta) \zeta_1^m \zeta_2^2 \zeta_3^2 d\sigma(\zeta) = b_3^m \frac{m! (3m)!}{(4m+1)!}. \]

Since \( b_3 \neq 0 \), the CIE condition (1.2) implies that

\[ \gamma_m = \frac{(m+1)(3m+3)(3m+2)(3m+1)}{(4m+3)(3m+2)(4m+3)} |b_3|^2 \]
\[ + \frac{(m+1)(m+4)(m+3)(m+2)(m+1)}{(4m+5)(4m+4)(4m+3)(4m+2)}. \]

This is contradictory to the fact that \( \gamma_m \to 1 \) as \( m \to \infty \). See [1, page 135].

Case 4. \( b_3 \neq 0, b_2 \neq 0 \): We may assume \( b_3 > 0 \) by a unitary transformation. (b.1) and (b.3) then reduce respectively to

(b.1)' \quad 5b_2 + 2(b_2^2 + 2b_4) + 5b_2b_4 = 0,
(b.3)' \quad 20b_2 + 20b_2^2 + 40b_2b_4 + 140 |b_4|^2 = \tilde{T}_1. \]

From (b.0) and (b.3)', we should have

(4.5) \quad \overline{b}_4 = \text{real} \quad \text{and} \quad b_2 \overline{b}_4 = \text{real}. \]

If we set \( b_2 = \rho \omega \) with \( \rho > 0 \) and \( |\omega| = 1 \) and set \( t = b_2 \rho \omega \) (=real), then

(4.6) \quad b_4 = \frac{t}{\rho^2 \omega^2}, \]
and
so $\omega^3 = \pm 1$. The equations (b. 0), (b. 1)', (b. 2), (b. 3)' and (b. 4) then can be written respectively as follows.

(b. 0)" $140 + 10 \rho^2 + 5 b_3^2 + 2 t + 4 \frac{t^2}{\rho^4} = \tilde{\gamma}_1$,

(b. 1)" $5 \rho + 2 \left( \rho^2 + 2 \frac{t}{\rho^2} \right) \omega^3 + 5 \frac{t}{\rho} = 0$,

(b. 2)" $60 + 12 \rho^2 + 24 \frac{t}{\rho^2} + 30 b_3^2 + 30 b_3^2 \frac{t}{\rho^2} \omega^3 + 60 \frac{t^2}{\rho^4} = \tilde{\gamma}_1$,

(b. 3)" $20 + 20 \rho^2 + 20 b_3^2 + 40 \frac{t}{\rho} \omega^3 + 140 \frac{t^2}{\rho^4} = \tilde{\gamma}_1$,

(b. 4)" $\frac{2 \rho^4}{t} + 4 + 5 b_3^2 + \omega^3 + 10 \rho^2 + 35 b_3^2 + 140 \frac{t^2}{\rho^4} = \tilde{\gamma}_1$.

We consider the cases $\omega^3 = 1$ and $\omega^3 = -1$ separately.

Subcase 1. $\omega^3 = 1$: From (b. 1)", we have

(4. 8) $t = - \frac{\rho^3 (2 \rho + 5)}{5 \rho + 4}$.

If we eliminate $\tilde{\gamma}_1$ from (b. 0)" and (b. 3)" and substitute (4. 8) in place of $t$, we get

(4. 9) $15 (5 \rho + 4)^2 b_3^2 = -20 \rho^5 - 460 \rho^4 - 1840 \rho^3 + 240 \rho^2 + 4800 \rho + 1920$.

From (b. 3)", (b. 4)" and (4. 8), we have

(4. 10) $5 (\rho + 11) (5 \rho + 4) b_3^2 = -60 \rho^4 - 420 \rho^3 - 560 \rho^2 + 560 \rho + 320$.

From (b. 0)", (b. 2)" and (4. 8), we again have

(4. 11) $5 (5 \rho + 4) (13 \rho - 10) b_3^2 = -20 \rho^5 - 340 \rho^4 - 1000 \rho^3 + 1360 \rho^2 + 3680 \rho + 1280$.

If we eliminate $b_3^2$ either from (4. 9) and (4. 10) or from (4. 9) and (4. 11), we have

(4. 12) $\rho^5 - 11 \rho^4 - 6 \rho^3 + 328 \rho^2 - 288 \rho^2 - 2160 \rho - 864 = 0$.

The equation (4. 12) can be factored as

(4. 13) $(\rho + 2) (\rho - 6)^3 (\rho^2 + 5 \rho + 2) = 0$.

Since $\rho > 0$, $\rho = 6$. (4. 10) then reduces to

$2890 b_3^2 = -184960$,

which is impossible since $b_3^2 > 0$. Therefore the case $\omega^3 = 1$ cannot happen.

Subcase 2. $\omega^3 = -1$: From (b. 1)", we have

$(5 \rho - 4) t = \rho^3 (2 \rho - 5)$,

which implies $5 \rho - 4 \neq 0$ and
Homogeneous polynomials satisfying Cauchy integral equalities

(4.14) \[ t = \frac{\rho^3(2\rho-5)}{5\rho-4}. \]

If we eliminate \( \tilde{t}_1 \) and \( t \) from (b.0)'', (b.3)'' and (4.14), we have
\[ (4.15) \quad 15(5\rho-4)^2 b_3^2 = 20\rho^5 - 460\rho^4 + 1840\rho^3 + 240\rho^2 - 4800\rho + 1920. \]

Eliminating \( \tilde{t}_1 \) and \( t \) from (b.3)''', (b.4)''' and (4.14), we get
\[ (4.16) \quad 5(\rho-11)(5\rho-4) b_3^2 = -60\rho^4 + 420\rho^3 - 560\rho^2 - 560\rho + 320. \]

If we eliminate \( b_3^2 \) from (4.15) and (4.16), we have
\[ (4.17) \quad \rho^6 + 11\rho^5 - 6\rho^4 - 328\rho^3 - 288\rho^2 + 2160\rho - 864 = 0. \]

The equation (4.17) can be factored as
\[ (4.18) \quad (\rho-2)(\rho+6)^3(\rho^2-5\rho+2) = 0. \]

Therefore we have either \( \rho=2 \) or \( \rho^2=5\rho-2 \). If \( \rho^2=5\rho-2 \), then the right hand side of (4.15) becomes zero; so \( b_3^2=0 \), a contradiction. Therefore \( \rho=2 \). We then have
\[ t = -\frac{4}{3}, \]
\[ b_3^2 = \frac{64}{27}, \quad \text{or} \quad b_3 = \frac{8}{3\sqrt{3}} \]
\[ b_4 = -\frac{1}{3} \omega^2. \]

We have then
\[ \pi(z) = z_1^4 + 2\omega z_1^2 z_2^2 + \frac{8}{3\sqrt{3}} z_1 z_2^3 - \frac{1}{3} \omega^2 z_1^4 \]
\[ = z_1^4 - 2z_1^2(\omega^2 z_2)^2 + \frac{8}{3\sqrt{3}} z_1 (\omega^2 z_2)^3 - \frac{1}{3} (\omega^2 z_1)^4, \]
which can be transformed to the monomial \( \frac{16}{3\sqrt{3}} z_1^3 z_2 \) by a suitable unitary transformation as in the table (3.4). This completes the proof of the Theorem 1.1.

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References

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