APPLICATIONS OF THE GENERALIZED EVALUATION SUBGROUPS ON CONVERSES OF THE LEFSCHETZ FIXED POINT THEOREM

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1. Introduction.
Let $X$ be a compact connected polyhedron and let $f : X \to X$ be a self map of $X$. Let $MF(f)$ stand for the least number of fixed points of self maps homotopic to $f$, $N(f)$ the Nielsen number of $f$, and $L(f)$ the Lefschetz number of $f$. We always have $N(f) \leq MF(f)$.

The celebrated Lefschetz fixed point theorem says that $L(f) \neq 0$ implies that every map homotopic to $f$ has a fixed point, i.e., $MF(f) > 0$. Its converse statement, "$L(f) = 0$ implies $MF(f) = 0$" is not always true even for homeomorphisms of closed manifolds, as shown by example in [Mc]. It is desirable to understand under what restrictions on the space or the self map the converse does hold true.

In [J1, J2], Jiang showed the following theorem as a converse of the Lefschetz fixed point theorem:

THEOREM. Let $X$ be a compact connected polyhedron without global separating points. Suppose $X$ satisfies the condition $\pi_1(X, x_0) = G(X, x_0) (= J(X))$. Then the Lefschetz number $L(f) = 0$ iff $f$ is homotopic to a fixed point free map.

THEOREM. Let $X$ be a compact connected polyhedron without global separating points. Suppose $\pi_1(X, x_0)$ is finite and the universal covering space $\tilde{X}$ has the same rational homology as $X$. Then for any $f : X \to X$, $L(f) = 0$ iff $f$ is homotopic to a fixed point free map.

It is also desirable to understand under what restrictions on the space which does not satisfy $\pi_1(X, x_0) = G(X, x_0)$ the converse does hold true.

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The purpose of this paper is to give a partial solution of the above question using the generalized evaluation subgroups of the fundamental group.

2. Notation and terminology.

Let $X$ be a topological space with $x_0$ as a base point. A homotopy $H : X \times I \to X$ is called a cyclic homotopy [Go] if

\[ H(x, 0) = H(x, 1) = x. \]

In another notation, $h_t$ is a cyclic homotopy if $h_0 = h_1 = 1_X$, where $1_X$ denotes the identity map of $X$. If $h_t$ is a cyclic homotopy, the path given by $\alpha : I \to X$ such that $\alpha(t) = h_t(x_0)$ is called the trace of $h_t$.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy form a subgroup $G(X, x_0)$ of the fundamental group which is called the evaluation subgroup [Go].

In [WK], the author and Kim defined the generalized evaluation subgroup $G(X, A, x_0)$ of the fundamental group as follows; Let $(X, A)$ be a topological pair and $i : A \to X$ be the inclusion. Consider the class of continuous functions $H : A \times I \to X$ such that

\[ H(x_0, 0) = H(x_0, 1) = i(x). \]

Then the map $h : I \to X$ defined by $h(s) = H(x_0, s)$ represents an element $[h]$ in $\pi_1(X, x_0)$. The set of all elements $[h] \in \pi_1(X, x_0)$ obtained in the above manner from some $H$ is denoted by $G(X, A, x_0)$. Thus for every $[h] \in G(X, A, x_0)$, there is at least one map $H : A \times I \to X$ such that $[H(x_0, _)] = [h]$. $H$ is called an affiliated map to $[h]$ with respect to $A$.

Let $A$ be locally compact and regular, and $X^A$ be the space of mappings from $A$ to $X$ with compact open topology. The map $p : X^A \to X$ given by $p(g) = g(x_0)$ is continuous. Thus $p$ induces a homomorphism

\[ p_* : \pi_1(X^A, i) \to \pi_1(X, x_0). \]

In this case, the image of $p_*$ is $G(X, A, x_0)$. Thus $G(X, A, x_0)$ is called the generalized evaluation subgroup of the fundamental group $\pi_1(X, x_0)$. 
It is easy to show that \( J(X) = G(X) \) is a subgroup of \( G(X, A) \).

3. Main results:

In the following theorem, we substitute the generalized evaluation subgroup for the evaluation subgroup in the converses of the Lefschetz fixed point theorem.

**Theorem 1.** Let \( X \) be a compact connected polyhedron without global separating points. Suppose there exists a compact connected subpolyhedron \( A \) of \( X \) such that \( A \) satisfies \( \pi_1(X, x_0) = G(X, A, x_0) \) and also satisfies either of the following:

1. if \( X \) has no local separating points, \( G(X, A, x_0) \) is abelian, or
2. if \( X \) has a local separating point, \( A \) has a deformation retract homeomorphic to \( S^1 \).

Then for any map \( f : X \to X \) such that \( f(X) \subset A \), \( L(f) = 0 \) iff \( f \) is homotopic to a fixed point free map.

**Proof.** Case 1. If \( X \) has no local separating points, then \( \pi_1(X, x_0) \) is an abelian group by the hypothesis. For any map \( f : X \to X \) such that \( f(X) \subset A \), we have

\[
\pi_1(X, x_0) = G(X, A, x_0) \subset G(X, f(X), x_0).
\]

Thus \( G(X, f(X), x_0) = \pi_1(X, x_0) \). Let \([h]\) be any element of \( G(X, f(X), x_0) \). Then there exists a homotopy \( H : f(X) \times I \to X \) such that

\[
H( ,0) = i = H( ,1) \quad \text{and} \quad H[x_0, ] = [h].
\]

Since \( x_0 \in f(X) \), there exists an element \( z \in X \) such that \( f(z) = x_0 \). Define a homotopy \( K : X \times I \to X \) by \( K = H(f x 1) \).

Then \( K \) is a continuous function and

\[
K(x, 0) = H(f(x), 0) = i f(x) = f(x),
K(x, 1) = H(f(x), 1) = i f(x) = f(x),
K(z, t) = H(f(z), t) = H(x_0, t) = h(t).
\]

Thus we have \([h]\) \( J(f, z) \). This means that \( f_*(\pi_1(X, z)) \subset J(f, z) \).

By Theorem 2.4.2 \( J_2 \), we obtain that \( L(f) = 0 \) implies \( N(f) = 0 \).
Since \( \pi_1(X, x_0) \) is an abelian group, \( X \) is not a surface of negative Euler characteristic. Therefore, \( X \) is a compact connected polyhedron and not a surface of negative Euler characteristic. If we use Theorem 1.6.3 \([J_2]\), we have \( MF(f) = N(f) \). By these two results, we have that \( L(f) = 0 \) implies \( MF(f) = 0 \). The converse is clear.

Case 2. Let \( X \) be a compact connected polyhedron with a local separating point which is not a global separating point. Since \( X \) has a compact connected subspace \( A \) which has a deformation retract homeomorphic to \( S^1 \), we have the inclusion \( i : S^1 \to A \) and the deformation retraction \( r : A \to S^1 \). Let \( f : X \to X \) be a self map such that \( f(X) \subset A \) and \( f_A : A \to A \) be its restriction. Consider

\[
g = r \circ f_A \circ i : S^1 \to S^1
\]

then \( g \) and \( f_A \) are of the same homotopy type. By homotopy type invariance of the Nielsen number (Theorem 1.5.3 \([J_2]\)), we have \( N(g) = N(f_A) \). For \( S^1 \), we know that \( g \) can be homotoped to a map \( k \) with exactly \( N(g) \) fixed points. Thus, on \( A \), the map \( f_A \) (homotopic to \( i \circ g \circ r \)) can be homotoped to \( i \circ k \circ r \) with exactly \( N(g) \) fixed points. We denote this homotopy by \( G \). Consider that \( A \) is an ANR and \( H' : (X \times 0) \cup (A \times I) \to A \) such that \( H'_{X \times 0} = f, H'_{A \times I} = G \), there exists a homotopy \( H : X \times I \to A \) such that \( H = H' \) on \( (X \times 0) \cup (A \times I) \). Let \( f' = i \circ H( , 1) : X \to X \). Then \( f'(X) \subset A \) and \( f'_A = i \circ k \circ r \). Since \( (X, A) \) is a pair of compact connected polyhedron and \( f : X \to X \) satisfies \( f(X) \subset A \), we have \( N(f_A) = N(f) \) (Corollary 1.5.5 \([J_2]\)).

Now \( f' \) and \( f'_A = i \circ k \circ r \) have the same fixed points. Thus \( f' \) has exactly \( N(f'_A) \) fixed points. Since \( f \) is homotopic to \( f' \), we have \( MF(f) \leq \# \text{Fix}(f') = \# \text{Fix}(k) = N(g) = N(f_A) = N(f) \).

Otherwise, \( N(f) \leq MF(f) \) is clear. Thus we have \( MF(f) = N(f) \). By case 1, we already know that \( L(f) = 0 \) implies \( N(f) = 0 \). Therefore we have that \( L(f) = 0 \) iff \( MF(f) = 0 \).

**Theorem 2.** Let \( X \) be a compact connected polyhedron without global separating points. Suppose \( \pi_1(X, x_0) \) is finite and \( X \) has a subspace \( A \) such that \( G(X, A, x_0) = \pi_1(X, x_0) \). Then for any map \( f : X \to X \)
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X such that \( f(X) \subset A \), we have \( L(f) = 0 \) iff \( f \) is homotopic to a fixed point free map.

Proof. Since \( \pi_1(X, x_0) \) is finite, \( X \) can not be a surface of negative Euler characteristic and \( X \) can not have a local separating point which is not a global one (Lemma 2.6.4 [J2]). So, according to Theorem 1.6.3 [J2], we have \( N(f) = MF(f) \) for any \( f : X \to X \). Since \( G(X, A, x_0) = \pi_1(X, x_0) \), we know that \( L(f) = 0 \) implies \( N(f) = 0 \). Thus we obtain the result.

References


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