A REMARK ON THE
HARDY–LITTLEWOOD–SOBOLEV–THEOREM

E. G. KWON

1. In the n-dimensional Euclidean space $E^n$, the maximal function
$Mf(x)$ of an integrable function $f(x)$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dy,$$

where $m(B(x,r))$ denotes the n-dimensional volume of the ball $B(x,r) = \{y \in E^n; |x-y| < r\}$ and $dy = dy_1 dy_2 \cdots dy_n$. Also the Riesz potentials are defined for $f(x)$ and $\alpha > 0$ by

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{E^n} |y|^{-n+\alpha} f(x-y) dy, \quad x \in E^n$$

with a constant $\gamma(\alpha) = \sqrt{\pi}^n 2^n \frac{\Gamma(\alpha/2)}{\Gamma(n/2-\alpha/2)}$. See [1. p.117].

The Hardy–Littlewood–Sobolev theorem (of fractional integration) says that if $f(x) \in L^p(E^n)$, $1 < p < \infty$, and $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ then

$$\|I_\alpha f\|_q \leq A_{p,q}\|f\|_p.$$  

Here $\|f\|_p$ denotes the usual $L^p(E^n)$ norm of $f(x)$ and $A_{p,q}$ denotes a constant depending only on $p$ and $q$ (and $n$) [1. p.119]. Compared with the Bessel potentials, it is known that the Riesz potentials leads to less favourable behavior as $|x| \to \infty$ [1. p.131]. Also if $f(x) \in L^p(E^n)$ and $f(x)$ is continuous in a deleted neighborhood of 0 then by a successive use of the intermediate value theorem one verifies that

$$|x|^n|f(x)|^p \sim \int_{|x|}^{2|x|} \cdots \int_{|x|}^{2|x|} |f(y)|^p dy_1 \cdots dy_n,$$

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which tends to 0 as \( |x| \to 0 \). Our question on this point is that how much the \( L^q \) behavior of \( I_\alpha f \) is affected by the decreasing rapidity of \( f(x) \) as \( |x| \to \infty \) or as \( |x| \to 0 \).

**Theorem.** Let \( 1 < p < \infty \), \( 0 < s \leq \infty \), and \( 0 < \alpha < \beta < n \). Suppose that \( f(x) \in L^p(\mathbb{E}^n) \) and

\[
F_\beta(x) = \text{ess sup}_y |x - y|^\beta |f(x - y)| \in L^s(\mathbb{E}^n),
\]

then

\[
(1) \quad \|I_\alpha f\|_q \leq C_{\alpha, \beta, p} \|f\|_p^{1-\delta} F_\beta^\delta,
\]

where

\[
(2) \quad \delta = \alpha/\beta \quad \text{and} \quad 1/q = (1-\delta)/p + \delta/s.
\]

**Corollary.** Let \( 0 < \alpha < \beta < n \), \( 1 < p < \infty \), \( q(1 - \frac{\alpha}{\beta}) = p \), \( f(x) \in L^p(\mathbb{E}^n) \) and \( |x|^\beta |f(x)| \) be essentially bounded. Then \( I_\alpha f \in L^q(\mathbb{E}^n) \).

2. For the proof of Theorem we let

\[
E = \{x ; Mf(x) < \infty \quad \text{and} \quad F_\beta(x) < \infty\}
\]

and

\[
t(x, f) = [F_\beta(x)/Mf(x)]^{1/\beta}.
\]

Then we divide \( \|I_\alpha f\|_q^2 \) into two parts;

\[
(3) \quad \|I_\alpha f\|_q^2 = \int_{\mathbb{E}^n} |I_\alpha f(x)|^q dx
\]

\[
= \left[ \int_{2|x| \leq t(x, f)} + \int_{2|x| > t(x, f)} \right] |I_\alpha f(x)|^q dx
\]

\[= (I) + (II).\]
First, to estimate (1) fix $x \in E$ such that $2|x| \leq t = t(x, f)$. Then since $|x - y| \geq \frac{|y|}{2}$ if $|y| > t$ in this case, we have

\[
\int_{|y| > t} |y|^{-n+\alpha} |f(x - y)|dy \leq F_\beta(x) \int_{|y| > t} |y|^{-n+\alpha} |x - y|^{-\beta}dy \\
\leq 2^\beta F_\beta(x) \int_{|y| > t} |y|^{-n+\alpha-\beta}dy \\
= 2^\beta (\beta - \alpha)^{-1} w t^{\alpha-\beta} F_\beta(x).
\]

Here $w$ is the volume of the unit sphere $S^{n-1} = \{\zeta \in E^n : |\zeta| = 1\}$. On the other hand, if we temporarily set

\[
\Omega(r) = \Omega(r, x) = r^{n-1} \int_{S^{n-1}} |f(x - r\zeta)|d\sigma(\zeta),
\]

where $d\sigma$ is the element of volume on $S^{n-1}$, then by use of the integration by parts we obtain

\[
\int_{|y| \leq t} |y|^{-n+\alpha} |f(x - y)|dy \\
= \int_0^t r^{-n+\alpha} \Omega(r)dr \\
= t^{-n+\alpha} \int_0^t \Omega(r)dr + (n - \alpha) \int_0^t r^{-n+\alpha-1} \left[ \int_{|y| < r} |f(x - y)|dy \right]dr \\
\leq \alpha^{-1} n V t^{\alpha} M f(x),
\]

where $V$ is the volume of the unit ball $\{x ; |x| < 1\}$.

Combining (4) and (5), we can majorize (1) ;

\[
(I) \leq A_{\alpha, \beta}^q \int_{2|x| \leq t(x, f)} M f(x)^q (1 - \delta) F_\beta(x)^q \delta dx,
\]

where $A_{\alpha, \beta} = \frac{1}{\gamma(\alpha)} \left[ \frac{n V}{\alpha} + \frac{2^\beta w}{(\beta - \alpha)} \right]$. 

3. Next, it is not difficult to see from [1. p.118] that

$$\int_{E^n} |y|^{-n+\alpha}|x-y|^{-\beta} dy = \frac{\gamma(\alpha)\gamma(n-\beta)}{\gamma(n+\alpha-\beta)}|x|^{\alpha-\beta}.$$ 

Thus,

$$\int_{2|x|>t} \left[ \int_{E^n} |y|^{-n+\alpha}|f(x-y)|dy \right]^q dx$$

$$\leq \gamma(\alpha)^{-q} \int_{2|x|>t} \left[ \int_{E^n} |y|^{-n+\alpha}|x-y|^{-\beta} dy \right]^q dx$$

$$= \gamma(n-\beta)^q\gamma(n+\alpha-\beta)^{-q} \int_{2|x|>t} F_\beta(x)^q |x|^{-q(\beta-\alpha)} dx$$

$$\leq B_{\alpha,\beta}^q \int_{E^n} Mf(x)^q(1-\delta)F_\beta(x)^{q\delta} dx,$$

where $B_{\alpha,\beta} = 2^{\beta-\alpha}\gamma(n-\beta)\gamma(n+\alpha-\beta)^{-1}$.

Therefore combining (6), (7), and (1),

$$\int_{E^n} |I_\alpha f(x)|^q dx \leq C_{\alpha,\beta}^q \int_{E^n} Mf(x)^q(1-\delta)F_\beta(x)^{q\delta} dx,$$

where $C_{\alpha,\beta} = A_{\alpha,\beta} + B_{\alpha,\beta}$. Applying Hölder’s inequality we finally obtain

$$\|I_\alpha f\|_q \leq C_{\alpha,\beta}\|Mf\|^{(1-\delta)}_p\|F_\beta\|^{\delta}_s.$$ 

Now the required result follows from the Maximal theorem [1. p.5].

4. Let us see that our exponents condition (2) on $q$ and $\delta$ are appropriate. For the purpose assume (1) and change $f(x)$ with its dilation defined by $\tau_\nu f(x) = f(\nu x)$, $\nu > 0$. Noting that

$$\|I_\alpha(\tau_\nu f)\|_q = \nu^{-\frac{n}{q}-\alpha}\|I_\alpha f\|_q,$$

$$\|\tau_\nu f\|_p = \nu^{-\frac{n}{p}}\|f\|_p,$$
[1. p.118] and

$$\sup_{y} |x-y|^\beta |\tau_\nu f(x-y)| = \nu^{-\beta} F_\beta(\nu x) = \nu^{-\beta} \tau_\nu F_\beta(x),$$

we have by (1),

$$\nu^{-\frac{n}{q} - \alpha} \|I_\alpha f\|_q \leq C_{\alpha, \beta, p} \nu^{-\frac{n(1-\delta)}{p} - \frac{n\delta}{s} - \beta\delta} \|f\|_p^{1-\delta} \|F_\beta\|_s^\delta$$

for all $\nu > 0$. Thus we should have

$$\frac{n}{q} + \alpha = \left[ \frac{1-\delta}{p} + \frac{\delta}{s} \right] n + \beta\delta. \tag{8}$$

If (8) holds independently of $n$, then (8) is equivalent to (2).

References


Department of Mathematics Education
Andong National University
Andong 760–749, Korea