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A Note on Bordism Stiefel-Whitney and Pontrjagin Numbers*

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It is well-known that the cobordism (bordism) theory can be applied to the classifying problems in topology ([1],[2] and [3]). Recently, an impotant tool in the Atiyah-Singer index theory is the cobordism theory ([5]). In this note, we shall prove some relations between characteristic classes and bordism groups (Theorem 5 and Corollary 6).

Let (X, A) be a topological pair. An *orineted singular n-manifold over* (X, A) is a pair (M, f), where M is an oriented, compact and n-dimensional C^∞ real manifold and $f: M \longrightarrow X$ is a continuous map such that for the boundary ∂M of M $f(\partial M) \subset A$. Note that if $A=\phi$(empty) then $\partial M=\phi$.

Definition 1. Let (M, f) be an oriented singular n-manifold over (X, A). If there is an oriented, compact, $n+1$-dimensional C^∞ real manifold W and a continuous F : $W \longrightarrow X$ such that

(i) there is a compact n-dimensional submanifold M' of ∂W such that $F(\overline{\partial W - M'}) \subset A$,

(ii) $(M', F|M')$ and (M, f) are isomorphic, i.e., there exists a continuous map $g: M' \longrightarrow M$ such that $F|M' = f \circ g$, then (M, f) is said to *bordant to zero,* written $(M, f) \sim 0$.

Let (M_1, f_1) and (M_2, f_2) be oriented singular n-manifolds over (X, A). It $(M_1 \cup -M_2, f_1 \cup f_2) \sim 0$($\cup$ means the disjoint union) then (M_1, f_1) and (M_2, f_2) are bordant to each other, where $-M_2$ is the manifold with the inverse orientation of M_2, $f_1 \cup f_2 | M_1 = f_1$ and $f_1 \cup f_2 | (-M_2) = f_2$, written $(M_1, f_1) \sim (M_2, f_2)$. Note that "$\sim$" is an equivalence relation.

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We put

$$O_n(X, A) = \text{the family of oriented singular } n\text{-manifolds over } (X, A)$$

and also

$$\Omega_n(X, A) = O_n(X, A)/\sim,$$

which is called the *n-dimensional bordism group* of (X, A). For each integer $n \geq 0$ $\Omega_n(X, A)$ is an abelian group ([4]).

Moreover,

$$\Omega_*(X, A) = \bigoplus_{n \geq 0} \Omega_n(X, A) \quad (\text{direct sum})$$

is the *oriented bordism ring* of (X, A) and is a graded additive group over the *Thom oriented bordism ring*

$$\Omega_* = \bigoplus_{n \geq 0} \Omega_n$$

([4], [7] and [8]). Ω is a generalized homology functor ([4]).

Definition 2. For topological pair (X, A) the natural map

$$\mu : \Omega_n(X, A) \longrightarrow H_n(X, A : \mathbf{Z}) \quad (\mathbf{Z} = \{\text{integers}\})$$

is defined by

$$\mu([M, f]) = f_* \sigma(M, \partial M)$$

where $[M, f] (\in \Omega_n(X, A))$ is the class of the oriented singular n-manifold over (X, A) and $\sigma(M, \partial M)$ is the fundamental homology class in $H_n(M, \partial M : \mathbf{Z})$.

The mapping μ is called the *Thom homomorphism*. Using this Thom homomorphism the following *Conner-Floyd* theorem is proved ([3], [7] and [8]) :

"Let X be a finite *CW*-complex and $H_*(X : \mathbf{Z})$ be free. If there exists a homomorphism

$$\theta : H_*(X ; \mathbf{Z}) \longrightarrow \Omega_*(X)$$

with degree 0 such that $\mu \circ \theta$ is the identity map of $H_*(X ; \mathbf{Z})$, then the Ω_*-isomorphism

$$\theta_* : \Omega_* \otimes H_*(X ; \mathbf{Z}) \xrightarrow{\cong} \Omega_*(X)$$

is induced from θ, where θ_* is of degree 0."

For a topological pair (X, A) we can construct the unoriented bordism ring $\mathcal{N}_*(X, A)$ which is a graded abelian group over the unoriented Thom bordism ring $\mathcal{N}_*$ ([4], [7] and [8]). Each element of $\mathcal{N}_*(X, A)$ will be denoted as $[M, f]_2$. The Thom homomorphism

$$\mu : \mathcal{N}_n(X, A) \longrightarrow H_n(X, A ; \boldsymbol{Z}_2)$$

is defined by

$$\mu([M, f]_2) = f_* \sigma(M, \partial M)_2$$

where $\sigma(M, \partial M)_2 \in H_n(M, \partial M : \boldsymbol{Z}_2)$ is the fundamental homology class of $(M, \partial M)$ with respect to $\boldsymbol{Z}_2$. Since $H_*(X ; \boldsymbol{Z}_2)$ is $\boldsymbol{Z}_2$-free the Conner-Floyd theorem above is described as follows ([3], [4] and [7]) :

"Let X be a finite CW-complex and a homomorphism with degree 0

$$\theta : H_*(X : \boldsymbol{Z}_2) \longrightarrow \mathcal{N}_*(X)$$

satisfy that $\mu \circ \theta$ is the identity map. Then the $\mathcal{N}_*$-isomorphism

$$\theta_* : \mathcal{N}_* \otimes H_*(X : \boldsymbol{Z}_2) \longrightarrow \mathcal{N}_*(X)$$

is induced from θ."

Definition 3. (i) For a compact n-dimensional C^∞ real manifold M without boundary let

$$\omega_k(M) \in H^k(M : \boldsymbol{Z}_2)$$

be the k^{th} Stiefel-Whitney class of the tangent bundle over M([6]). For non-negative integers $k, k_1, \cdots, k_r$ with $k + k_1 + \cdots + k_r = n$ and for each $x \in H^k(X : \boldsymbol{Z}_2)$ and $[M, f]_2 \in \mathcal{N}_n(X)$

$$<\omega_{k_1}(M) \cdots \omega_{k_r}(M) \ f^*(x), \ \sigma(M)_2> (\in \boldsymbol{Z}_2)$$

is called a *bordism Stiefel-Whitney number* of $f : M \longrightarrow X$ associated with $k, k_1, \cdots$, and k_r, where $<,>$ is the Kronecker product and $\omega_{k_1}(M)\omega_{k_2}(M) = \omega_{k_1}(M) \cup \omega_{k_2}(M)$ is the cup product.

(ii) Let M be an oriented compact n-dimensional C^∞ real manifold without boundary and $p_k(M) \in H^{4k}(M : \boldsymbol{Z})$ be the k^{th} Pontrjagin class. For continuous map $f : M \longrightarrow X$

and $x \in H^k(X : \boldsymbol{Z})$

$$<p_{k_1}(M) \cdots p_{k,s}(M)\ f^*(x),\ \sigma(M)>$$

is called a *bordism Pontrjagin number* of $f : M \longrightarrow X$ associated with non-negative integers $k, k_1, \cdots, k_s$ such that

$$k + 4(k_1 + \cdots + k_s) = n.$$

We have the following lemma ([4], [8]) :

Lemma 4. (i) (Thom) For $[M]_2 \in \mathcal{N}_n$ $[M]_2 = 0$ if and only if every bordism Stiefel-Whitney number of M is zero.

(ii) (Thom and Wall) For $[M] \in \Omega_n$ $[M] = 0$ if and only if every bordism Stiefel-Whitney and Pontrjagin numbers are zero.

Theorem 5. Let X be a topological space.

(i) If $[M_1, f_1]_2 \sim [M_2, f_2]_2$ in $\mathcal{N}_*(X)$ then bordism Stiefel-Whitney numbers of $f_1 : M_1 \longrightarrow X$ and $f_2 : M_2 \longrightarrow X$ are equal each other, respectively.

(ii) If $[M_1, f_1] \sim [M_2, f_2]$ in $\Omega_*(X)$ then bordism Stiefel-Whitney numbers and Pontrjagin numbers of $f_1 : M_1 \longrightarrow X$ and $f_2 : M_2 \longrightarrow X$ are equal each other, respectively.

Proof. (i) in $\mathcal{N}_*(X)$ $[M_1, f_1]_2 \sim [M_2, f_2]_2$ means that there exist a compact $(n+1)$-dimensional C^∞ real manifold B^{n+1} and a continuous map $F : B^{n+1} \longrightarrow X$ such that $\partial B^{n+1} = M_1 \cup M_2$ $F|M_1 = f_1$ and $F|M_2 = f_2$. In the long homology exact sequence of $(B^{n+1}, \partial B^{n+1} = M_1 \cup M_2)$:

$$\cdots \longrightarrow H^{n+1}(B^{n+1},\ \partial B^{n+1} : \boldsymbol{Z}_2) \xrightarrow{\partial_*} H_n(\partial B^{n+1} : \boldsymbol{Z}_2) \xrightarrow{i_*} H_n(B^{n+1} : \boldsymbol{Z}_2) \longrightarrow \cdots$$

$$\partial_*(\sigma(B^{n+1},\ \partial B^{n+1})_2) = \sigma(\partial B^{n+1})_2 = \sigma(M_1)_2 \oplus \sigma(M_2)_2$$

and

$$i_*(\sigma(M_1)_2 \oplus \sigma(M_2)_2) = i_*\ \partial_*(\sigma(B^{n+1},\ \partial B^{n+1})_2) = 0$$

where ∂_* is the boundary operator and $i : M_1 \cup M_2 = \partial(B^{n+1}) \longrightarrow B^{n+1}$ is the inclusion map. (Note that $\sigma(M_1 \cup M_2)_2 = \sigma(M_1)_2 \oplus \sigma(M_2)_2$ because of that $H_n(M_1 \cup M_2 : \boldsymbol{Z}_2) = H_n(M_1 : \boldsymbol{Z}_2) \oplus H_n(M_2 : \boldsymbol{Z}_2)$.)

Moreover, since

$$\omega_k(M_1 \cup M_2) = \omega_k(M_1) \oplus \omega_k(M_2) \in H^k(M_1 : \boldsymbol{Z}_2) \oplus H^k(M_2 : \boldsymbol{Z}_2),$$

for $k+k_1+\cdots+k_r=n$ ($k, k_i \geq 0$: integers) and $X \in H^k(X : \mathbf{Z}_2)$

$$\begin{aligned}
&<(\omega_{k_1}(M_1)\cdots\omega_{k_r}(M_1) \oplus \omega_{k_1}(M_2)\cdots\omega_{k_r}(M_2))\ (f_1^*(x) \oplus f_2^*(x)),\ \sigma(M_1)_2 \oplus \sigma(M)_2)_2> \\
&=<\omega_{k_1}(M_1)\cdots\omega_{k_r}(M_1) f_1^*(x),\ \omega(M_1)_2> + <\omega_{k_r}(M_2)\cdots\omega_{k_r}(M_2) f_2^*(x),\ \sigma(M_2)_2)_2> \\
&=<(i|M_1)^*\ (\omega_{k_1}(B_1)\cdots\omega_{k_r}(B_1))\ F^*(x),\ \sigma(M_1)_2> \\
&+<(i|M_2)^*\ (\omega_{k1}(B_2)\cdots\omega_{k_r}(B_2))\ F^*(x),\ \sigma(M_2)_2> \\
&=0
\end{aligned}$$

because $(i|M_1)_* \sigma(M_1)_2 = 0 = (i|M_2)_* \sigma(M_2)_2$. Since in $\mathbf{Z}_2$ $1=-1$ we have

$$<\omega_{k_1}(M_1)\cdots\omega_{k_r}(M_1)\ f_1^*(x),\ \sigma(M_1)_2> = <(\omega_{k_1}(M_2)\cdots\omega_{k_r}(M_2)\ f_2^*(x),\ \sigma(M_2)_2>.$$

(ii) In $\Omega_*(X)$ $[M_1, f_1] \sim [M_2, f_2]$ means that there exist an oriented and compact $(n+1)$-dimensional C^∞ real manifold B^{n+1} and a continuous map $F : B^{n+1} \longrightarrow X$ such that $\partial B^{n+1} = M_1 \cup -M_2$, $F|M_1 = f_1$ and $F|M_2 = f_2$. Since

$$\sigma(\partial B^{n+1}) = \sigma(M_1) \oplus \sigma(-M_2) = \sigma(M_1) \oplus -\sigma(M_2)$$

by the same way as above we obtain

$$\begin{aligned}
&<p_{k_1}(M_1)\cdots p_{k_s}(M_1)\ f_1^*(y),\ \sigma(M_1)> \\
&\quad -<p_{k_1}(M_2)\cdots p_{k_s}(M_2)\ f_2^*(y),\ \sigma(M_2)> = 0,
\end{aligned}$$

where $k+4(k_1+\cdots+k_s)=n$ and $y \in H^k(X : \mathbf{Z})$.

Similarly, for $k+k_1+\cdots+k_r=n$ we also obtain

$$<\omega_{k_1}(M_1)\cdots\omega_{k_r}(M_1)\ f_1^*(x),\ \sigma(M_1)> = <\omega_{k_1}(M_2)\cdots\omega_{k_r}(M_2)\ f_2^*(x),\ \sigma(M_2)>,$$

where $x \in H^k(X : \mathbf{Z})$. ///

Corollary 6. Let X be a finite CW-complex. For $[M, f]_2 \in \mathcal{N}_*(X)$ $[M, f]_2 = 0$ if and only if every bordism Stiefel-Whitney number of $f : M \longrightarrow X$ is zero.

Proof. Since $[M, f]_2 \sim 0$ implies that there exist a compact $(n+1)$-dimensional C^∞ real manifold B^{n+1} and a continuous map $F : B^{n+1} \longrightarrow X$ such that $\partial B^{n+1} = M$ and $F|\partial B^{n+1} = f$, by the same way as in the proof of (i) of Theorem 5, for each $k+k_1+\cdots+k_r=n$

$$<\omega_{k_1}(M)\cdots\omega_{k_r}(M)\ f^*(x),\ \sigma(M)_2> = 0$$

for every $x \in H^k(X : \mathbf{Z}_2)$.

We assume that every bordism Stiafel-Whitney number of $f : M \longrightarrow X$ is zero.

Since $H_n(X:\boldsymbol{Z}_2)$ is $\boldsymbol{Z}_2$-free for all $n\geq 0$ we take a basis $\{c_{n,i}\}$ of $H_n(X:\boldsymbol{Z}_2)$. Let $\{c^*_{n,i}\}$ be the dual basis of $H^n(X:\boldsymbol{Z}_2)$, i.e.,

$$<c^*_{n,i},\ c_{n,j}>=\delta_{ij}$$

Since the Thom homomorphism $\mu:\mathcal{N}_*(X)\longrightarrow H_*(X:\boldsymbol{Z}_2)$ is always an epimorphism ([8]), for each $c_{n,i}\in H_*(X:\boldsymbol{Z}_2)$ there exists an element $[M_i^n, f_i]_2\in\mathcal{N}_*(X)$ satisfying $\mu([M_i^n, f_i]_2)=C_{n,i}$, Then $\mathcal{N}_*(X)$ is a free $\mathcal{N}_*$-abelian group with a basis $\{[M_i^n, f_i]_2\}_{n,i}$ by the Conner-Floyd theorem described above (a homomorphism

$$\begin{array}{ccc}\theta : H_*(X:\boldsymbol{Z}_2) & \longrightarrow & \mathcal{N}_*(X)\\ \Cup & & \Cup\\ c_{n,i} & \longmapsto & [M_i^n, f_i]_2\end{array}$$

with degree 0, where $\mu([M_i^n, f_i]_2)=c_{n,i}$.). Thus we can write as

$$[M,f]_2=\sum_{m,i}[V_i^{n-m}]_2\,[M_i^m, f_i]_2$$

where $[V^{n-m}]_2\in\mathcal{N}_{n-m}\subset\mathcal{N}_*$. Then $M=\bigcup_{m,i} V_i^{n-m}\times M_i^m$
(disjoint union) $f=\bigcup_{m,i}\bar{f}_i$ where

$$\begin{array}{ccc}\bar{f}_i : V_i^{n-m}\times M_i^m & \longrightarrow & X\\ \Cup & & \Cup\\ (v,y) & \longmapsto & f_i(y)\end{array}$$

By the cross product

$$\begin{array}{ccc}X : H^p(V_i^{n-m}:\boldsymbol{Z}_2)\otimes H^q(M_i^m:\boldsymbol{Z}_2) & \longrightarrow & H^{p+q}(V_i^{n-m}\times M_i^m:\boldsymbol{Z}_2)\\ \Cup & & \Cup\\ a\otimes b & \longmapsto & a\times b\end{array}$$

for each $x\in H_*(X:\boldsymbol{Z}_2)$ we have $\bar{f}_i^*(x)=1\times f_i^*(x)$. In particular,

$$\omega_l(V_i^{n-m}\times M_i^m)=\sum_{j+k=l}\omega_j(V_i^{n-m})\times\omega_k(M_i^m)\qquad(*)$$

and

$$\sigma(V_i^{n-m}\times M_i^m)_2=\sigma(V_i^{n-m})_2\times\sigma(M_i^m)_2.$$

Since $[V_i^{n-m}]_2=0$ implies that $[V_i^{n-m}]_2\cdot[M_i^m, f_i]_2=0$
we assume that $[M,f]_2\neq 0$ and put

$$m_0=\text{the greatest } m \text{ such that } [V_i^{n-m}]_2\neq 0.$$

Then for $m_0+k_1+\cdots+k_r=n$ we have the following by our assumption :

$$0=\langle\omega_{k_1}(M)\cdots\omega_{k_r}(M)\ \ f^*(c^*_{m_0,i_0}),\ \sigma(M)_2\rangle$$

$$=\sum_i\langle\omega_{k_1}(V_i^{n-m_0}\times M_i^{m_0})\cdots\omega_{k_r}(V_i^{n-m_0})\ (1\times f_i^*(c^*_{m_0,i_0})),\ \sigma(V_i^{n-m_0})_2\times\sigma(M_i^{m_0})_2\rangle$$

$$=\sum_i\langle\omega_{k_1}(V_i^{n-m_0})\cdots\omega_{k_r}(V_i^{n-m_0})\times f_i^*(c^*_{m_0,i_0}),\ \sigma(V_i^{n-m_0})_2\times\sigma(M_i^{m_0})_2\rangle \text{(by(*) above)}$$

$$=\sum_i\langle\omega_{k_1}(V_i^{n-m_0})\cdots\omega_{k_r}(V_i^{n-m_0}),\ \sigma(V_i^{n-m_0})_2\rangle\langle f_i^*(c^*_{m_0,i_0}),\ \sigma(M_i^{m_0})_2\rangle$$

$$=\langle\omega_{k_1}(V_{i_0}^{n-m_0})\cdots\omega_{k_r}(V_{i_0}^{n-m_0}),\ \sigma(V_{i_0}^{n-m_0})_2\rangle.$$

Thus each bordism Stiefel-Whitney number of $V_{i_0}^{n-m_0}$ is zero.

Thus by (i) of Lemma 4 $[V_{i_0}^{n-m_0}]_2=0$ which contradicts to $[V_{i_0}^{n-m_0}]_2\neq 0$.

Therefore $[M,f]_2=0$. ///

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