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Some properties of general local cohomology*

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In §1 and §2 of this note we introduce the theory of general local cohomology modules and Artinian property of certain general local cohomology modules.

And in §3, we study the Attached prime ideals of $H_a^n(A)$ and the Matlis duality $\mathrm{Ass}(M)=\mathrm{Att}(D(M))$. Finally we prove the Theorem (3.3);

Suppose that A is a local ring of dimension n and Φ is a system of ideals of A. Then

$$\begin{aligned}\mathrm{Att}_A(H_\Phi^n(A)) &= \mathrm{Ass}(\mathrm{Hom}_A(H_\Phi^n(A),\ E)) \\ &= \mathrm{Ass}(H_\Phi^n(A)) \cap \{m\}.\end{aligned}$$

1. Preliminaries.

All rings considered in this paper will be commutative and noetherian and will have non-zero identity. Such a ring will be denoted by A and $\mathscr{C}_A$ will denote the category of all A-modules and all A-homomorphisms between them.

Let Φ be a non-empty set of ideals of A. We call Φ a *system of ideals* of A if whenever $a, b \in \Phi$, then thers is an ideal c in such that $c \subset ab$.

For every A-module M we define

$$L_\Phi(M) = \{x \in M \mid ax = 0 \text{ for some } a \in \Phi\} = \bigcup_{a \in \Phi} (0 : a)_M.$$

It is clear that $L_\Phi(M)$ is a submodule of M. Also for any A-homomorphism $f : M \longrightarrow N$, we define

$$L_\Phi(f) : L_\Phi(M) \longrightarrow L_\Phi(N)$$

to be the restriction of f to the submodule $L_\Phi(M)$ of M.

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Then $L_\Phi : \mathscr{C}_A \longrightarrow \mathscr{C}_A$ is an additive, covariant, A-linear and left exact functor. We call L_Φ *the general local cohomology functor* from $\mathscr{C}_A$ to $\mathscr{C}_A$ with respect to the system Φ.

Moreover it is easy to see that $L_\Phi(M/L_{\Phi(M)})=0$ for any A-module M. And whenever M is a submodule of N then $M \cap L_\Phi(N)=L_\Phi(M)$. In (7), we say that L_Φ is a *torsion radical* on A. And we may define the *torsion theory* $(\mathscr{T}, \mathscr{F})$ over A such that

$$\mathscr{T}=\{M\in\mathscr{C}_A \mid L_\Phi(M)=M\}, \quad \mathscr{F}=\{M\in\mathscr{C}_A \mid L_\Phi(M)=0\}$$

which corresponds to the torsion radical L_Φ on A.

Suppose that Φ is a system of ideals of A. Define the relation $\leq$ on Φ as follows;

$$a\leq b \text{ iff } a\supset b$$

It is clear that this relation is a partial order on Φ, now let $a. b\in\Phi$, there is an ideal $c\in\Phi$ such that $c\subset ab$. Hence $a\leq c$ and $b\leq c$.

So Φ is a directed set. Now for any $a, b\in\Phi$ with $a\leq b$, the natural A-homomorphism $A/b \longrightarrow A/a$ induces the A-homomorphism

$$\pi_a{}^b : Ext_A{}^n(A/a, \mathrm{M}) \longrightarrow Ext_A{}^n(A/b, M)$$

for are A-module M and each integer $n\geq 0$.

Hence we have the direct system of A-modules and A-homomorphisms

$$\{Ext_A{}^n(A/a, M),\ \pi_a{}^b\} \text{ over } \Phi.$$

We may form the direct limit

$$\varinjlim_{a\in\Phi} Ext_A{}^n(A/a, M).$$

for each $n\geq 0$.

On the other hand, We denote the n-th right derived functor of L_Φ by R^nL_Φ for each integer $n\geq 0$.

Proposition 1.1. Let Φ be a system of ideals of A and M an A-module. For each integer $n\geq 0$.

$$R^nL_\Phi(M)\cong \varinjlim_{a\in\Phi} Ext_A{}^n(A/a, M).$$

Proof. We can apply the Theorem 10 of (10) to the connected right sequences of covariant functors from $\mathscr{C}_A$ to $\mathscr{C}_A$;

$$\{\varinjlim_{a\in\Phi} Ext_A{}^n(A/a,-)\}_{n\geq 0} \text{ and } \{R^n L_\Phi\}_{n\geq 0}.$$

It is clear that, if $n>0$ and E is an injective A-module then

$$\varinjlim_{a\in\Phi} Ext_A{}^n(A/a,E)=0=R^n L_\Phi(E).$$

And, it is sufficient to show that $L_\Phi(M)$ and $\varinjlim_{a\in\Phi} \mathrm{Hom}_A(A/a,M)$ are isomorphic for an A-module M. This can be established by a argument for any $a\in\Phi$,

$$\mathrm{Hom}_A\ (A/a,M)\cong(0:a)_M$$

as A-modules.

We denote the module in Proposition (1.1) by $H_\Phi{}^i(M)$ for any A-module M. We call it *the general local cohomology module of* M *with respect to* Φ. In particular, when Φ consists of the powers of an ideal a of A, we denote $H_a{}^i(M)$ for any A-module M.

2. Artinian property of certain general local cohomology modules.

In this section, A will denote a local ring with maximal ideal m, and $\hat{A}$ will denote the m-adic completion of A.

Proposition 2.1. Let $f: A\longrightarrow\hat{A}$ be the natural homomorphism and Φ a system of ideals of A. Then the set

$$\Phi^f=\{a\hat{A}\,|\,a\in\Phi\}$$

is a system of ideals of $\hat{A}$ and, moreover,

(1) $H^i_{\Phi^f}(\hat{A})\cong H^i_\Phi(A)\otimes_A\hat{A}$.

(2) $H_\Phi{}^i(A)$ is an Artinian A-module iff $H^i_{\Phi^f}(\hat{A})$ is an Artinian $\hat{A}$-module.

Proof. Φ^f is a system of ideals for its elements are extended ideals of Φ.

(1). By definition,

$$H^i_{\Phi^f}(\hat{A})\cong\varinjlim_{a\hat{A}\in\Phi^f} H^i_{a\hat{A}}(\hat{A}).$$

But by (3),

$$H^{i}_{\mathfrak{a}\hat{A}}(\hat{A})\cong H^{i}_{\mathfrak{a}}(A)\otimes_{A}\hat{A}.$$

The result follows from the commuativity of direct limits and tensor products.

(2) Let $H^{i}_{\Phi}(A)$ be an Artinian A-module then $H^{i}_{\Phi}(A)$ has a natural structure as an $\hat{A}$-module and, as such, $H^{i}_{\Phi}(A)\otimes_{A}\hat{A}\cong H^{i}_{\Phi}(A)$. But by (1), $H^{i}_{\Phi}(A)\otimes_{A}\hat{A}\cong H^{i}_{\Phi^{f}}(\hat{A})$.

So $H^{i}_{\Phi^{f}}(\tilde{A})$ is an Arinian $\hat{A}$-module.

Conversely it is easy to see that if $H^{i}_{\Phi^{f}}(\hat{A})$ is Arinian $\hat{A}$-module then $H^{i}_{\Phi}(A)$ is an Artinian A-module.

Suppose that $f: A\longrightarrow B$ is a ring homomorphism and $\mathscr{T}$ is a torsion class over B, We denote *the idempotent filter* corresponding to $\mathscr{T}$ by $F(\mathscr{T})$. Then if $\mathscr{T}_{*}$ is the largest torsion class over A which has direct image $\mathscr{T}$, then we have

$$F(\mathscr{T}_{*})=\{a\,|\,a:\text{ideal of } A,\ aB\in F(\mathscr{T})\}$$

Proposition 2.2. Suppose that A is a local ring of dimension n, and Φ is a system of ideals of A.

Then $H^{n}_{\Phi}(A)$ is an Artinian A-module.

Proof. By Proposition (2.1). we may assume that A is complete. Hence by the Cohen structure theorem A is the homomorphic image of a complete local Gorenstein ring B of dimension n.

Let $f: B\longrightarrow A$ be the surjective homomorphism. Suppose that $\mathscr{T}$ is the torsion class corresponding to Φ. There exists a torsion class $\mathscr{T}_{*}$ over B which has direct image $\mathscr{T}$. Now if we let $\psi=F(\mathscr{T}_{*})$ then $\Phi=\psi^{f}=\{bA\,|\,b\in\psi\}$. We may regard A as a B-module by means of f and form the B-module $H^{n}_{\psi}(A)$. On the other hand we may use f to regard $H^{n}_{\Phi}(A)$ as a B-module.

By (1), there is an isomorphism of B-modules

$$H^{n}_{\Phi}(A)\cong H^{n}_{\psi}(A)$$

from which we see that it is sufficient to prove that $H^{n}_{\psi}(A)$ is an Artinian B-module Now set $C=\ker f$, so that A and B/C are isomorphic B-modules.

It follows that

$$H_{\phi}^{n}(A)\cong H_{\phi}^{n}(B/C)\cong H_{\phi}^{n}(B)\otimes_{B}B/C\cong H_{\phi}^{n}(B)/CH_{\phi}^{n}(B).$$

From this, it is sufficient to show that $H_{\phi}^{n}(B)$ is an Artinian B-module. This can be done by considering a minimal injective resolution for B and applying the functor L_{ϕ} to it and using the fact that if r is the maximal ideal of B then $E_{B}(B/r)$ is an Artinian B-module.

Corollary 2.3. Suppose that A is a local ring with dimension n and a is an ideal of A. Then $H_{a}^{n}(A)$ is an Artinian A-module.

3. Attached prime ideals of certain local cohomlogy modules.

An A-module S is said to be *secondary* iff $S\neq 0$ and, for all $x\in A$, the A-endomorphism of S given by multiplication by x is either surjective or nilpotent.

If S is a secondary A-module, then $r(0\,;S)$ is a prime ideal of A; if $r(0\,;S)=p$, we say that S is p-secondary.

Suppose M is a non-zero A-module. We say that M has a *secondary representation* iff there exists a finite number of secondary submodules $S_1, S_2, \cdots S_r$ of M such that

$$M=S_1+S_2+S_3+\cdots+S_r.$$

A reduced secondary representation for M is a secondary representation

$$M=S_1+S_2+\cdots+S_r.$$

with S_i, p_i-secondary $(i=1,2,\cdots,r)$ such that

(1) no S_i is redundant in the sum.

(2) $p_1, p_2, \cdots, p_r$ are all distinct.

These p_i are called *Attached prime ideals* of M and we denote

$$\mathrm{Att}(M)=\{p_1, p_2, \cdots, p_r\}$$

By (8), we know that any non-zero Artinian A-module M has a reduced sscondary representation and $\mathrm{Att}(M)=\{p\in\mathrm{Spec}(A)\mid$ there exists a quotiont module N of M such that $(0\,;N)=p\}$.

Proposition 3.1. Let a be a proper ideal of the local ring A, denote $\dim(A)$ by n. Then

$$\mathrm{Att}_{\hat{A}}\ (H_a{}^n(A)) = \{q \in \mathrm{Spec}(\hat{A})\ |\dim\ \hat{A}/q = n,\ \dim\ (\hat{A}/(a\hat{A}+q)) = 0\}$$

Proof. We may assume A is complete. First of all, suppose that $q \in \mathrm{Att}_{\hat{A}}(H_a{}^n(A))$. Then $H_a{}^n(A) \neq qH_a{}^n(A)$. Whence $H_a{}^n(A) \otimes_A A/q \neq 0$. It thus follows that $H_a{}^n(A/q) \neq 0$.

We may now use (12) to see that, over the complete local integral domain A/q, the local cohomology omdule $H^n_{(a+q)/q}(A/q) \neq 0$. It thus follows from (12) that $\dim(A/q) = n$, while the local Lichtenbaum-Hartshone theorem (14) shows that $\dim(A/(a+q)) = 0$.

Conversely, suppose that q is a prime ideal of A of dimension n for which

$$\dim(A/a+q) = 0.$$

Thus, in the n-dimensional complete local integral domain A/q, the ideal $a+q/q$ is m/q-primary.

Thus $H^n_{(a+q)/q}(A/q) = H^n_{m/q}(A/q)$. So, $H^n_{(a+q)/q}(A/q)$ is an Artinian A/q-module. And $H_a{}^n(A/q)$ has annihilator equal to q. Since $H_a{}^n(A/q)$ is a homomorphic image of $H_a{}^n(A)$, and it follows that $q \in \mathrm{Att}(H_a{}^n(A))$.

Let E denote the injective envelope of the residue field k of the local ring A, and let D denote *the Matlis duality functor*

$$\mathrm{Hom}_A(-, E) : \mathscr{C}_A \longrightarrow \mathscr{C}_A.$$

Whenever M is a non-zero finitely generated A-module, it is easy to see that $D(M)$ is Artinian.

Proposition 3.2. Let M be a non-zero finitely generated A-module. Then $\mathrm{Ass}(M) = \mathrm{Att}(D(M))$ where p is an associated prime of M if there exists $x \in M$ such that $\mathrm{Ann}(x) = p$.

Proof. Let $p \in \mathrm{Ass}(M)$. Then M has a cyclic submodule with annihilator equal to p. Since D is an exact additive functor on $\mathscr{C}_A$, M and $D(M)$ have the same annihilator. So $D(M)$ has a quotient module with annilator equal to p. Hence, by (8), $p \in \mathrm{Att}(D(M))$.

Conversely, suppose that $p \in \mathrm{Att}(D(M))$. Thus p is a prime ideal of A, and there is a quotient module of $D(M)$ which has annihilator equal to p. Thus $D(D(M))$ has a submodule which has annihilator equal to p. Now

$$D(D(M))=\mathrm{Hom}(\mathrm{Hom}(M,E),E)\cong M\otimes\mathrm{Hom}(E,E).$$

But $\mathrm{Hom}(E,E)$ is a A-isomorphic to $\hat{A}$, the completion of A. It follows that $M\otimes\hat{A}$ has an A-submodule W with $(0\,;W)=p$. By (4), $p\in\mathrm{Ass}(M)$.

Theorem 3.3. Suppose that A is a local ring of dimension n, and Φ is a system of ideals of A. Then

$$\mathrm{Att}_A(H_\Phi^n(A))=\mathrm{Ass}(\mathrm{Hom}_A(H_\Phi^n(A),E))=\mathrm{Ass}(H_\Phi^n(A))\cap\{m\}.$$

Proof. By Proposition (3.2) we obtain the first equality. By (4),

$$\mathrm{Ass}(\mathrm{Hom}_A(M,N))=\mathrm{Ass}(M)\cap\mathrm{Supp}(N).$$

for a finitely generated A-module M and an A-module N. We can obtain the second equality.

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