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A Note on K-Groups in Topology

Sunghee Kang

Jeonju Technical Junior College, Chonju 560-110, Korea

1. Introduction

This paper is mainly concerned with the K-theory which is fundamental and crucial to the study of index theory. Early in 1960, English Mathematician M.F. Atiyah formulated K-groups for topological space ([2]), and then the K-groups in algebraic systems were regulated by H. Bass approximately in 1965. [4]

Thus we have two branches in K-theory, one is algebraic K-theory developed by J. Milnor ([11]) and the other one is geometric K-theory which has been elaborately established by J.F. Adams, L. Hodgkin and M. Karoubi, etc([1], [6], [8], [13]).

The purpose of this paper is to epitomize some of results which has been obtained in seminar on K-theory performed during the last two semesters under my academic advisor. The contents of the paper is as follows.

In Section 2, we outline the formulation of K-groups and then prove some basic properties (Lemma 2.2 and Lemma 2.3).

In Section 3, we deal with the relative K-groups. The main results are Theorem 3.5 and Theorem 3.6. In Theorem 3.5, we shall prove that for the additive functor $\varphi_n : \varepsilon(X) \longrightarrow \varepsilon(X)$ $(E \longmapsto E\oplus\cdots\oplus E(n\text{-times}))$ if we put $K(\varphi_n)=K^{-1}(X : \boldsymbol{Z}/n)$ and $K^{-1}(X,Y; \boldsymbol{Z}/n)=\text{Coker } (K^{-1}(\boldsymbol{P} : \boldsymbol{Z}/n)\longrightarrow K^{-1}(X/Y : \boldsymbol{Z}/n))$ then $K^{-1}(X, Y : \boldsymbol{Z}/n)\longrightarrow K^{-1}(X : \boldsymbol{Z}/n)\longrightarrow K^{-1}(Y : \boldsymbol{Z}/n)$ is exact, where X is compact, Y is a closed subset of X and $\boldsymbol{P}$ is an one point space. Theorem 3.6 proves that for a Banach algebra A

$$K^{-1}(X,Y : A)\longrightarrow K^{-1}(X : A)\longrightarrow K^{-1}(Y: A)$$

is exact.

In Section 4, we study cup-product in K-groups. In particular, we prove in Theorem 4.3 that for locally compact spaces X and Y and a n-fold covering $\Pi : X\longrightarrow Y$, we have $\Pi_*(\Pi^*(y)\cdot x)=y\cdot \Pi_*(x)$, where $x\in K(X)$ and $y\in K(Y)$.

2. Preliminaries

Let M be an abelian monoid, and let $F(M)$ be the free abelian group with basis $\{[m] \mid m \in M\}$. We take a subgroup $D(M)$ which is generated by linear combinations of the form $[m+n]-[m]-[n]$, and put

$$S(M)=F(M)/D(M).$$

Then $S(M)$ is an abelian group with addition

$$[m]+[n]=[m+n],$$

which is called the *symmetrization* of M. It is clear that the inverse of $[m] \in S(M)$ is $-[m]=[-m]$, where we have to note that it $[m] \in F(M)$, then $[-m] \in F(M)$. In the product $M \times M$ we shall consider two equivalence relations:

$$(m,n) \sim (m',n') \Longleftrightarrow \exists p \in M \cdot\ni\cdot\ m+n'+p=n+m'+p$$

and

$$(m,n) \approx (m',n') \Longleftrightarrow \exists p,q \in M \cdot\ni\cdot\ (m,n)+(p,p)=(m',n')+(q,q)$$
$$\text{i.e., } m+p=m'+q \text{ and } n+p=n'+q.$$

Then we have

$$S(M) \cong M \times M/\sim \cong M \times M/\approx.$$

(**Proof**) Let $[m,n]$ be the equivalent class of (m,n) in $M \times M/\sim$. Then $[m,m]=0$ and $[m,n]+[n,m]=0$. That is, if we put $[m,n]=[m]-[n]$, then $[m,n]$ is the element $[m]-[n]$ of $S(M)$. Hence we can easily prove that $S(M) \cong M \times M/\sim$.

Next we shall prove that $M \times M/\sim \cong M \times M/\approx$. Let $\{m,n\}$ be the equivalent class of (m,n) in $M \times M/\approx$. Then for all $m \in M$ $\{m,m\}=\{0,0\}$ is the zero point of $M \times M/\approx$. Thus we can denote such that $\{m,n\}=\{m\}-\{n\}$. The map γ defined by

$$\begin{array}{ccc} \gamma : M \times M/\sim & \longrightarrow & M \times M/\approx \\ \Cup & & \Cup \\ [m,n] & \longmapsto & \{m,n\} \end{array}$$

is a group homomorphism, because of that if $[m,n]=[m',n']$. then $\{m,n\}=\{m',n'\}$. (Note that $\{m,n\}+\{n',m'\}=\{m+n',\ n+m'\}=\{0,0\}$ because that $m+n'+p=n+m'+p$ by $[m,n]=[m',n']$. That is $\{m,n\}=-\{n',m'\}=\{m',n'\}$). It is clear that γ is an isomorphism. ///

We define the monoid homomorphism $s: M \longrightarrow S(M)$ by $s(m)=[m,0]=[m]$. Then the symmetrization $S(M)$ satisfies the universal property such that for an abelian group A and a monoid homomorphism $f: M \longrightarrow A$ there exists a unique group homomorphism $\tilde{f}: S(M) \longrightarrow A$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ s\ } & S(M) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle E!\tilde{f}} & \\ & A & \end{array}$$

is commutative([7], [9]).

Example 2. 1. Let A be a commutative ring with 1, and let $\mathscr{P}(A)$ be the category consisting of all finitely generated projective A-modules and A-module homomorphisms. Then $\mathscr{P}(A)$ is an abelian monoid with direct sum as its addition.
We put $K(A)=S(\mathscr{P}(A))$

(i) If $(A, \mathfrak{M})$ is a local ring, then $K(A) \cong \boldsymbol{Z}$(the ring of all integers).

Proof. We prove that any minimal basis of M is a basis of M. Since $M/\mathfrak{M}M=M\otimes_A k$ is a vector space over $k=A/\mathfrak{M}$, it suffices to prove that, if $x_1, \ldots, x_n \in M$ are such that their images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{M}M$ are linearly independent over k, then they are linearly independent over A. If M is projective, then it is flat. The two following conditions are equivalent:$\cdots\cdots(A)$

(1) M is A-flat

(2) If $a_i \in A$, $x_i \in M$ $(1 \leq i \leq r)$ and $\sum_{i=1}^{r} a_i x_i=0$, then there exist an integer s, elements $b_{ij} \in A$ and $y_j \in M (1 \leq j \leq s)$ such that $\sum_i a_i b_{ij}=0$ for all j and $x_i=\sum_j b_{ij} y_j$ for all i.
Now we use induction on n. When $n=1$, put $ax_1=0$. Then, by (A) there exist an integer s, elements $b_{1j} \in A$ and $y_j \in M(1 \leq j \leq s)$ such that $ab_{1j}=0$ for all $j=1, \ldots, s$ and $x_1=\sum_{j=1}^{s} b_{1j} y_j$. Since $\bar{x}_1 \neq 0$ in $M/\mathfrak{M}M$, there exists an element $b_{1j} \in \mathfrak{M}$. Assume $b_{11} \notin \mathfrak{M}$.

Then b_{11} is a unit in A and $ab_{11}=0$. Hence $a=0$. Suppose $n>1$ and $\sum_{i=1}^{n} a_i x_i=0$. Also, by (A), there exist an integer s, elements $b_{ij} \in A$ and $y_j \in M(1 \leq j \leq s)$ such that $\sum_i a_i b_{ij}=0$ for all $j=1, \ldots, s$ and $x_i=\sum_j b_{ij} y_j$ for all $i=1, \ldots, n$. Since $\bar{x}_n \neq 0$ in $M/\mathfrak{M}M$ we have $b_{nj} \notin \mathfrak{M}$ for at least one j. Then since b_{nj} is a unit in A, we have

$$a_n=\sum_{i=1}^{n-1}(-b_{ij}/b_{nj})a_i.$$

Therefore, if $c_i=-b_{ij}/b_{nj}$,

$$0=\sum_{i=1}^{n} a_i x_i=a_1(x_1+c_1x_n)+\ldots+a_{n-1}(x_{n-1}+c_{n-1}x_n).$$

Since $\bar{x}_1+\bar{c}_1\bar{x}_n,\ldots,\bar{x}_{n-1}+\bar{c}_{n-1}\bar{x}_n$ are linearly independent over k,

we have $a_1=\ldots=a_{n-1}=0$ and $a_n=\sum_{i=1}^{n-1} c_i a_i=0$. ///

(ii) If A is a principal ideal domain, then $S(\mathscr{P}(A))=K(A)\cong \boldsymbol{Z}$.

Proof. Since A is a principal ideal domain it is a Dedekind domain (or A is a field). For two ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of A if there exist elements a_1 and a_2 of A such that $a_2\mathfrak{a}_1=a_1\mathfrak{a}_2$, then we say that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ belong to the *same ideal class*. Then, the ideal classes of A, which is denoted by $\mathscr{C}(A)$, is an abelian group under multiplication([4], [11]). In particular, the identity of $\mathscr{C}(A)$ is the class of principal ideals and $K(A)\cong \boldsymbol{Z}\oplus\mathscr{C}(A)$ ([11]). In our case, since A is a principal ideal domain $\mathscr{C}(A)=\{0\}$, and thus $K(A)\cong\boldsymbol{Z}$. ///

Throughout this paper, by a topological space we mean a Hausdorff topological space.

Let X be a topological space, and let $\mathscr{E}_k(X)$ be the category consisting of all k-vector bundles over X with finite ranks and bundle morphisms, where $k=\boldsymbol{R}$ (reals) or $\boldsymbol{C}$ (complexes). We shall sometimes put $\mathscr{E}(X)=\mathscr{E}_k(X)$. It is easy to prove that $\mathscr{E}(X)$ is an abelian monoid with the Whitney sum of bundles ([5], [9]). Moreover, $\mathscr{E}(X)$ is an additive category ([9]).

For a topological space X we define

$$K_k(X)=S(\mathscr{E}_k(x)),$$

and we call $K_k(X)$ the *real* (when $k=\boldsymbol{R}$) (complex when $k=\boldsymbol{C}$) *K-group* over X.

Let us put

$\boldsymbol{T}_{op}$=the category consisting of topological spaces and continuous maps between topological spaces

and

$\boldsymbol{A}_b$=the category of all abelian groups and group homomorphisms.

Then $K(=K_k): \mathbf{T}_{op} \longrightarrow \boldsymbol{A}_b$ is a cofunctor ([2]). In fact, for a continuous maps $f: X \longrightarrow Y$ and for a vector bundle E over Y let f^*E be the fiber product. Then, for each element $[E]-[F]$ of $K(Y)$

$$f^*([E]-[F])=[f^*E]-[f^*F].$$

Let $\boldsymbol{P}$ be a topological space of one point. It is clear that $K(\boldsymbol{P}) \cong \boldsymbol{Z}$. Moreover, we have continuous maps $i: \boldsymbol{P} \longrightarrow X$ and $j: X \longrightarrow \boldsymbol{P}$ such that $ji(\boldsymbol{P})=\boldsymbol{P}$. Thus the functors

$$i^*: K(X) \longrightarrow K(\boldsymbol{P}) \text{ and } j^*: K(\boldsymbol{P}) \longrightarrow K(X)$$

satisfy $i^*j^*=1_{k(P)}$. We put

$$\tilde{K}(X)=\mathrm{Ker}(i^*: K(X) \longrightarrow K(\boldsymbol{P})).$$

Then the sequence of groups

$$0 \longrightarrow \tilde{K}(X) \longrightarrow K(X) \longrightarrow K(\boldsymbol{P}) \longrightarrow 0$$

is split and exact. Thus, we have

$$K(X) \cong \tilde{K}(X) \oplus \boldsymbol{Z}(K(\boldsymbol{P}) \cong \boldsymbol{Z}).$$

$\tilde{K}(X)$ is called the *reduced K-group* of X([2], [10]).

For a topological space X and a finite dimensional vector bundle E over X we put

$$\Gamma(X,E)=\text{the set of all continuous sections } X \longrightarrow E.$$

Lemma 2.2. If X is paracompact and Y is a closed subset of X, then the restrictions homomorphism

$$\Gamma(X,E) \longrightarrow \Gamma(Y,E_Y) \ (f \longmapsto f|Y)$$

is surjective, where $E_Y=E|_Y$.

Proof. We first the case where E is trivial, i.e., $E \cong X \times k^n (k=\boldsymbol{R}$ or $\boldsymbol{C})$. Then

$$\Gamma(X,E)=\mathrm{Cont}(X,k^n)=\text{the set of all continuous functions } X \longrightarrow k^n.$$

Similarly, $\Gamma(Y,E_Y)=\mathrm{Cont}\ (Y,k^n)$. But the restrictions

$$\begin{array}{ccc} \mathrm{Cont}(X,k^n) & \longrightarrow & \mathrm{Cont}(Y,k^n) \\ \Uparrow & & \Uparrow \\ f & \longmapsto & f|Y \end{array}$$

is surjective due to the Tietze extension theorem ([12]).

In general case, let us take a locally finite open cover $\{U_i|i\in\Lambda\}$ of X such that E_{U_i} is trivial. Since X is paracompact and Hausdorff, we have an open cover $\{V_i|i\in\Lambda\}$ such that $\bar{V}_i\subset U_i$ for all $i\in\Lambda$, where $\bar{V}_i$ is the closure of V_i. We put $W_i=\bar{V}_i\cap Y$. Since $\Gamma(\bar{V}_i,E_{\bar{V}_i})\longrightarrow\Gamma(W_i,E_{W_i})$ is surjective, for each section $t\in\Gamma(Y,E_Y)$ if we put $t_i=t|W_i$ then there exists a section $s_i\in\Gamma(\bar{V}_i,E_{\bar{V}_i})$ such that $s_i|W_i=t_i$.

Let (α_i) be a partition of unity associated with the cover $\{V_i|i\in\Lambda\}$. We put

$$s'_i(x)=\begin{cases}\alpha_i(x)\ s_i(x) & {}^\forall x\in\bar{V}_i\\ 0 & {}^\forall x\in X-\bar{V}_i.\end{cases}$$

Then $s'_i\in\Gamma(X,E)$ which is zero over all but a finite number of the V_j.
Therefore $\sum_{i\in\Lambda}s'_i(x)$ is actually a finite sum on a neighborhood of each $x\in X$.
We put $s'(x)=\sum_{i\in\Lambda}s'_i(x)$ for all $x\in X$, then $s'\in\Gamma(X,E)$.
For each $y\in Y$ we have

$$\begin{aligned}s'(y)&=\sum_{i\in\Lambda}\alpha_i(y)\ s_i(y)=\sum_{i\in\Lambda}\alpha_i(y)\ t(y)\\ &=(\sum_{i\in\Lambda}\alpha_i(y))\ t(y)=t(y).\end{aligned}$$

Thus $s'|Y=t$. ///

Lomma 2.3. If X is compact and E is a vector bundle over X, then there exists a vector bundle E' over X such that $E\oplus E'$ is trivial.

Proof. Take a finite open cover $\{U_i|i=1,\dots,r\}$ of X such that $E_{U_i}(i=1,\dots,r)$ is trivial, i.e.,

$$E_{U_i}\cong U_i\times k^{n_i}\ (n_i\text{: nonnegative integer}).$$

Let $\{\alpha_i\}$ be a partiton of unity of X associated with the cover $\{U_i|i=1,\dots,r\}$. Since E_{U_i} is trivial we have n_i linear independent continuous sections $s_i^1,\dots,s_i^{n_i}$ of E_{U_i} $(i=1,\dots,r)$. Then these sections can be extended to X by the partition $\{\alpha_i\}$ such that $\alpha_is_i^1,\dots,\alpha_is_i^{n_i}$, where $\alpha_is_i^j$ is zero outside $U_i(j=1,\dots,n_i)$ and these are linearly independent sections of E_{V_i}, $V_i=\alpha_i^{-1}((0,1])$. Then for each $x\in V_i$ $\alpha_i(x)s_i^1(x),\dots,$ and $\alpha_i(x)s_i^{n_i}(x)$ generate E_x as a vector space. Put $n=\sum_{i=1}^{r}n_i$. Then we have an epimorphism

$$\eta: X\times k^n\longrightarrow E$$

such that for each $(x;\ \lambda_1,\dots,\lambda_n)\in V_i\times k^n\subset X\times k^n$

$$\eta(x;\ \lambda_1,\dots,\lambda_n)=\sum_{j=1}^{n_i}\alpha_i(x)S_i^j(x)$$

Hence, for each $x\in X$.

$$\eta_x : x\times k^n\longrightarrow E_x$$

is surjective. Therefore we have a bundle morphism

$$\xi : E\longrightarrow X\times k^n$$

such that $\eta\cdot\xi=1_E$ ([9]). In consequence, we have the split exact sequence of bundles over X :

$$0\longrightarrow \text{Ker } \eta\longrightarrow X\times k^n\longrightarrow E\longrightarrow 0.$$

Thus $E\oplus\text{Ker } \eta=X\times k^n$. ///

3. Relative K—Groups

Let X be a compact space. For each finite dimensional (k)-vector bundle E over $X(k=\boldsymbol{R}$ or $\boldsymbol{C})$ we shall define a Banach space topology on $\Gamma(X,E)$ as follows.

(i) When $E=X\times k$: $\Gamma(X,E)=\text{Cont}(X,k)=A$ is a Banach algebra with sup norm :

$$^{\forall}s\in\Gamma(X,E)\qquad \|s\|=\sup_{x\in X}|s(x)|\quad ([9]).$$

When $E=X\times k^n$: Since $\Gamma(X,E)\cong A^n$, we define a norm of A^n such that

$$^{\forall}(s_1,\dots,s_n)\in A^n\qquad \|(s_1,\dots,s_n)\|=\|s_1\|+\dots+\|s_n\|.$$

Then A^n is a Banach space([10], [14]).

(ii) When E is arbitrary. By Lemma 2.3 we have a vector bundle E' over X such that $E\oplus E'\cong X\times k^n$. Then

$$A^n\cong\Gamma(X,E\oplus E')\cong\Gamma(X,E)\oplus\Gamma(X,E').$$

Thus we have a surjective A-module homomorphism

$$u : A^n\longrightarrow\Gamma(X,E).$$

Let us endow the quotient topology on $\Gamma(X,E)$ by u. Note that the quotient topology of $\Gamma(X,E)$ is independent of the choice of u ([9]). In this case, $\Gamma(X,E)$ is a Banach space with the quotient topology.

Definition 3.1. (i) Let $\mathscr{C}$ be an additive category. For each pair $E,F\in\mathrm{Obj}(\mathscr{C})$ (the class of objects of $\mathscr{C}$) if $\mathrm{Hom}_{\mathscr{C}}(E,F)$ has a Banach space topology such that for each $G\in\mathrm{Obj}(\mathscr{C})$ the composition of morphisms :

$$\mathrm{Hom}_{\mathscr{C}}(E,F)\times\mathrm{Hom}_{\mathscr{C}}(F,G)\longrightarrow\mathrm{Hom}_{\mathscr{C}}(E,G)$$

is bilinear and continuous, then $\mathscr{C}$ has a *Banach structure. A Banach category* is an additive category provided with a Banach structure.

(ii) Let $\mathscr{C}$ and $\mathscr{C}'$ be additive categories, and let $\varphi:\mathscr{C}\longrightarrow\mathscr{C}'$ be an additive functor. If every object of $\mathscr{C}'$ is a direct summand of an object of the form $\varphi(E)$ $(E\in\mathrm{Obj}(\mathscr{C}))$, then φ is said to be *quasi-surjective.* If $\mathscr{C}$ and $\mathscr{C}'$ are Banach categories and the map

$$\mathrm{Hom}_{\mathscr{C}}(E,F)\longrightarrow\mathrm{Hom}_{\mathscr{C}'}(\varphi(E),\varphi(F))$$

is linear and continuous, then φ is called a *Banach functor*, where $E,F\in\mathrm{Obj}(\mathscr{C})$.

Let X be a compact space, and let Y be a closed subset of X. Then $\mathscr{E}(X)$ and $\mathscr{E}(Y)$ are Banach categories, and the restriction functor

$$\varphi:\mathscr{E}(X)\longrightarrow\mathscr{E}(Y)\ \ (E\longmapsto E_Y)$$

is a Banach functor ([2]), [9]). Moreover, the functor φ is a quasi-surjective, because of that for each $E\in\mathrm{Obj}(\mathscr{E}(Y))$ there exists $E'\in\mathrm{Obj}(\mathscr{E}(Y))$ such that $E\oplus E'$ is a trivial bundle over Y by Lemma 2.3 and thus

$$\varphi(X\times k^n)=Y\times k^n\cong E\oplus E'.$$

Definition 3.2. Let $\varphi:\mathscr{C}\longrightarrow\mathscr{C}'$ be a quasi-surjective functor. We put

$$\Gamma(\varphi)=\{(E,F,\alpha)\,|\,E,F\in\mathrm{Obj}(\mathscr{C})\text{ and }\alpha:\varphi(E)\cong\varphi(F)\}.$$

For two (E,F,α), $(E',F',\alpha')\in\Gamma(\varphi)$ if there are isomorphisms $f:E\cong E'$ and $g:F\cong F'$ in $\mathrm{Morph}(\mathscr{C})$ (=the class of morphisms of $\mathscr{C}$) such that

$$\begin{array}{ccc}\varphi(E) & \xrightarrow{\ \alpha\ } & \varphi(F)\\ {\scriptstyle\varphi(f)}\downarrow & & \downarrow{\scriptstyle\varphi(g)}\\ \varphi(E') & \xrightarrow{\ \alpha'\ } & \varphi(F')\end{array}$$

is commutative, then (E,F,α) and (E',F',α') are said to be *isomcrphic*, written $(E,F,\alpha)\cong(E',F',\alpha')$.

A triple $(E,F,\alpha)\in\Gamma(\varphi)$ is called *elementary* if $E=F$ and $\alpha:\varphi(E)\cong\varphi(E)$ is homotopic to $1_{\varphi(E)}$ within the automorphisms of $\varphi(E)$.

For triples (E,F,α) and (E',F',α') in $\Gamma(\varphi)$ we define the *addition* as

$$(E,F,\alpha)+(E',F',\alpha')=(E\oplus E',\ F\oplus F',\ \alpha\oplus\alpha').$$

We introduce the equivalence relation "$\sim$" on $\Gamma(\varphi)$ as follows.
$\sigma\sim\sigma'$ in $\Gamma(\varphi)\Longleftrightarrow$there exist elementaries τ and τ' in $\Gamma(\varphi)$ such that

$$\sigma+\tau\cong\sigma'+\tau'.$$

We put $K(\varphi)=\Gamma(\varphi)/\sim$ and use the following notation:

$$\begin{array}{ccc}\Gamma(\varphi) & \xrightarrow{\ d\ } & K(\varphi)\\ \Cup & & \Cup\\ (E,F,\alpha) & \longmapsto & d(E,F,\alpha).\end{array}$$

We have the following properties ([9]).

Property 3.3. With the above notations:

(i) $K(\varphi)$ is an abelian group with addition

$$d(E,F,\alpha)+d(E',F',\alpha')=d(E\oplus E',\ F\oplus F',\ \alpha\oplus\alpha')$$

(ii) $d(E,F,\alpha)+d(F,E,\alpha^{-1})=0$

$$d(E,F,\alpha)+d(F,G,\beta)=d(E,G,\beta\alpha)$$

(iii) $d(E,F,\alpha)=d(E,F,\alpha')\Longleftrightarrow\alpha$ and α' are homotopic within the isomorphisms from $\varphi(E)$ to $\varphi(F)$.

Let $\varphi:\mathscr{C}\longrightarrow\mathscr{C}'$ be a quasi-surjective functor. Since $\mathscr{C}$ and $\mathscr{C}'$ are abelian monoids with direct sum of objects as additions, we get the abelian groups

$$S(\mathscr{C})=K(\mathscr{C}),\ S(\mathscr{C}')=K(\mathscr{C}')$$

where $S(\mathscr{C})$ is the symmetrization of $\mathscr{C}$. We define the group homomorphisms as follows:

$$\begin{array}{ccccccc} i: K(\varphi) & \longrightarrow & K(\mathscr{C}), & \quad & j: K(\mathscr{C}) & \longrightarrow & K(\mathscr{C}')\\ \Cup & & \Cup & & \Cup & & \Cup\\ d(E,F,\alpha) & \longmapsto & [E]-[F] & & [E]-[F] & \longmapsto & [\varphi(E)]-[\varphi(F)].\end{array}$$

Then there is an exact sequence of abelian groups:

$$K(\varphi)\xrightarrow{i}K(\mathscr{C})\xrightarrow{j}K(\mathscr{C}'),$$

and moreover if there exists a functor $\psi:\mathscr{C}'\longrightarrow\mathscr{C}$ such that $\varphi\psi$ is isomorphic to $1_{\mathscr{C}'}$, then

$$0\longrightarrow K(\varphi)\xrightarrow{i}K(\mathscr{C})\xrightarrow{j}K(\mathscr{C}')\longrightarrow 0$$

is split and exact ([2], [9], [10]).

Property 3.4. Let X be a compact space and $A=C_k(X)=\{f:X\longrightarrow k\mid f$ is continuous$\}$ ($k=\boldsymbol{R}$ or $\boldsymbol{C}$). Then Γ: $\mathscr{E}(X)\longrightarrow\mathscr{P}(A)$ $(E\longmapsto\Gamma(X,E))$ induces an equivalence of categories $\mathscr{E}(X)\sim\mathscr{P}(A)$ (See Example 2.1) ([9], [10]).

Let X be a compact space, and let Y be a closed subset of X. As before, the restriction functor

$$\varphi:\mathscr{E}(X)\longrightarrow\mathscr{E}(Y)$$

is a quasi-surjective Banach functor. We put $K(\varphi)=K_k(X,Y)$(Sometimes $K(X,Y)$) and we call it the *relative K-group* with respect to X and Y. We shall define

$$\begin{array}{rccl}\varphi_n: & \mathscr{E}(X) & \longrightarrow & \mathscr{E}(X)\\ & \Cup & & \Cup\\ & E & \longmapsto & E\oplus\cdots\oplus E\ (n\text{-times}),\end{array}$$

and put $K^{-1}(X:\boldsymbol{Z}/n)=K(\varphi_n)$ (Note that φ_n is also a quasi-surjective Banach functor). For an one point space $\boldsymbol{P}$ we define

$$K^{-1}(X,Y:\boldsymbol{Z}/n)=\mathrm{Coker}(K^{-1}(\boldsymbol{P}:\boldsymbol{Z}/n)\longrightarrow K^{-1}(X/Y:\boldsymbol{Z}/n))$$

where Y is closed in X.

Theorem 3.5. Under the above situations

(i) for every $x\in K^{-1}(X:\boldsymbol{Z}/n)$ $x\oplus\cdots\oplus x=nx=0$

(ii) there is an exact sequence:

$$K^{-1}(X/Y:\boldsymbol{Z}/n)\longrightarrow K^{-1}(X:\boldsymbol{Z}/n)\longrightarrow K^{-1}(Y:\boldsymbol{Z}/\mathrm{n}).$$

Proof. (i) By Definition 3.2 each elemelt of $K^{-1}(X:\boldsymbol{Z}/n)$ is of the form $d(E,F,\alpha)$, where $E,F\in\mathrm{Obj}(\mathscr{E}(X))$ and

$$\alpha : E^n = E\oplus\ldots\oplus E \xrightarrow{\cong} F^n = F\oplus\ldots\oplus F \ (n\text{-times})$$

Thus, if we put $x = d(E, F, \alpha)$ then

$$x\oplus\ldots\oplus x(n\text{-times}) = d(E^n,\ F^n, \alpha\oplus\ldots\oplus\alpha(n\text{-times}))$$

Thus, we have to prove that $d(E^n,\ F^n,\ \alpha\oplus\ldots\oplus\alpha(n\text{-times})) = 0$.
Consider the triple$(E^n,\ E^n,\ 1_{E^{n^2}})$ and the following commutative diagram which implies that $d(E^n,\ E^n,\ \alpha\oplus\ldots\oplus\ \alpha(n\text{-times})) = 0$:

$$\begin{array}{ccc} E^n\oplus\ldots\oplus E^n(n\text{-times}) & \xrightarrow{\alpha\oplus\ldots\oplus\alpha} & F^n\oplus\ldots\oplus F^n(n\text{-times}) \\ {\scriptstyle 1_{E^n\oplus\ldots\oplus E^n}}\downarrow & & \downarrow{\scriptstyle \alpha^{-1}\oplus\ldots\oplus\alpha^{-1}} \\ E^n\oplus\ldots\oplus E^n(n\text{-times}) & \longrightarrow & E^n\ \oplus\ldots\oplus E^n(n\text{-times}). \end{array}$$

(ii) By Definition 3.2 we have

$$\begin{array}{ccc} \mathrm{K}^{-1}(X : \boldsymbol{Z}/\mathrm{n}) & \longrightarrow & K^{-1}(Y : \boldsymbol{Z}/n) \\ \Cup & & \Cup \\ d(E, F, \alpha) & \longmapsto & d(E_Y, F_Y, \alpha_Y), \end{array}$$

where $\alpha | Y = \alpha_Y : E_Y\oplus\ldots\oplus E_Y(n\text{-times}) \xrightarrow{\cong} F_Y\ \oplus\ldots\oplus\ F_Y(n\text{-times})$.

For each element

$$d(E', F', \alpha') \in K^{-1}(X/Y : \boldsymbol{Z}/n)$$

it is clear that $E', F' \in \mathrm{Obj}(\mathscr{E}(X/Y))$ by the above definition. Moreover, $\alpha' : (E')^n \xrightarrow{\cong} (F')^n$ and for $\{y_o\} \in X/Y$ (the base point) $\alpha'_{\{y_o\}} : (E'_{y_o})^n \xrightarrow{\cong} (F'_{y_o})^n$ is isomorphic to an elementary. For the canonical projection $j : X \longrightarrow X/Y$

$$d(j^*(E'),\ j^*(F'),\ j^*(\alpha)) \in K^{-1}(X : \boldsymbol{Z}/n)$$

and it is clear that $d(j^*(E')|Y,\ j^*(F')|Y,\ j^*(\alpha)|Y)$ is isomorphic to an elementary.

Next, for an element $d(E, F, \alpha) \in K^{-1}(X : \boldsymbol{Z}/n)$ we assume that

$$d(E_Y, F_Y, \alpha_Y) = 0 \text{ in } K^{-1}(Y : \boldsymbol{Z}/n).$$

Then $\alpha_Y : (E_Y)^n \xrightarrow{\cong} (F_Y)^n$ is isomorphic to an elementary in $\Gamma(\psi'_n)$
where $\psi'_n : \mathscr{E}(Y) \longrightarrow \mathscr{E}(Y)\ (E' \longmapsto E'^n)$
Thus, we may consider that $\alpha_Y \simeq 1_{(E_Y)^n}$ i.e., $d(E_Y, E_Y, \alpha_Y) = 0$. Since Y is closed in X which is compact, we have a closed subset V of X such that $Y \subset V \subset X$ and $\alpha_V = 1_{(E_V)^n}$,

Thus, by clutching, E_{X-Y} and the trivial bundle of rank n(=the rank of E_Y) over V/Y we have a vector bundle E' over X/Y and an isomorphism

$\beta : (E')^n \longrightarrow (E')^n$ such that

$$\begin{array}{ccc} K^{-1}(X/Y : \boldsymbol{Z}/n) & \longrightarrow & K^{-1}(X : \boldsymbol{Z}/n) \\ \Cup & & \Cup \\ d(E', E', \beta) & \longmapsto & d(E, E, \alpha) \end{array}$$

Hence we have the exact sequence

$$K^{-1}(X/Y : \boldsymbol{Z}/n) \longrightarrow K^{-1}(X : \boldsymbol{Z}/n) \longrightarrow K^{-1}(Y : \boldsymbol{Z}/n). \quad ///$$

Let A and B be Banach algebras. If there is a ring homomorphism $\varphi' : A \longrightarrow B$ ($\varphi'(1) = 1$), then there exists a Banach functor

$$\begin{array}{ccc} \varphi : \mathscr{P}(A) & \longrightarrow & \mathscr{P}(B) \\ \Cup & & \Cup \\ M & \longmapsto & M \otimes_A B \end{array}$$

Let X be a compact space, and let Y be a closed subset of X. Put

$$A(X) = \{f : X \longrightarrow A \mid f \text{ is continuous}\}$$

then $A(X)$ is a Banach algebra with sup norm. Since the restriction

$$A(X) \longrightarrow A(Y) \quad (f \longmapsto f \mid Y)$$

is a ring homomorphism we have a Banach functor

$$\begin{array}{ccc} \varphi : \mathscr{P}(A(X)) & \longrightarrow & \mathscr{P}(A(Y)) \\ \Cup & & \Cup \\ M & \longmapsto & M \otimes_{A(X)} A(Y). \end{array}$$

We put $K(\varphi) = K(X, Y : A)$.

Theorem 3.6. With the above notations.

(i) $K(X, Y : \boldsymbol{R}) = K_{\boldsymbol{R}}(X, Y)$, $K(X, Y : \boldsymbol{C}) = K_{\boldsymbol{C}}(X, Y)$

(ii) If we define

$$K^{-1}(X, Y : A) = K(X \times B^1,\ X \times S^0 \cup Y \times B^1 : A)$$

then we have the exact sequence of abelian groups :

$$K^{-1}(X, Y : A) \longrightarrow K^{-1}(X : A) \longrightarrow K^{-1}(Y : A),$$

where $B^1 = [0, 1]$ and $S^0 = \{0, 1\}$.

Proof. (i) At first, we want to prove that for each $E\in \mathrm{Obj}(\mathscr{E}(X))$ ($\mathscr{E}=\mathscr{E}_R$ or $\mathscr{E}_C$) $\Gamma(X,E)\otimes_{k(X)} k(Y)\cong\Gamma(Y,E_Y)$. For each $f\in\Gamma(X,E)$ and $h\in k(Y)$ we shall put $f\otimes h=f\cdot h$. Then for each $y\in Y$

$$(f\cdot h)\ (y)=f(y)\cdot h(y)$$

and thus $f\otimes h=f\cdot h\in\Gamma(Y,E_Y)$. Hence $\Gamma(X,E)\otimes_{k(X)} k(Y)\subseteq\Gamma(Y,E_Y)$. Conversely, by Lemma 2.2 for each $f\in\Gamma(Y,E_Y)$ there exists $\bar{f}\in\Gamma(X,E)$ such that $\bar{f}|Y=f$. This implies that $\Gamma(Y,E_Y)\subseteq\Gamma(X,E)\otimes_{k(X)} k(Y)$.
Hence we have

$$\Gamma(X,E)\otimes_{k(X)} k(Y)\cong\Gamma(Y,E_Y).$$

We have to recall that

$$\mathscr{P}(A(X))\sim\mathscr{E}(X),\quad \mathscr{P}(A(Y))\sim\mathscr{E}(Y)$$

if $A=\boldsymbol{R}$ or $\boldsymbol{C}$ (see Property 3.4). Take an element $d(E,F,\alpha)\in K_k(X,Y)$ ($k=\boldsymbol{R}$ or $\boldsymbol{C}$), then we have an element

$$d(\Gamma(X,E),\ \Gamma(X,F),\ \tilde{\alpha})\in K(X,Y;\ k)$$

where

$$\tilde{\alpha}:\Gamma(X,E)\otimes_{k(X)} k(Y)(\cong\Gamma(Y,E_Y))\cong\Gamma(X,F)\otimes_{k(X)}k(Y)(\cong\Gamma(Y,F_Y))$$

is induced from the isomorphism $\alpha: E_Y\cong F_Y$.

Conversely, for each $d(E',F',\alpha)\in K(X,Y:k)$, there exist $E,F\in\mathrm{Obj}(\mathscr{E}(X))$ such that

$$\Gamma(X,E)=E',\quad \Gamma(X,F)=F',\quad \tilde{\alpha}: E_Y\cong F_Y,$$

where $\tilde{\alpha}$ is induced from α such that

$$\begin{array}{lcl} E'\otimes_{k(X)}k(Y)=\Gamma(X,E)\otimes_{k(X)}k(Y)=\Gamma(Y,E_Y) & & E_Y \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\alpha\big\downarrow\cong & \Longrightarrow & \tilde{\alpha}\big\downarrow\cong \\ F'\otimes_{k(X)}k(Y)=\Gamma(X,F)\otimes_{k(X)}k(Y)=\Gamma(Y,F_Y) & & F_Y \end{array}$$

Since $d(E,F,\tilde{\alpha})\in K_k(X,Y)$,

$$K(X,Y:k)\cong K_k(X,Y).$$

(ii) We have to note that

$$K^{-1}(X : A) = K(X \times B^1,\ X \times S^0 : A).$$

We shall prove that

$$K(X \times B^1, X \times S^0 \cup Y \times B^1 : A) \longrightarrow K(X \times B^1, X \times S^0 : A) \longrightarrow K(Y \times B^1, Y \times S^0 : A)$$

is exact. Recall that A is a Banach algebra and

$$A(X \times B^1) = \{f : X \times B^1 \longrightarrow A \mid f \text{ is continuous}\}$$

For an element $d(E, F, \alpha) \in K(X \times B^1,\ X \times S^0 : A) = K^{-1}(X : A)$
we assume that

$$0 = d(E \otimes_{A(X \times B^1)} A(Y \times B^1), F \otimes_{A(X \times B^1)} A(Y \times B^1), \alpha \mid Y \times S^0) \in K(Y \times B^1, Y \times S^0 : A)$$

where

$$\begin{aligned}\alpha \mid Y \times S^0 : E \otimes_{A(X \times B^1)} A(Y \times B^1) \otimes_{A(Y \times B^1)} A(Y \times S^0) \\ = E \otimes_{A(X \times B^1)} A(Y \times S^0) \cong F \otimes_{A(X \times B^1)} A(Y \times S^0).\end{aligned}$$

By Lemma 2.2, we have an isomorphism

$$\beta : E \otimes_{A(X \times B^1)} A(X \times S^0 \cup Y \times B^1) \cong F \otimes_{A(X \times B^1)} A(X \times S^0 \cup Y \times B^1)$$

such that

$$\beta \mid X \times S^0 = \alpha,$$

because that $X \times S^0 \cup Y \times B^1$ is compact and

$$X \times S^0 \text{ is closed in } X \times S^0 \cup Y \times B^1.$$

Therefore

$$d(E, F, \beta) \in K^{-1}(X, Y : A)$$

such that

$$d(E, F, \beta \mid X \times S^0) = d(E, F, \alpha).$$

Next, for each $d(E, F, \beta) \in K^{-1}(X, Y : A)$, since

$$\beta|Y\times B^1 : E\otimes_{A(X\times B^1)} A(Y\times B^1)\cong F\otimes_{A(X\times B^1)} A(Y\times B^1)$$

we have the commutative diagram

$$\begin{array}{ccc} E\otimes_{A(X\times B^1)}A(Y\times B^1) & \xrightarrow{\beta|Y\times B^1} & F\otimes_{A(X\times B^1)} A(Y\times B^1) \\ {\scriptstyle 1_E\otimes_{A(X\times B^1)}A(Y\times B^1)}\downarrow & & \downarrow{\scriptstyle 1_E\otimes_{A(X\times B^1)}A(Y\times B^1)\ (\beta|Y\times B^1)^{-1}} \\ E\otimes_{A(X\times B^1)}A(Y\times B^1) & \longrightarrow & E\otimes_{A(X\times B^1)}A(Y\times B^1,) \end{array}$$

and thus

$$d(E\otimes_{A(X\times B^1)} A(Y\times B^1),\ F\otimes_{A(X\times B^1)} A(Y\times B^1),\ \beta|Y\times S^0)=0$$

in $K^{-1}(Y;A)$. ///

4. Products in K-Theory

Let X and Y be compact spaces. For the projections

$$\Pi_1 : X\times Y\longrightarrow X((x,y)\longmapsto x)$$
$$\Pi_2 : X\times Y\longrightarrow Y((x,y)\longmapsto y),$$

vector bundle E over X and a vector bundle F over Y, we define the *external tensor product* $E\boxtimes F$ of E and F, which is a vector bundle over $X\times Y$, by $E\boxtimes F=\Pi_1^*(E)\otimes\Pi_2^*(F)$. Thus for each $(x,y)\in X\times Y$ $(E\boxtimes F)_{(x,y)}=E_x\otimes F_y$. The correspondence $(E,F)\longrightarrow E\boxtimes F$ induces a functor $\varphi : \mathscr{E}(X)\times\mathscr{E}(Y)\longrightarrow\mathscr{E}(X\times Y)$ such that

$$\varphi(E\oplus E',F)=\varphi(E,F)\oplus\varphi(E',F),\quad \varphi(E,F\oplus F')=\varphi(E,F)\oplus\varphi(E,F').$$

From this functor we can define a bilinear group homomorphism

$$\varphi_* : K(X)\times K(Y)\longrightarrow K(X\times Y)$$

by $\varphi_*([E]-[E'],\ [F]-[F'])=[\varphi(E,F)]+[\varphi(E',F')]$

$$-[\varphi(E,F')]-[\varphi(E',F)] \quad\cdots\cdots\cdots\cdots(*_1)\quad ([9]).$$

For each $x\in K(X)$ and $y\in K(Y)$ we put

$$\varphi_*(x,y)=\!=x\cup y$$

Sometimes φ_* is called the cup product in K-theory. The diagonal map

$$\triangle : X\longrightarrow X\times X\ \ (x\longmapsto(x,x))$$

defines the group homomorphism

$$\triangle^* : K(X\times X)\longrightarrow K(X).$$

Thus $\triangle^*\varphi_* : K(X)\times K(X)\longrightarrow K(X\times X)\longrightarrow K(X)$ is a group homomorphism. We put for $x, x'\in K(X)$

$$\triangle^*\varphi_*(x, x')=\triangle^*(x\cup x')=x\cdot x'=xx'\in K(X) \cdots\cdots\cdots\cdots (*_2)$$

With this operator $K(X)$ is provided with a commutative ring structure ([2], [9]).

Lemma 4.1. For locally compact spaces X and Y there is the cup-product

$$K(X)\times K(Y)\longrightarrow K(X\times Y)$$

which satisfies the associativity and the commutativity.

Proof. Let $\dot{X}$ and $\dot{Y}$ be the one point compactifications of X and Y respectively. Then, we have the exact sequence of abelian groups:

$$K(\dot{X}\times\dot{Y}\times\boldsymbol{R})\longrightarrow K((\dot{X}\vee\dot{Y})\times\boldsymbol{R})\longrightarrow K(\dot{X}\times\dot{Y}-\dot{X}\vee\dot{Y})$$
$$\longrightarrow K(\dot{X})\times\dot{Y})\longrightarrow K(\dot{X}\vee\dot{Y})$$

where $\dot{X}\vee\dot{Y}=\{\infty\}\times\dot{Y}\cup\dot{X}\times\{\infty\}$ ([9]). Note that $X\times Y\approx\dot{X}\times\dot{Y}-\dot{X}\vee\dot{Y}$.
Thus, we have the exact sequence

$$K(\dot{X}\times\dot{Y}\times\boldsymbol{R})\longrightarrow K((\dot{X}\vee\dot{Y})\times\boldsymbol{R})\longrightarrow K(X\times Y)$$
$$\longrightarrow K(\dot{X}\times\dot{Y})\longrightarrow K(\dot{X}\vee\dot{Y})$$

Since we can prove that

$$K(\dot{X}\times\dot{Y}\times\boldsymbol{R})\longrightarrow K((\dot{X}\vee\dot{Y})\times\boldsymbol{R})$$

is surjective ([9]), we have the exact sequence

$$0\longrightarrow K(X\times Y)\longrightarrow K(\dot{X}\times\dot{Y})\longrightarrow K(\dot{X}\vee\dot{Y}).$$

Moreover, since

$$K(X)=\mathrm{Ker}(K(\dot{X})\longrightarrow K(\{\infty\}))\ ([9]),$$

we have the inclusion $i : K(X)\longrightarrow K(\dot{X})$. Similarly, there is the inclusion

$j: K(Y) \longrightarrow K(\dot{Y})$.

Consider the following diagram:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ K(X)\times K(Y) & \xrightarrow{\theta} & K(X\times Y) \\ {\scriptstyle i\times j}\downarrow & \cup=\varphi_* & \downarrow \\ K(\dot{X})\times K(\dot{Y}) & \longrightarrow & K(\dot{X}\times\dot{Y}) \\ & {\scriptstyle\gamma}\nwarrow & \downarrow \\ & & K(\dot{X}\vee\dot{Y}) \end{array}$$

(exact)

where γ is induced from the inclusions $\dot{X}\longrightarrow\dot{X}\vee\dot{Y}$ ($x\longrightarrow x\times\{\infty\}$) and $\dot{Y}\longrightarrow\dot{X}\vee\dot{Y}$ ($y\mapsto\{\infty\}\times y$). In fact, γ is injective, because of that for each $([E]-[T])\in K(\dot{X}\vee\dot{Y})$

$$\gamma([E]-[T])=([E|_{\dot{X}\times\{\infty\}}]-[T|_{\dot{X}\times\{\infty\}}])\times([E|_{\{\infty\}\times\dot{Y}}]-[T|_{\{\infty\}\times\dot{Y}}]$$

and thus

$$\gamma([E]-[T])=0\Longrightarrow[E]=[T],\ \text{i.e.,}\ E\cong T.$$

We shall construct the map θ in the above diagram.

For $x\in K(X)$ and $y\in K(Y)$

$$i(x)\times j(y)\in K(\dot{X})\times K(\dot{Y})$$

Thus $i(x)\cup j(y)\in K(\dot{X}\times\dot{Y})$. We want to prove that the restriction of $i(x)\cup j(y)$ to $K(\dot{X}\vee\dot{Y})$ is zero. By the definition of $K(X)$ above $i(x)|_{\{\infty\}}=0$. Similary, $j(y)|_{\{\infty\}}=0$. Therefore, we have $i(x)\cup j(y)\,|\dot{X}\times\{\infty\}=0=i(x)\cup j(y)\,|\,\{\infty\}\times\dot{Y}$

This means that the restriction of $i(x)\cup j(y)$ to $K(\dot{X}\vee\dot{Y})$ is zero. From the exact sequence

$$0\longrightarrow K(X\times Y)\xrightarrow{\kappa}K(\dot{X}\times\dot{Y})\longrightarrow K(\dot{X}\vee\dot{Y})$$

we have an element $x\cup y\in K(X\times Y)$ such that $\kappa(x\cup y)=i(x)\cup j(y)$.

We define $\theta(x,y)=x\cup y$. Since $i(x)\cup j(y)=j(y)\cup i(x)$ in $K(\dot{X}\times\dot{Y})$ it is clear that $x\cup y=y\cup x$ in $K(X\times Y)$. Moreover, for another locally compact space Z, $z\in K(Z)$

and the inclusion $l : K(Z) \longrightarrow K(\dot{Z})$

$$\text{since} \quad (i(x) \cup j(y)) \cup l(z) = i(x) \cup (j(y) \cup l(z)) \quad ([9]).$$

we have also

$$(x \cup y) \cup z = x \cup (y \cup z). \quad ///$$

Theorem 4.2. Let X and Y be finite CW-complexes, and let $X^{(n)}$ be the n^{th} skeleton of X. If we put

$$K_{(n)}(X) = \text{Ker } (K(X) \longrightarrow K(X^{n-1})),$$

then

$$\cup \mid K_{(n)}(X) \times K_{(p)}(Y) : K_{(n)}(X) \times K_{(p)}(Y) \longrightarrow K_{(n+p)}(X \times Y).$$

Proof. Since X and Y are finite CW-complexes, X and Y are compact. We put

the cellular decomposition of $X = \{e_\lambda : \lambda = 1, \ldots, m\}$,
the cellular decomposition of $Y = \{e'_{\lambda'} : \lambda' = 1, \ldots, n\}$.

Then the cellular decomposition of $X \times Y$ is $\{e_\lambda \times e'_{\lambda'} \mid \lambda = 1, \ldots, m \text{ and } \lambda' = 1, \ldots, n\}$.
By our definition

$$[E] - [F] \in K_{(n)}(X) \Longleftrightarrow E\mid_{X^{n-1}} \cong F\mid_{X^{n-1}}$$
$$[E'] - [F'] \in K_{(p)}(Y) \Longleftrightarrow E'\mid_{Y^{p-1}} \cong F'\mid_{Y^{p-1}}.$$

Since

$$([E]-[F]) \cup ([E']-[F']) = [\varphi(E, E')] + [\varphi(E, F')] - \{[\varphi(E, F')] + [\varphi(F, E')]\}$$

(for notations see $(*_1)$ above), where $\varphi(E, E') = E \boxtimes E'$.

For each point $(u, v) \in X \times Y$ consider

$$[E \boxtimes E']_{(u,v)} + [F \boxtimes F']_{(u,v)} - \{[E \boxtimes F']_{(u,v)} + [F \boxtimes E']_{(u,v)}\}.$$

Taking $(u, v) \in (X \times Y)^{n+p-1}$ we have the following two cases:

1) $u \in e_\lambda$, dim $e_\lambda \leq n-1$ and $v \in e'_{\lambda'}$, dim $e'_{\lambda'} \leqq n+p-1-\text{dim } e_\lambda$

2) $u \in e_\lambda$, dim $e_\lambda \leqq n+p-1-\text{dim } e'_{\lambda'}$, and $v \in e'_{\lambda'}$, dim $e'_{\lambda'} \leq p-1$.

In case 1) :

$$\begin{array}{ll} [E\otimes E']_{(u,v)}=E_u\otimes E'_v & [F\otimes F']_{(u,v)}=F_u\otimes F'_v \\ \wr\| & \wr\| \\ [F\otimes F']_{(u,v)}=F_u\otimes E'_v & [E\otimes F']_{(u,v)}=E_u\otimes F'_v. \end{array}$$

Thus, $([E]-[F])\cup([E']-[F'])\in K_{(n+p)}(X\times Y)$.

In case 2):

$$\begin{array}{ll} [E\otimes E']_{(u,v)}=E_u\otimes E'_v & [F\otimes F']_{(u,v)}=F_u\otimes F'_v \\ \wr\| & \wr\| \\ [E\otimes F']_{(u,v)}=E_u\otimes F'_v & [F\otimes E']_{(u,v)}=F_u\otimes E'_v \end{array}$$

Thus $([E]-[F])\cup([E']-[F'])\in K_{(n+p)}(X\times Y)$. ///

Let $\Pi : X\longrightarrow Y$ be a n-fold covering such that X and Y are locally compact (Note that X and Y are path-connected). For each finite dimensional (k)-vector bundle E over X ($k=\boldsymbol{R}$ or $\boldsymbol{C}$) the vector bundle $F=\Pi_*(E)$ over Y is defined as follows.

$$\Pi_*(E)_v=F_v=\bigoplus_{u\in\pi^{-1}(v)} E_u \qquad (v\in Y).$$

Let U be an open subset of Y. Then we can put

$$\Pi^{-1}(U)=V_1\cup\ldots\cup V_n$$

where $U\approx V_i$ ($i=1,\ldots,n$ and $\approx$: homeomorphic) and $i\neq j\Longrightarrow V_i\cap V_j=\phi$. The topology of F_U is induced by the bijection

$$F_U=E_{V_1}\oplus\ldots\oplus E_{V_n}\cong(E_{V_1})^n$$

Theorem 4.3. With the above notations

(i) F is a well-defined vector bundle over Y

(ii) $$\begin{array}{ccc} \Pi_* : K(X) & \longrightarrow & K(Y) \\ \Uparrow & & \Uparrow \\ [E] & \longmapsto & [\Pi_*(E)] \end{array}$$

is a group homomorphism, and for $x\in K(X)$ and $y\in K(Y)$

$$\Pi_*(\Pi^*(y)\cdot x)=y\cdot\Pi_*(x)$$

(for notation $y\cdot\Pi_*(x)$ see($*_2$) above).

Proof. (i) It suffices to prove that the locally triviality of $\Pi_*(E)=F$. For each $v\in Y$ we shall take an open neighborhood U of v such that

$$\Pi^{-1}(U)=V_1\cup\ldots\cup V_n,\quad E_{V_i}=V_i\times k^m(i=1,\ldots,n)$$

($k=\boldsymbol{R}$ or $\boldsymbol{C}$), where $i\neq j\Longrightarrow V_i\cap V_j=\phi$ and $U\approx V_i$ ($i=1,\ldots,n$).

By our definition above

$$\Pi_*(E)_U = F_U \cong U \times (k^n)^n = U \times k^{nm}.$$

(ii) Note that for locally compact spaces X and Y we have defined the cup-product

$$K(X) \times K(Y) \longrightarrow K(X \times Y)$$

by Lemma 4.1. Since $[E_1], [E_2] \in K(X) \Longrightarrow [E_1] + [E_2] = [E_1 \oplus E_2]$,
we have the following : For each $v \in Y$

$$\begin{aligned}\Pi_*(E_1 \oplus E_2)_v &= \bigoplus_{u \in \pi^{-1}(v)} (E_1 \oplus E_2)_u \\ &= \bigoplus_{u \in \pi^{-1}(v)} ((E_1)_u \oplus (E_2)_u) \\ &= (\bigoplus_{u \in \pi^{-1}(v)} (E_1)_u) \oplus (\bigoplus_{u \in \pi^{-1}(v)} (E_2)_u) \\ &= \Pi_*(E_1)_v \oplus \Pi_*(E_2)_v \\ &= (\Pi_*(E_1) \oplus \Pi_*(E_2))_v.\end{aligned}$$

Hence

$$\Pi_*([E_1 \oplus E_2] = \Pi_*([E_1]) + \Pi_*(E_2]).$$

That is, Π_* is a group homomorphism.

To prove that $\Pi_*(\Pi^*(y) \cdot x) = y \cdot \Pi_*(x)$ we shall prove that for each $v \in Y$

$$(\Pi_*(\Pi^*(y) \cdot x))_v = (y \cdot \Pi_*(x))_v.$$

By our definition,

$$\begin{aligned}(\Pi_*(\Pi^*(y) \cdot x))_v &= \bigoplus_{u \in \pi^{-1}(v)} (\Pi^*(y) \cdot x)_u \\ &= \bigoplus_{u \in \pi^{-1}(v)} y_v \otimes x_u.\end{aligned}$$

where we have to note that $\Pi_*(y)_u = y_v$ $(\Pi(u) = v)$, and

$$\begin{aligned}(y \cdot \Pi_*(x))_v = y_v \otimes \Pi_*(x)_v &= y_v \otimes (\bigoplus_{u \in \pi^{-1}(v)} x_u) \\ &= \bigoplus_{u \in \pi^{-1}(v)} y_v \otimes x_u.\end{aligned}$$

Hence $\Pi_*(\Pi^*(y) \cdot x) = y \cdot \Pi_*(x)$. ///

Let $\Pi : X \longrightarrow Y$ be a principal covering with finite group G, i.e., $X/G \approx Y$ and G acts freely on X.

Proposition 4.4. Under the above circumstances we also assume that X and Y are locally compact.
Then for each $x \in K(X)$

$$(\Pi^* \cdot \Pi_*)\ (x) = \sum_{g \in G} \rho(g)^*(x)$$

where $\rho(g)^* : K(X) \longrightarrow K(X)$ is the automorphism of $K(X)$ induced by the action of $g \in G$.

Proof. For each $u \in X$ we have to note that

$$^{\forall} g \in G \quad \Pi(g(u)) = \Pi(u)$$

Thus, by our definition

$$^{\forall} v \in Y \quad (\Pi_*(x))_v = \sum_{\substack{g \in G \\ \pi(u) = v}} x_{g(u)}$$

Thus, if $\Pi(u) = v$ then

$$(\Pi^*(\Pi_*(x)))_u = (\Pi_*(x))_v = \sum_{\substack{g \in G \\ \pi(u) = v}} x_{g(u)}.$$

On the other hand, for $\rho(g) : X \longrightarrow X\ (u \longmapsto \rho(g)(u) = g(u))$, which is the action of G on X,

$$\Big(\sum_{g \in G} \rho^*(g)\ (x)\Big)_u = \bigoplus_{g \in G} x_{g(u)} = \sum_{g \in G} x_{g(u)}$$

and thus for each $u \in X$

$$\Big(\sum_{g \in G} \rho^*(g)\ (x)\Big)_u = (\Pi^* \cdot \Pi_*(x))_u. \quad ///$$

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