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Some Remarks on Modules of Generalized Fractions and Gorenstein Rings.

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1. Introduction

The concepts of a module of generalized fractions were formulated by R.Y. Sharp and H. Zakeri and introduced in their paper [19] in 1982. Since then many mathematicians have studied the relations of Cousin Complexes [14], Local Cohomology [21], Krull Dimension [4], Balanced Big Cohen-Macaulay Module [20] with modules of generalized fractions. Especially, R.Y. Sharp and H. Zakeri proved the following:

If A is a Noetherian ring, then

$$U_i^{-i}A \cong \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}(A) \\ ht(\mathfrak{p})=i-1}} (U_i^{-i}A)_{\mathfrak{p}}$$

for some triangular subsets U_i. In this paper, we will prove that if A is a Gorenstein local ring then

$$U_i^{-i}A \cong \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}(A) \\ ht(\mathfrak{p})=i-1}} E(A/\mathfrak{p}).$$

The detailed contents of this paper can be described as follows:

In section 2, we will define a module of generalized fractions and discuss some known properties of modules of generalized fractions which will be used later in this paper.

In section 3, we will describe the properties of poor A-sequence.

In section 4, which is the main part of this paper, we will prove the main Theorem 4.7 and its corollaries 4.8 and 4.9, noticing that every system of parameters over a Gorenstein local ring A is a poor A-sequence.

2. Modules of Generalized Fractions.

Throughout this paper, A denotes a commutative ring with identity and M denotes an A-module and $D_n(A)$ denotes the set of all $n\times n$ lower triangular matrices with entries in A. For $H\in D_n(A)$ we shall use $|H|$ to denote the determinant of H, diag $(u_1, u_2, \cdots, u_n)$ will denote the diagonal matrix with diagonal elements $u_1, u_2, \cdots, u_n$. We shall use H^T to denote the transpose of a matrix H. $\boldsymbol{N}$ denotes the set of all natural numbers.

Definition 2.1. Let A be a commutative ring and let $n\in \boldsymbol{N}$. A subset U of $A^n=A\times A\times\cdots\times A$ is said to be a triangular subset of A^n if the following hold:

(i) U is non-empty.

(ii) Whenever $(u_1, u_2,\cdots, u_n)\in U$, we have $(u_1^{\alpha_1}, u_2^{\alpha_2}, \cdots, u_n^{\alpha_n})\in U$ for all choices of positive integers $\alpha_1, \alpha_2, \cdots, \alpha_n$.

(iii) Whenever $(u_1, u_2, \cdots, u_n)\in U$ and $(v_1, v_2,\cdots,v_n)\in U$, there exists $(w_1, w_2, \cdots, w_n)\in U$ such that

$$w_i\in(Au_1+Au_2+\cdots+Au_i)\cap(Av_1+Av_2+\cdots+Av_i)$$

so that there are lower triangular matrices $H, K\in D_n(A)$ such that

$$H(u_1, u_2, \cdots, u_n)^T=(w_1, w_2, \cdots, w_n)^T=K(v_1, v_2, \cdots, v_n)^T.$$

Example 2.2. Let A be a Noetherian ring. We adopt the convention whereby the ideal A of A has height ∞. For each $i\in \boldsymbol{N}$, we set

$$U_i=\{(u_1,u_2,\cdots,u_i)\in A^i\,|\,ht(Au_1+Au_2+\cdots+Au_j)\geq j \text{ for all } j=1,2,\cdots,i\}.$$

Then U_i is a triangular subset of A^i for each $i\in \boldsymbol{N}$.

Proof. It is obvious that U_1 is a triangular subset of $A^1=A$. Suppose, inductively, that U_j is a triangular subset of A^j for $1\leq j<i$. Of course, $U_i\neq\phi$ since $(1, 1, \cdots, 1)\in U_i$. For every prime ideal $\mathfrak{p}$ and every $\alpha_j\in \boldsymbol{N}$, $\sum_{r=1}^{j}Au_r\subseteq\mathfrak{p}$ if and only if $\sum_{r=1}^{j}Au_r^{\alpha_r}\subseteq\mathfrak{p}$. Hence it follows that if $(u_1, u_2, \cdots, u_i)\in U_i$ then $(u_1^{\alpha_1}, u_2^{\alpha_2}, \cdots, u_i^{\alpha_i})\in U_i$ also for every $\alpha_j\in \boldsymbol{N}$. For $(u_1, u_2, \cdots, u_i)$, $(v_1, v_2, \cdots, v_i)\in U_i$, there exists $(w_1, w_2, \cdots, w_{i-1})\in U_{i-1}$ such that

$$w_j\in\Big(\sum_{r=1}^{j}Au_r\Big)\cap\Big(\sum_{r=1}^{j}Av_r\Big)$$

for $1\leq j\leq i-1$ because that $(u_1, u_2, \cdots, u_{i-1})$ $(v_1, v_2, \cdots, v_{i-1})$ are contained in U_{i-1}.

In the case of $\sum_{r=1}^{i-1} Aw_r=A$, if we put $w_i=u_iv_i$ then $(w_1, w_2, \cdots, w_i)\in U_i$ such that

$$w_j\in\left(\sum_{r=1}^{j}Au_r\right)\cap\left(\sum_{r=1}^{j}Av_r\right) \text{ for } j=1,2,\cdots,i.$$

In the case of $\sum_{r=1}^{i-1} Aw_r\neq A$, set

$$\Omega=\{\mathfrak{p}\in \operatorname{Spec}(A) \mid \sum_{r=1}^{i-1} Aw_r\subseteq\mathfrak{p} \text{ and } ht(\mathfrak{p})=i-1\}.$$

Then we have

$$\left(\sum_{r=1}^{i}Au_r\right)\cap\left(\sum_{r=1}^{i}Av_r\right)\subsetneq\mathfrak{p} \text{ for all } \mathfrak{p}\in\Omega.$$

Since Ω is finite, we have

$$\left(\sum_{r=1}^{i}Au_r\right)\cap\left(\sum_{r=1}^{i}Av_r\right)\subsetneq\bigcup_{\mathfrak{p}\in\Omega}\mathfrak{p},$$

and so there exists $w_i\in\left(\sum_{r=1}^{i}Au_r\right)\cap\left(\sum_{r=1}^{i}Av_r\right)$ such that $w_i\notin\bigcup_{\mathfrak{p}\in\Omega}\mathfrak{p}$.
This implies that $(w_1,w_2,\cdots,w_i)\in U_i$. Hence U_i is a triangular subset of A^i for all $i\in N$. ///

Example 2.3. Let A be a Noetherian local ring of dimension n with the maximal ideal $\mathfrak{M}$. It is well known that n is the smallest number of non-zero elements required to generate an $\mathfrak{M}$-primary ideal [12; Chapter Ⅳ. Theorem 1]. A set of n elements which generates an $\mathfrak{M}$-primary ideal is called "a system of parameters of A". In this case, the set

$$U=\{(u_1,u_2,\cdots,u_n)\in A^n \mid u_1,u_2,\cdots,u_n \text{ is a system of parameters of } A\}$$

is a triangular subset of A^n.

Proof. Since $\dim A=ht(\mathfrak{M})=n$, there exist $u_1,u_2,\cdots,u_n\in\mathfrak{M}$ such that $ht\left(\sum_{r=1}^{n}Au_r\right)=n$. This implies that $(u_1,u_2,\cdots,u_n)\in U$. It is obvious that if $(u_1,u_2,\cdots,u_n)\in U$ then $(u_1^{\alpha_1},u_2^{\alpha_2},\cdots,u_n^{\alpha_n})\in U$ for every positive integers $\alpha_1,\alpha_2,\cdots,\alpha_n$ because that $\sum_{r=1}^{n}Au_r$ and $\sum_{r=1}^{n}Au_r^{\alpha_r}$ have the same radical. Let $(u_1,u_2,\cdots,u_n)\in U$, $(v_1,v_2,\cdots,v_n)\in U$ and $w_1=u_1v_1$.

Then $\dim(A/Aw_1) \geq n-1$.

Assume that $\dim(A/Aw_1)=n$. There are prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_n$ such that

$$Aw_1 \subset \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n.$$

So $Au_1 \subseteq \mathfrak{p}_0$ or $Av_1 \subseteq \mathfrak{p}_0$, and thus $\dim(A/Au_1)=n$ or $\dim(A/Av_1)=n$. But this is a contradiction. Hence $\dim(A/Aw_1)=n-1$, and so $w_1=u_1v_1$ is a subset of a system of parameters of A.

Suppose that, for $1 \leq i < n$, we have a subset $\{w_1, w_2, \cdots, w_i\}$ of a system of parameters of A such that

$$w_j \in \left(\sum_{r=1}^{j} Au_r\right) \cap \left(\sum_{r=1}^{j} Av_r\right)$$

for all $j=1, 2, \cdots, i$. Set

$$\Omega = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \sum_{r=1}^{i} Aw_r \subseteq \mathfrak{p} \text{ and } \dim(A/\mathfrak{p})=n-i\}.$$

Then it follows from [12; Chapter Ⅳ. Theorem 2] that

$$\left(\sum_{r=1}^{i+1} Au_r\right) \cap \left(\sum_{r=1}^{i+1} Av_r\right) \subsetneq \mathfrak{p}$$

for all $\mathfrak{p} \in \Omega$. Hence

$$\left(\sum_{r=1}^{i+1} Au_r\right) \cap \left(\sum_{r=1}^{i+1} Av_r\right) \subsetneq \bigcup_{\mathfrak{p} \in \Omega} \mathfrak{p}$$

so there exists $w_{i+1} \in \left(\sum_{r=1}^{i+1} Au_r\right) \cap \left(\sum_{r=1}^{i+1} Av_r\right)$ such that $w_{i+1} \notin \bigcup_{\mathfrak{p} \in \Omega} \mathfrak{p}$.

Since $\sum_{r=1}^{i} Aw_r \subseteq \sum_{r=1}^{i+1} Aw_r$, we have

$$n-(i+1) \leq \dim\left(A/\sum_{r=1}^{i+1} Aw_r\right) \leq \dim\left(A/\sum_{r=1}^{i} Aw_r\right) = n-i$$

by [12; Chapter Ⅳ. Theorem 2]. On the other hand, $\dim\left(A/\sum_{r=1}^{i+1} Aw_r\right) < n-i$ because that $w_{i+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \Omega$. Thus $\dim\left(A/\sum_{r=1}^{i+1} Aw_r\right) = n-i-1$. Therefore, by [12; Chapter Ⅳ. Theorem 2], $\{w_1, w_2, \cdots, w_{i+1}\}$ is a subset of a system of parameters of A. Therefore U is a triangular subset of A^n. ///

Example 2.4. Let U_n be a triangular subset of A^n. Then it is easily proved that $U_{n+1} = \{(u_1, u_2, \cdots, u_n, 1) \in A^{n+1} \mid (u_1, u_2, \cdots, u_n) \in U_n\}$ is a triangular subset of A^{n+1}, and that, for $i<n$, the set

$$U_i=\{(u_1,u_2,\cdots,u_i)\in A^i \mid \text{there exist } u_{i+1},u_{i+2},\cdots,u_n \text{ such that } (u_1,\cdots,u_i,u_{i+1},\cdots,u_n)\in U_n\}$$

is also a triangular subset of A^i. We call U_i the restriction of U_n to A^i.

In [19], R.Y. Sharp and H. Zakeri proved the following proposition.

Proposition 2.5. (i) Let U be a triangular subset of A^n, and let $(u_1,u_2,\cdots,u_n)\in U$, $(v_1,v_2,\cdots,v_n)\in U$ and suppose that

$$H(u_1,u_2,\cdots,u_n)^T=(v_1,v_2,\cdots,v_n)^T$$

for some $H\in D_n(A)$. Then

$$|H|u_i\in\sum_{r=1}^{i}Av_r$$

for all $i=1,2,\cdots,n$.

(ii) Let $(u_1,u_2,\cdots,u_n)\in U$, $(v_1,v_2,\cdots,v_n)\in U$ and suppose that there exist $H,K\in D_n(A)$ such that

$$H(u_1,u_2,\cdots,u_n)^T=(v_1,v_2,\cdots,v_n)^T=K(u_1,u_2,\cdots,u_n)^T.$$

Then

$$|DH|-|DK|\in\sum_{r=1}^{n-1}Av_r^2,\ \text{where } D=\mathrm{diag}(v_1,v_2,\cdots,v_n).$$

Using Proposition 2.5, we can show the following proposition[19].

Proposition 2.6. Let M be an A-module and let U be a triangular subset of A^n. Consider the relation $\sim$ on $M\times U$ defined as follows;

For $m,n\in M$, and $(u_1,u_2,\cdots,u_n)$, $(v_1,v_2,\cdots,v_n)\in U$, we define

$$(m,(u_1,u_2,\cdots,u_n))\sim(n,(v_1,v_2,\cdots,v_n))$$

when there exist $(w_1,w_2,\cdots,w_n)\in U$ and $H,K\in D_n(A)$ such that

$$H(u_1,u_2,\cdots,u_n)^T=(w_1,w_2,\cdots,w_n)^T=K(v_1,v_2,\cdots,v_n)^T$$

and $(|H|m-|K|n)\in\left(\sum_{r=1}^{n-1}Aw_r\right)M$.

Then $\sim$ is an equivalence relation on $M\times U$.

Definition 2.7. With the above notations, define the formal symbol $\frac{m}{(u_1, u_2, \cdots, u_n)}$ to be the equivalence class of $\sim$ containing $(m, (u_1, u_2, \cdots, u_n))$. Let $U^{-n}M$ denote the set of all equivalence classes of $\sim$.

R.Y. Sharp and H. Zakeri have proved that the set $U^{-n}M$ has an A-module structure under the following operations;

$$\text{(i)}\quad \frac{m}{(u_1, u_2, \cdots, u_n)} + \frac{n}{(v_1, v_2, \cdots, v_n)} = \frac{|H|m + |K|n}{(w_1, w_2, \cdots, w_n)}$$

where $H(u_1, u_2, \cdots, u_n)^T = (w_1, w_2, \cdots, w_n)^T = K(v_1, v_2, \cdots, v_n)^T$.

$$\text{(ii)}\quad a\frac{m}{(u_1, u_2, \cdots, u_n)} = \frac{am}{(u_1, u_2, \cdots, u_n)}, \text{ where } a \in A$$

This module $U^{-n}M$ is called a module of generalized fractions of M [13].

Definition 2.8. A triangular subset U' of A^n is said to be expanded if whenever $(u_1, u_2, \cdots, u_n) \in U'$, it is the case that $(u_1, u_2, \cdots, u_i, 1, \cdots 1) \in U'$ for all i with $0 \leq i < n$. For a triangular subset U of A^n, let $\bar{U}$ be the set of all sequences $(v_1, v_2, \cdots, v_n) \in A^n$ for which there exist $0 \leq i \leq n$ and $(u_1, u_2, \cdots, u_n) \in U$ such that

$$v_j = \begin{cases} u_j & \text{for } j = 1, 2, \cdots, i. \\ 1 & \text{for } j = i+1, \cdots, n \end{cases}$$

Then we can easily show that $\bar{U}$ is a triangular subset of A^n. This set $\bar{U}$ is called the expansion of U.

Proposition 2.9. Let U be an expanded triangular subset of A^n. Then, for $m \in M$ and $(u_1, u_2, \cdots, u_n) \in U$, the following hold;

(i) $\frac{u_n m}{(u_1, u_2, \cdots, u_n)} = \frac{m}{(u_1, u_2, \cdots, u_{n-1}, 1)}$ in $U^{-n}M$.

(ii) if $m \in \left(\sum_{r=1}^{n-1} Au_r\right)M$, then $\frac{m}{(u_1, u_2, \cdots, u_{n-1}, u_n)} = 0$.

(iii) if $\frac{u_n m}{(u_1, u_2, \cdots, u_n)} = 0$ in $U^{-n}M$, then $\frac{m}{(u_1, u_2, \cdots, u_n)} = 0$.

Proof. (i) If we put $I_n = \text{diag}(1, 1, \cdots, 1)$, $K = \text{diag}(1, \cdots, 1, u_n) \in D_n(A)$ then

$$I_n(u_1, u_2, \cdots, u_n)^T = (u_1, u_2, \cdots, u_n)^T = K(u_1, u_2, \cdots, u_{n-1}, 1)^T$$
$$|I_n| u_n m - |K| m = 0 \in \left(\sum_{r=1}^{n-1} Au_r\right)M.$$

Hence the result follows.

(ii) If $I_n=\mathrm{diag}(1,1,\cdots,1)$, $D=\mathrm{diag}(u_1,u_2,\cdots,u_n)\in D_n(A)$, then

$$I_n(u_1,u_2,\cdots,u_n)^T=(u_1,u_2,\cdots,u_n)^T=D(1,1,\cdots,1)^T$$
$$|I_n|m=m\in\left(\sum_{r=1}^{n-1}Au_r\right)M.$$

Hence the assertion holds.

(iii) There exist $(w_1,w_2,\cdots,w_n)\in U$ and $H=(h_{ij})\in D_n(A)$ such that

$$H(u_1,u_2,\cdots,u_n)^T=(w_1,w_2,\cdots,w_n)^T$$

and

$$|H|u_nm\in\left(\sum_{r=1}^{n-1}Aw_r\right)M.$$

Hence

$$\prod_{r=1}^{n-1}h_{rr}\left(w_n-\sum_{r=1}^{n-1}h_{nr}u_r\right)m\in\left(\sum_{r=1}^{n-1}Aw_r\right)M.$$

Therefore, by (i) of Proposition 2.5,

$$h_{11}h_{22}\cdots h_{n-1n-1}w_nm\in\left(\sum_{r=1}^{n-1}Aw_r\right)M.$$

Hence, by (ii),

$$\frac{h_{11}\cdots h_{n-1n-1}w_nm}{(w_1,w_2,\cdots,w_{n-1},w_n^2)}=0\text{ in }U^{-n}M,$$

and thus, by (i),

$$\frac{h_{11}h_{22}\cdots h_{n-1n-1}h_{nn}m}{(w_1,w_2,\cdots,w_{n-1},w_n)}=0.$$

Therefore

$$\frac{m}{(u_1,u_2,\cdots,u_n)}=0.\quad ///$$

Notation 2.10. We shall use the symbol $\mathscr{U}$ to denote a family of sets $\{U_i\mid i\in\boldsymbol{N}\}$ such that

(i) U_i is a triangular subset of A^i for all $i\in\boldsymbol{N}$,

(ii) whenever $(u_1,u_2,\cdots,u_i)\in U_i$ with $1<i\in\boldsymbol{N}$ then $(u_1,u_2,\cdots,u_{i-1})\in U_{i-1}$,

(iii) whenever $(u_1,u_2,\cdots,u_i)\in U_i$ with $1\leq i\in\boldsymbol{N}$ then $(u_1,u_2,\cdots,u_i,1)\in U_{i+1}$,

(iv) $(1) \in U_1$.

(Note that if each U_i is expanded, then (ii), (iii) and (iv) can be replaced by the following single condition;

(v) for all $i \in N$, U_i is the restriction of U_{i+1} to A^i).

Given such a family $\mathscr{U}$ and an A-module M we can construct a complex $\mathscr{C}(\mathscr{U}, M)$ of A-modules and A-homomorphisms,

$$\mathscr{C}(\mathscr{U}, M) : 0 \longrightarrow M \xrightarrow{d^0} U_1^{-1}M \xrightarrow{d^1} U_2^{-2}M \xrightarrow{d^2} \cdots \longrightarrow U_i^{-i}M \xrightarrow{d^i} U_{i+1}^{-i-1}M \longrightarrow \cdots$$

where

$$d^0(m) = m/1 \text{ and } d^i(m/(u_1, u_2, \cdots, u_i)) = m/(u_1, u_2, \cdots, u_i, 1).$$

We note that $\mathscr{C}(\mathscr{U}, M)$ is a complex, by (ii) of Proposition 2.9.

Lemma 2.11. In a complex $\mathscr{C}(\mathscr{U}, M)$, each A-module $U_i^{-i}M$ is an essential extension of $Im\, d^{i-1}$ for all $i \in N$.

Proof. For each $\frac{m}{(u_1, u_2, \cdots, u_i)} \in U_i^{-i}M$, there exists $u_i \in A$ such that

$$0 \neq \frac{u_i m}{(u_1, u_2, \cdots, u_i)} = \frac{m}{(u_1, u_2, \cdots, u_{i-1}, 1)} \in Im\, d^{i-1}$$

by (i) and (iii) of Proposition 2.9. Therefore $U_i^{-i}M$ is an essential extension of $Im\, d^{i-1}$. ///

Proposition 2.12. Let A be a Noetherian ring and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$, and let $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$. Then

(i) $E(A/\mathfrak{p})$ is indecomposable.

(ii) If M is an indecomposable injective A-module, then there exists a prime ideal $\mathfrak{p}$ such that $M \cong E(A/\mathfrak{p})$.

(iii) If $\mathfrak{q} \subseteq \mathfrak{p}$, then $E(A/\mathfrak{q})$ is the injective envelope of $(A/\mathfrak{q})_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}$. That is,

$$E_A(A/\mathfrak{q}) \cong E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}).$$

Proof. (i) Let M_1 and M_2 be non-zero submodules of $E(A/\mathfrak{p})$. Since $M_1 \cap A/\mathfrak{p}$ and $M_2 \cap A/\mathfrak{p}$ are non-zero ideals of $A/\mathfrak{p}$ by definition of $E(A/\mathfrak{p})$, we have

$$0 \neq (M_1 \cap A/\mathfrak{p})(M_2 \cap A/\mathfrak{p}) \subseteq (M_1 \cap A/\mathfrak{p}) \cap (M_2 \cap A/\mathfrak{p}) \subseteq M_1 \cap M_2.$$

This implies that $E(A/\mathfrak{p})$ has no direct summand.

(ii) Since $M\neq 0$ there exists a prime ideal $\mathfrak{p}$ in $\mathrm{Ass}(M)$ [9]. Since M is injective there exists an A-homomorphism $g: E(A/\mathfrak{p})\longrightarrow M$ satisfying the following commutative diagram:

$$\begin{array}{ccc} A/\mathfrak{p} & \longrightarrow & E(A/\mathfrak{p}) \\ \downarrow & \swarrow g & \\ M & & \end{array}$$

Assume that $g(b)=0$ where $0\neq b\in E(A/\mathfrak{p})$. Then there exists $0\neq a\in A$ such that $0\neq ab\in A/\mathfrak{p}$. Hence, by the above commutative diagram, we have

$$0\neq ab=g(ab)=ag(b)=0$$

which is a contradiction. Thus g is a monomorphism. Hence $E(A/\mathfrak{p})$ is a direct summand of M because that $E(A/\mathfrak{p})$ is an injective A-module. Therefore $M\cong E(A/\mathfrak{p})$ since M is indecomposable.

(iii) We first note that $E(A/\mathfrak{q})$ can be regarded as an $A_{\mathfrak{p}}$-module and $E(A/\mathfrak{q})$ contains $(A/\mathfrak{q})_{\mathfrak{p}}$. Hence $A/\mathfrak{q}\subseteq(A/\mathfrak{q})_{\mathfrak{p}}\subseteq E(A/\mathfrak{q})$. Since $E(A/\mathfrak{q})$ is the injective envelope of $A/\mathfrak{q}$, it is also an essential extension of $(A/\mathfrak{q})_{\mathfrak{p}}$. For any A-module M, M is injective as an $A_{\mathfrak{p}}$-module if and only if M is injective as an A-module. Therefore $E(A/\mathfrak{q})$ is the injective envelope of $(A/\mathfrak{q})_{\mathfrak{p}}\cong A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}$. ///

3. Poor A-Sequences.

Let M be an A-module. For any subset $\{x_1, x_2, \cdots, x_n\}\subseteq A$, the set of all zero divisors of $M/\left(\sum_{r=1}^{n}Ax_r\right)M$ is denoted by $Z\left(M/(\sum_{r=1}^{n}Ax_r)M\right)$.

Definition 3.1. Under the above situation, if

$$x_i\notin Z\left(M/(\sum_{r=1}^{i-1}Ax_r)M\right)$$

for all $i=1,2,\cdots,n$ then $x_1, x_2, \cdots, x_n$ is called a poor A-sequence on M. If we take A insteady of M, the sequence $x_1, x_2, \cdots, x_n$ is simply called a poor A-sequence.

Proposition 3.2. Let S be a multiplicative closed subset of A. If $x_1, x_2, \cdots, x_n$ is a poor A-sequence on M, then $x_1^*, x_2^*, \cdots, x_n^*$ is also a poor As-sequence on Ms, where x_i^* is the image of x_i under the canonical map $A\longrightarrow As$.

Proof. Suppose that $x_i^* \in Z\left(Ms/(\sum_{r=1}^{i-1} Asx_r^*)Ms\right)$ for some $1 \leq i \leq n$. Then there exist $m^* = m/s \in Ms$ and $m_j^* = m_j/s_j \in Ms$ $(j=1, 2, \cdots, i-1)$ such that

$$x_i^* m^* = \sum_{r=1}^{i-1} x_r^* m_r^* \cdots\cdots\cdots\cdots\cdots\cdots (※)$$

where $s, s_j \in S$. We multiply (※) by $s s_1 s_2 \cdots s_{i-1}$ and then we get an equation for elements of M. This yields that

$$x_i \in Z\left(M/(\sum_{r=1}^{i-1} Ax_r)M\right).$$

But this contradicts to our assumption. Hence the assertion holds. ///

Proposition 3.3. Let $x_1, x_2, \cdots, x_n$ be a poor A-sequence on M. Then

$$x_i \notin Z(M/(Ax_1 + \cdots + Ax_{i-1} + Ax_{i+1} + \cdots + Ax_n)M).$$

Proof. Suppose that there exists $m \in M - (Ax_1 + \cdots + Ax_{i-1} + Ax_{i+1} + \cdots + Ax_n)M$ such that

$$x_i m \in (Ax_1 + \cdots + Ax_{i-1} + Ax_{i+1} + \cdots + Ax_n)M.$$

Then there exists a subset $\{m_{11}, m_{21}, \cdots, m_{i-1\,1}, m_{i+1\,1}, \cdots m_{n1}\} \subseteq M$ such that

$$x_i m = x_1 m_{11} + \cdots + x_{i-1} m_{i-1\,1} + x_{i+1} m_{i+1\,1} + \cdots + x_n m_{n1}.$$

Since $x_n \notin Z\left(M/(\sum_{r=1}^{n-1} Ax_r)M\right)$, we have $m_{n1} \in \left(\sum_{r=1}^{n-1} Ax_r\right)M$. Hence there exists a subset $\{m_{12}, m_{22}, \cdots, m_{n-1\,2}\} \subseteq M$ such that

$$m_{n1} = x_1 m_{12} + x_2 m_{22} + \cdots + x_{n-1} m_{n-1\,2}.$$

Thus

$$x_i m = x_1(m_{11} + x_n m_{12}) + \cdots + x_{i-1}(m_{i-1\,1} + x_n m_{i-1\,2}) + x_i x_n m_{i2} + \cdots + x_{n-1}(m_{n-1\,1} + x_n m_{n-1\,2}).$$

That is,

$$x_i(m - x_n m_{i2}) = x_1(m_{11} + x_n m_{12}) + \cdots + x_{n-1}(m_{n-1\,1} + x_n m_{n-1\,2}).$$

Since $x_{n-1} \notin Z\left(M/(\sum_{r=1}^{n-2} Ax_r)M\right)$, we have

$$m_{n-1\,1} + x_n m_{n-1\,2} \in \left(\sum_{r=1}^{n-2} Ax_r\right)M.$$

By repeating the similar arguments, we obtain

$$x_i(m-(x_n m_{i2}+x_{n-1}m_{i3}+\cdots+x_{i+1}m_{i\ n-i+1}))\in\Big(\sum_{r=1}^{i-1}Ax_r\Big)M.$$

In this case, $m-(x_n m_{i2}+x_{n-1}m_{i3}+\cdots+x_{i+1}m_{i\ n-i+1})\notin\Big(\sum_{r=1}^{i-1}Ax_r\Big)M$ because that if $m-(x_n m_{i2}+x_{n-1}m_{i3}+\cdots+x_{i+1}m_{i\ n-i+1})\in\Big(\sum_{r=1}^{i-1}Ax_r\Big)M$ then $m\in(Ax_1+\cdots+Ax_{i-1}+Ax_{i+1}+\cdots+Ax_n)M$ and which is a contradiction. Hence $x_i\in Z\Big(M/(\sum_{r=1}^{i-1}Ax_r)M\Big)$, and which is a contradiction to our assumption. Therefore we have

$$x_i\notin Z(M/(Ax_1+\cdots+Ax_{i-1}+Ax_{i+1}+\cdots+Ax_n)M).\ ///$$

Lemma 3.4. Let $a, x_2, \cdots, x_n$ be a poor A-sequence on M and let $b\in A$. Suppose that

$$abm_1+x_2m_2+\cdots+x_nm_n=am_1'+x_2m_2'+\cdots+x_nm_n'$$

where $m_1, m_2, \cdots, m_n, m_1', m_2', \cdots, m_n'\in M$. Then

$$m_1'\in(Ab+Ax_2+\cdots+Ax_n)M.$$

Proof. We use the induction on n. Suppose that this lemma holds for $n-1$. Let

$$abm_1+x_2m_2+\cdots+x_nm_n=am_1'+x_2m_2'+\cdots+x_nm_n'$$

where $m_i, m_i'\in M$. Then since $x_n\notin Z(M/(Aa+Ax_2+\cdots+Ax_{n-1})M)$, we have

$$m_n'-m_n\in(Aa+Ax_2+\cdots+Ax_{n-1})M.$$

Hence there exist $m_1'', m_2'', \cdots, m_{n-1}''\in M$ such that

$$m_n'-m_n=am_1''+x_2m_2''+\cdots+x_{n-1}m_{n-1}''.$$

Thus

$$\begin{aligned}abm_1+x_2m_2+\cdots+x_{n-1}m_{n-1}&=am_1'+x_2m_2'+\cdots+x_{n-1}m_{n-1}'+x_n(m_n'-m_n)\\&=am_1'+x_2m_2'+\cdots+x_{n-1}m_{n-1}'+x_n(am_1''+x_2m_2''+\cdots+x_{n-1}m_{n-1}'')\\&=a(m_1'+x_nm_1'')+x_2(m_2'+x_nm_2'')+\cdots+x_{n-1}(m_{n-1}'+x_nm_{n-1}'').\end{aligned}$$

Therefore, by the inductive hypothesis,

$$m_1'\in(Ab+Ax_2+\cdots+Ax_n)M.\ ///$$

Lemma 3.5. Let $x_1, x_2, \cdots, x_n$ be elements of A, and suppose that $x_1 = ab$. Then the following statements are equivalent;

(i) $a, x_2, \cdots, x_n$ and $b, x_2, \cdots, x_n$ are poor A-sequences on M.

(ii) $x_1, x_2, \cdots, x_n$ is a poor A-sequence on M.

Proof. (i)$\longrightarrow$(ii) It is obvious that $x_1 = ab$ is a poor A-sequence on M. Suppose that

$$x_i m \in (Ax_1 + Ax_2 + \cdots + Ax_{i-1})M \subseteq (Aa + Ax_2 + \cdots + Ax_{i-1})M$$

for $2 \leq i \leq n$. Say,

$$x_i m = abm_1 + x_2 m_2 + \cdots + x_{i-1} m_{i-1}, \text{ where } m_j \in M \text{ for } j = 1, 2, \cdots, i-1.$$

Since $x_i \notin Z(M/(Aa + Ax_2 + \cdots + Ax_{i-1})M)$, we have

$$m = ay + x_2 y_2 + \cdots + x_{i-1} y_{i-1}, \text{ where } y, y_j \in M \text{ for } j = 1, 2, \cdots, i-1.$$

Then

$$ax_i y + x_2 x_i y_2 + \cdots + x_{i-1} x_i y_{i-1} = abm_1 + x_2 m_2 + \cdots + x_{i-1} m_{i-1}.$$

So, by Lemma 3.4,

$$x_i y \in (Ab + Ax_2 + \cdots + Ax_{i-1})M.$$

Since $x_i \notin Z(M/(Ab + Ax_2 + \cdots + Ax_{i-1})M)$

$$y = bz + x_2 z_2 + \cdots + x_{i-1} z_{i-1}, \text{ where } z, z_j \in M \text{ for } j = 1, 2, \cdots, i-1.$$

Hence

$$\begin{aligned} m &= a(bz + x_2 z_2 + \cdots + x_{i-1} z_{i-1}) + x_2 y_2 + \cdots + x_{i-1} y_{i-1} \\ &= abz + x_2(az_2 + y_2) + \cdots + x_{i-1}(az_{i-1} + y_{i-1}) \in (Ax_1 + Ax_2 + \cdots + Ax_{i-1})M. \end{aligned}$$

Therfore $x_1, x_2, \cdots, x_i$ is a poor A-sequence on M.

(ii)$\longrightarrow$(i). It is clear that a is a poor A-sequence. Suppose that

$$x_i m \in (Aa + Ax_2 + \cdots + Ax_{i-1})M, \text{ where } 2 \leq i \leq n.$$

Then $x_i bm \in (Ax_1 + Ax_2 + \cdots + Ax_{i-1})M$, and so $bm \in (Ax_1 + Ax_2 + \cdots + Ax_{i-1})M$. Hence

$$x_1 m = x_1 am_1 + x_2 m_2 + \cdots + x_{i-1} m_{i-1} \text{ for some } m_1, m_2, \cdots, m_{i-1} \in M.$$

So, by Lemma 3.4,

$$m \in (Aa + Ax_2 + \cdots + Ax_{i-1})M.$$

Therefore $a, x_2, \cdots, x_i$ is a poor A-sequence on M. ///

Corollary 3.6. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be arbitrary integers. Then the following statements are equivalent;

(i) the sequence $x_1, x_2, \cdots, x_n$ is a poor A-sequence on M.

(ii) the sequence $x_1^{\alpha_1}, x_2^{\alpha_2}, \cdots, x_n^{\alpha_n}$ is a poor A-sequence on M.

Proof. This is obvious by Lemma 3.5. ///

Lemma 3.7. Let U be a triangular subset of A^n each element of which is a poor A-sequence on M and S be a multiplicative closed subset of A. Let ϕ: $A \longrightarrow As$ denote the canonical ring homomorphism. Then the set

$$Us = \{(\phi(u_1),\ \phi(u_2), \cdots,\ \phi(u_n)) \in (As)^n \mid (u_1, u_2, \cdots, u_n) \in U\}$$

is also a triangular subset of $(As)^n$ each element of which is a poor As-sequence on Ms.

Proof. It is clear that if $(\phi(u_1), \phi(u_2), \cdots, \phi(u_n)) \in Us$ then $(\phi(u_1)^{\alpha_1}, \phi(u_2)^{\alpha_2}, \cdots, \phi(u_n)^{\alpha_n}) \in Us$ for every $\alpha_i \in N$. Moreover, for $H = (h_{ij}) \in D_n(A)$, we put $H' = (h_{ij}/1) \in D_n(As)$. Then, for $(\phi(u_1), \phi(u_2), \cdots, \phi(u_n)) \in Us$, $(\phi(v_1), \phi(v_2), \cdots, \phi(v_n)) \in Us$, there exist $H', K' \in D_n(As)$ and $(\phi(w_1), \phi(w_2), \cdots, \phi(w_n)) \in Us$ such that

$$H'(\phi(u_1),\ \phi(u_2), \cdots, \phi(u_n))^T = (\phi(w_1),\ \phi(w_2), \cdots, \phi(w_n))^T = K'(\phi(v_1),\ \phi(v_2), \cdots, \phi(v_n))^T$$

since $(u_1, u_2, \cdots, u_n)$, $(v_1, v_2, \cdots, v_n) \in U$. Hence Us is a triangular subset of $(As)^n$.

Next we shall prove that $\phi(u_1), \phi(u_2), \cdots, \phi(u_n)$ is a poor As-sequence on Ms for any $(\phi(u_1), \phi(u_2), \cdots, \phi(u_n)) \in Us$. Of course, $\phi(u_1) \notin Z(Ms)$. Let $1 < i \leq n$. Assume that

$$\phi(u_i)m/s \in (As\phi(u_1) + As\phi(u_2) + \cdots + As\phi(u_{i-1}))Ms$$

for some $m/s \in Ms$. Then there exist m_1/s_1, $m_2/s_2, \cdots, m_{i-1}/s_{i-1} \in Ms$ such that

$$\phi(u_i)m/s = \phi(u_1)m_1/s_1 + \phi(u_2)m_2/s_2 + \cdots + \phi(u_{i-1})m_{i-1}/s_{i-1}.$$

Thus we have

$$u_i(s_1 s_2 \cdots s_{i-1})m \in (Au_1 + Au_2 + \cdots + Au_{i-1})M.$$

Hence $s_1 s_2 \cdots s_{i-1} m \in (Au_1 + Au_2 + \cdots + Au_{i-1})M$ since $u_i \notin Z\left(M/(\sum_{r=1}^{i-1} Au_r)M\right)$.

This implies that

$$m/s \in (As\phi(u_1) + As\phi(u_2) + \cdots + As\phi(u_{i-1}))Ms.$$

Therefore $\phi(u_1)$, $\phi(u_2), \cdots, \phi(u_n)$ is a poor As-sequence on Ms. ///

Laim O'Caroll proved the following important theorem;

Theorem 3.8. Let A be a (Noetherian) ring and Let M be an A-module. With the Notation 2.10, the complex $\mathscr{C}(\mathscr{U}, M)$ is exact if and only if, for all $i \in \boldsymbol{N}$, each member of U_i is a poor A-sequence on M.

Proof. See [20], or see [13] when A is Noetherian.

4. The Gorenstein Ring.

Proposition 4.1. Let U be a triangular subset of A^n and S be a multiplicative closed subset of A. Let ϕ: $A \longrightarrow As$ denote the canonical ring homomorphism, and put

$$Us = \{(\phi(u_1),\ \phi(u_2), \cdots, \phi(u_n)) \in (As)^n \mid (u_1, u_2, \cdots, u_n) \in U\}.$$

Then Us is a triangular subset of $(As)^n$, and there is an As-isomorphism $\Psi : (U^{-n}M)s \longrightarrow (Us)^{-n}Ms$ which is given by

$$\Psi\left(\frac{m}{(u_1, u_2, \cdots, u_n)}/s\right) = \frac{m/s}{(\phi(u_1), \phi(u_2), \cdots, \phi(u_n))}.$$

Proof. By Lemma 3.7, the set Us is a triangular subset of $(As)^n$.

It is obvious that Ψ is surjective, and so it remains to prove that Ψ is injective. Assume

$$\frac{m/s}{(\phi(u_1), \phi(u_2), \cdots, \phi(u_n))} = 0$$

in $(Us)^{-n}Ms$. Thus there exist $(v_1, v_2, \cdots, v_n) \in U$ and $L \in D_n(As)$ such that

$$L(\phi(u_1),\ \phi(u_2), \cdots, \phi(u_n))^T = (\phi(v_1),\ \phi(v_2), \cdots, \phi(v_n))^T$$

and

$$|L|\, m/s \in \left[\sum_{r=1}^{n-1} As\phi(v_r)\right] Ms.$$

Now there exist $(w_1, w_2, \cdots, w_n) \in U$ and $H, K \in D_n(A)$ such that

$$H(u_1, u_2, \cdots, u_n)^T = (w_1, w_2, \cdots, w_n)^T = K(v_1, v_2, \cdots, v_n)^T \cdots\cdots\cdots (※)$$

Hence

$$\phi(H)(\phi(u_1), \phi(u_2), \cdots, \phi(u_n))^T = (\phi(w_1), \phi(w_2), \cdots, \phi(w_n))^T$$
$$= \phi(K)(\phi(v_1), \ \phi(v_2), \cdots, \phi(v_n))^T = \phi(K)L(\phi(u_1), \phi(u_2), \cdots, \phi(u_n))^T.$$

Let $D = \mathrm{diag}\ (w_1, w_2, \cdots, w_n)$ then we have, by (ii) of Proposition 2.5,

$$(|\phi(H)\phi(D)| - |\phi(D)\phi(K)L|)(m/s) \in \left[\sum_{r=1}^{n-1} As(\phi(w_r)^2)\right] Ms.$$

Since

$$|\phi(D)\phi(K)L|(m/s) \in |\phi(D)\phi(K)| \left[\sum_{r=1}^{n-1} As(\phi(v_r))\right] Ms$$
$$\subseteq |\phi(D)| \left[\sum_{r=1}^{n-1} As\phi(w_r)\right] Ms \text{ by (i) of Proposition 2.5,}$$
$$\subseteq \left[\sum_{r=1}^{n-1} As(\phi(w_r)^2)\right] Ms$$

we have

$$|\phi(D)\phi(H)|(m/s) \in \left[\sum_{r=1}^{n-1} As(\phi(w_r)^2)\right] Ms.$$

It follows that

$$t|DH|m \in \left(\sum_{r=1}^{n-1} Aw_r^2\right) M$$

for some $t \in S$. Since

$$DH(u_1, u_2, \cdots, u_n)^T = (w_1^2, w_2^2, \cdots, w_n^2)^T \text{ from } (※)$$

we have

$$\frac{tm}{(u_1, u_2, \cdots, u_n)} = 0.$$

Therefore

$$\frac{m}{(u_1, u_2, \cdots, u_n)} / s = 0. \quad ///$$

Let A be a Noetherian ring and let M be an A-module. A minimal injective resolution over M is an injective resolution

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^i \xrightarrow{d^i} \cdots$$

such that, for each $i \geq 0$, I^i is an injective envelope of ker d^i. We note that every minimal injective resolution over a module is isomorphic because that all injective envelopes of a module are isomorphic.

Let

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^i \xrightarrow{d^i} \cdots$$

be a minimal injective resolution over M. Then, for each multiplicative closed subset S of A,

$$0 \longrightarrow Ms \longrightarrow (I^0)s \longrightarrow (I^1)s \longrightarrow \cdots \longrightarrow (I^i)s \longrightarrow \cdots$$

is a minimal injective resolution over Ms.

Since every injective module is a direct sum of indecomposable injective modules [8], we can write I^i as

$$I^i = \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(A)} \mu^i(\mathfrak{p}, M) E(A/\mathfrak{p})$$

by (i) and (ii) of Proposition 2.12, where $\mu^i(\mathfrak{p}, M)$ is the number of copies in I^i which are isomorphic to $E(A/\mathfrak{p})$. Accordingly, by the above descriptions,

$$\mu^i(\mathfrak{p}, M) = \mu^i(\mathfrak{p}As, Ms) \quad \text{where } S \cap \mathfrak{p} = \{0\}.$$

If we put $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ then it follows that

$$\mu^i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$$

for any $\mathfrak{p} \in \mathrm{Spec}(A)$ [10].

Lemma 4.2. Let A be a Noetherian ring and let M be an A-module. With the above notations, we have

$$\mu^i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^i_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})} (\mathrm{Ext}^i_A(A/\mathfrak{p}, M))_{\mathfrak{p}}.$$

Proof. By replacing $A_{\mathfrak{p}}, M_{\mathfrak{p}}$ by A, M respectively, we may assume that A is a Noetherian local ring with the maximal ideal $\mathfrak{p}$. Let

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^i \xrightarrow{d^i} \cdots$$

be a minimal injective resolution over M. We have a complex

$$\cdots \longrightarrow \mathrm{Hom}_A(\widetilde{\mathfrak{K}}, I^{i-1}) \longrightarrow \mathrm{Hom}_A(\widetilde{\mathfrak{K}}, I^i) \longrightarrow \mathrm{Hom}_A(\widetilde{\mathfrak{K}}, I^{i+1}) \longrightarrow \cdots$$

where $\widetilde{\mathfrak{K}} = A/\mathfrak{p}$. We have to note that

$$\mathrm{Hom}_A(A/\mathfrak{p}, I^i) \cong T^i = \{x \in I^i \mid \mathfrak{p}x = 0\}$$

because that, for each $f \in \mathrm{Hom}_A(A/\mathfrak{p}, I^i)$, the map ϕ: $\mathrm{Hom}_A(A/\mathfrak{p}, I^i) \longrightarrow T^i$ defined by $\phi(f) = f(1+\mathfrak{p})$ is isomorphic. Let $x \in T^i$. Then $Ax \cong \widetilde{\mathfrak{K}}$ is a submodule of I^i. Hence $Ax \cap d(I^{i-1}) \neq \{0\}$ since I^i is an essential extension of $d(I^{i-1})$, and thus it follows from the fact $Ax \cong \widetilde{\mathfrak{K}}$ that $x \in d(I^{i-1})$. This yields that $T^i \subseteq d(I^{i-1})$, and so $d(T^i) = 0$. In consequence,

$$\mathrm{Ext}^i_A(\widetilde{\mathfrak{K}}, M) \cong T^i \cong \mathrm{Hom}_A(\widetilde{\mathfrak{K}}, I^i).$$

Therefore the result follows from the above descriptions. ///

Definition 4.3. [10] Let $(A, \mathfrak{M}, \widetilde{\mathfrak{K}})$ be a Noetherian local ring of dimension n, where $\mathfrak{M}$ is the maximal ideal and $\widetilde{\mathfrak{K}} = A/\mathfrak{M}$. Then A is called a Gorenstein local ring if it satisfies the following conditions;

(i) inj dim$A < \infty$.

(ii) inj dim$A = n$.

(iii) $\mathrm{Ext}^i_A(\widetilde{\mathfrak{K}}, A) = 0$ if $i \neq n$, and $\mathrm{Ext}^i_A(\widetilde{\mathfrak{K}}, A) \cong \widetilde{\mathfrak{K}}$ if $i = n$.

(iv) A is a Cohen-Macaulay ring and $\mathrm{Ext}^n_A(\widetilde{\mathfrak{K}}, A) \cong \widetilde{\mathfrak{K}}$.

Lemma 4.4. Let A be a Gorenstein local ring of dimension n. Then $A_\mathfrak{p}$ is a Gorenstein local ring for every prime ideal $\mathfrak{p}$ of A.

Proof. Since A is a Gorenstein local ring of dimension n, we have an injective resolution over A:

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0$$

Hence the sequence

$$0 \longrightarrow A_\mathfrak{p} \longrightarrow (I^0)_\mathfrak{p} \longrightarrow (I^1)_\mathfrak{p} \longrightarrow \cdots \longrightarrow (I^n)_\mathfrak{p} \longrightarrow 0$$

is an injective resolution over $A_\mathfrak{p}$, and so inj dim $A_\mathfrak{p} < \infty$. ///

Lemma 4.5. Let A be a Gorenstein local ring. If

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow$$

is a minimal injective resolution over A, then

$$I^i = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ ht(\mathfrak{p}) = i}} E(A/\mathfrak{p}).$$

That is,

$$\mu^i(\mathfrak{p}, A) = \delta_{i, ht(\mathfrak{p})},$$

where δ is a kronecker delta.

Proof. If $ht(\mathfrak{p}) = i$, then, by Definition 4.3 and Lemma 4.4,

$$\mathrm{Ext}^i_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), A_\mathfrak{p}) = \kappa(\mathfrak{p}),$$

where $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Hence, by Lemma 4.2,

$$\mu^i(\mathfrak{p}, A) = \mu^i(\mathfrak{p}A_\mathfrak{p},\ A_\mathfrak{p}) = 1.$$

On the other hand, if $\mathfrak{q}$ is a prime ideal with $ht(\mathfrak{q}) \neq i$ then

$$\mathrm{Ext}^i_{A\mathfrak{q}}(\kappa(\mathfrak{q}),\ A\mathfrak{q}) = 0, \text{ where } \kappa(\mathfrak{q}) = A\mathfrak{q}/\mathfrak{q}A\mathfrak{q}.$$

Hence $\mu^i(\mathfrak{q}, A) = \mu^i(\mathfrak{q}A\mathfrak{q},\ A\mathfrak{q}) = 0$. Therefore

$$I^i = \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(A)} \mu^i(\mathfrak{p}, A) E(A/\mathfrak{p}) = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ ht(\mathfrak{p}) = i}} E(A/\mathfrak{p}). \quad ///$$

It is well known that if $(A, \mathfrak{M})$ is a Cohen-Macaulay local ring, the following statements are equivalent [9]:

(i) the sequence $x_1, x_2, \cdots, x_r$ in $\mathfrak{M}$ is a poor A-sequence.

(ii) $ht\left(\sum_{r=1}^{i} Ax_r\right) = i$ for every $1 \leq i \leq r$.

(iii) $\{x_1, x_2, \cdots, x_r\}$ is a subset of a system of parameters of A.

Let A be a Gorenstein local ring of dimension n and we put

$$\overline{V}_n = \{(x_1, x_2, \cdots, x_n) \in A^n \mid x_1, x_2, \cdots, x_n \text{ is a system of parameters of } A\}.$$

Then $\overline{V}_n$ is a triangular subset of A^n each element of which is a poor A-sequence by Example 2.3 because every Gorenstein local ring is Cohen-Macaulay. In this case, we define a family $\mathscr{V} = \{V_i\}_{i \in N}$ as follows;

(1) If $i=n$, V_n is an expansion of $\bar{V}_n$.

(2) If $i<n$, V_i is a restriction of V_n to A^i.

(3) If $i>n$, $V_i=\{(x_1,x_2,\cdots,x_i)\in A^i \mid (x_1,x_2,\cdots,x_n)\in V_n$ and $x_j=1$ for all $j=n+1, n+2,\cdots,i\}$

Then the family $\mathscr{V}=\{V_i\}_{i\in N}$ satisfies the conditions (i)$\sim$(iv) of Notation 2.10, and so we may form the complex $\mathscr{C}(\mathscr{V},A)$. Futhermore, for any $(x_1,x_2,\cdots,x_i)\in V_i$, $ht\left(\sum_{r=1}^{j} Ax_r\right)\geq j$ for every $1\leq j\leq i$.

R.Y. Sharp and *H.* Zakeri proved the following theorem [20].

Theorem 4.6. Let A be a Noetherian ring and let $\mathscr{U}$ be a family $(U_i)_{i\in N}$ of triangular subsets in Example 2.2. Then

$$U_i^{-i}A\cong \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}(A)\\ ht(\mathfrak{p})=i-1}} (U_i^{-i}A)_{\mathfrak{p}}.$$

We shall now prove our main theorem.

Theorem 4.7. Let A be a Gorenstein local ring of dimension n and let $\mathscr{V}$ be a family $(V_i)_{i\in N}$ of the above-mentioned triangular subsets. Then

$$(V_i^{-i}A)_{\mathfrak{p}}\cong E(A/\mathfrak{p})$$

for every $\mathfrak{p}\in\mathrm{Spec}(A)$ with $ht(\mathfrak{p})=i-1$. In particular,

$$(V_1^{-1}A)_{\mathfrak{p}}\cong E(A/\mathfrak{p})\cong A_{\mathfrak{p}}$$

for every $\mathfrak{p}\in\mathrm{Spec}(A)$ with $ht(\mathfrak{p})=0$.

Proof. We shall prove the assertions by induction on $ht(\mathfrak{p})=i-1$.

If $\mathfrak{p}$ is a prime ideal of A with $ht(\mathfrak{p})=0$, then $A_{\mathfrak{p}}$ is a Gorenstein local ring of dimension zero by Lemma 4.4. Hence, by (ii) of Definition 4.3, we have a minimal injective resolution over $A_{\mathfrak{p}}$:

$$0\longrightarrow A_{\mathfrak{p}}\longrightarrow E(A_{\mathfrak{p}})\longrightarrow 0$$

Thus it follows from Lemma 4.5 and (iii) of Proposition 2.12 that

$$A_{\mathfrak{p}}\cong E(A_{\mathfrak{p}})\cong E(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})\cong E(A/\mathfrak{p}).$$

On the other hand, we obtain

$$(V_j^{-j}A)_{\mathfrak{p}}\cong(V_S)_j^{-j}A_{\mathfrak{p}}=0 \text{ for all } j\geq 2$$

by Proposition 4.1, Lemma 3.7 and Theorem 4.6. So the sequence

$$0 \longrightarrow A_{\mathfrak{p}} \longrightarrow (V_1^{-1}A)_{\mathfrak{p}} \longrightarrow 0$$

is exact by Theorem 3.8. Therefore

$$E(A/\mathfrak{p}) \cong A_{\mathfrak{p}} \cong (V_1^{-1}A)_{\mathfrak{p}}.$$

Suppose that the theorem has been proved for $ht(\mathfrak{p})=0, 1, \cdots, i-2$. Let $ht(\mathfrak{p})=i-1$. Since there are no prime ideals of $A_{\mathfrak{p}}$ with height$\geq i$, we have, for all $j \geq i+1$,

$$(Vs)_j^{-j}A_{\mathfrak{p}} \cong \bigoplus_{\substack{\mathfrak{q}A_{\mathfrak{p}} \in \mathrm{Spec}(A_{\mathfrak{p}}) \\ ht(\mathfrak{q}A_{\mathfrak{p}})=j-1}} [(Vs)_j^{-j}A_{\mathfrak{p}}]_{\mathfrak{q}A_{\mathfrak{p}}} = 0$$

by Lemma 3.7 and Theorem 4.6. Hence we have an exact sequence

$$0 \longrightarrow A_{\mathfrak{p}} \xrightarrow{d^0} (V_1^{-1}A)_{\mathfrak{p}} \xrightarrow{d^1} (V_2^{-2}A)_{\mathfrak{p}} \xrightarrow{d^2} \cdots \longrightarrow (V_{i-1}^{-i+1}A)_{\mathfrak{p}} \xrightarrow{d^{i-1}} (V_i^{-i}A)_{\mathfrak{p}} \longrightarrow 0 \cdots (※)$$

by Theorem 3.8. Moreover, by Lemma 2.11, $(V_j^{-j}A)_{\mathfrak{p}}$ is an essential extension of Im d^{j-1} for $j=1, 2, \cdots, i$.

By Proposition 4.1, Lemma 3.7, Theorem 4.6 and the inductive hypothesis we obtain the following:

$$(V_1^{-1}A)_{\mathfrak{p}} \cong (Vs)_1^{-1}A_{\mathfrak{p}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=0} [(Vs)_1^{-1}A_{\mathfrak{p}}]_{\mathfrak{q}A_{\mathfrak{p}}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=0} E(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}).$$

$$(V_2^{-2}A)_{\mathfrak{p}} \cong (Vs)_2^{-2}A_{\mathfrak{p}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=1} [(Vs)_2^{-2}A_{\mathfrak{p}}]_{\mathfrak{q}A_{\mathfrak{p}}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=1} E(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}).$$

....

$$(V_{i-1}^{-i+1}A)_{\mathfrak{p}} \cong (Vs)_{i-1}^{-i+1}A_{\mathfrak{p}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=i-2} [(Vs)_{i-1}^{-i+1}A_{\mathfrak{p}}]_{\mathfrak{q}A_{\mathfrak{p}}} \cong \bigoplus_{ht(\mathfrak{q}A_{\mathfrak{p}})=i-2} E(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}).$$

Hence $(V_1^{-1}A)_{\mathfrak{p}}$, $(V_2^{-2}A)_{\mathfrak{p}}, \cdots, (V_{i-1}^{-i+1}A)_{\mathfrak{p}}$ are injective $A_{\mathfrak{p}}$-modules. Since $A_{\mathfrak{p}}$ is a Gorenstein local ring of dimension $i-1$, we have an injective resolution over $A_{\mathfrak{p}}$:

$$0 \longrightarrow A_{\mathfrak{p}} \longrightarrow (V_1^{-1}A)_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (V_{i-1}^{-i+1}A)_{\mathfrak{p}} \longrightarrow I \longrightarrow 0.$$

This means that

$$(V_i^{-i}A)_{\mathfrak{p}} \cong I$$

and so the sequence (※) is a minimal injective resolution over $A_{\mathfrak{p}}$. Therefore, by Lemma 4.5,

$$(V_i^{-i}A)_{\mathfrak{p}} \cong \bigoplus_{\substack{\mathfrak{q}A_{\mathfrak{p}} \in \mathrm{Spec}(A_{\mathfrak{p}}) \\ ht(\mathfrak{q}A_{\mathfrak{p}})=i-1}} E(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = E(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \cong E(A/\mathfrak{p}). \quad ///$$

Corollary 4.8. Under the same assumptions as in Theorem 4.7, we have

$$V_i^{-i}A \cong \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}(A) \\ ht(\mathfrak{p})=i-1}} E(A/\mathfrak{p}).$$

Corollary 4.9. Under the same asumptions as in Theorem 4.7, the complex $\mathscr{C}(\mathscr{V}, A)$

$$0 \longrightarrow A \xrightarrow{d^0} V_1^{-1}A \xrightarrow{d^1} V_2^{-2}A \xrightarrow{d^2} \cdots \longrightarrow V_n^{-n} A \xrightarrow{d^n} V_{n+1}^{-n-1}A \longrightarrow 0$$

is a minimal injective resolution over A.

Proof. Since, for $j \geq n+2$, every element of V_j is of the form $(v_1, v_2, \cdots, v_n, 1, 1, \cdots, 1)$ by definition of V_j,

$$V_j^{-j}A = 0$$

by Proposition 2.9. Therefore the result immediately follows from Lemma 2.11 and Corollary 4.8. ///

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