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A Study on the Iterated Integrals of Differential Forms

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1. Introduction

Because of the importance of path spaces to analysis, geometry and other fields, it is desirable to develop a geometric integration theory or a de Rham theory for path spaces. Having in mind this goal, we are going to consider a large class of path space differential forms, which can be constructed from usual differential forms by a method of iterated integration.

The purpose of this paper is to present a process of iterating the integration of differential forms and to demonstrate its usefulness in relating the analytical and the topological aspects of differentiable spaces.

In detail, the concepts of this paper is described as follows. In section 2, we develop the general theory which is a background for the section 3 and 4. And the definitions of the iterated integral $\int w_1 \cdots w_r$ are given by two different way. In Proposition 2.10, we shall prove that the two definitions agree.

Section 3 is devoted mainly to establishing de Rham type theorems.
Let X and Y be differentiable spaces and $h : Y \longrightarrow X$ differentiable map and let A_X and A_Y be differential graded subalgebra of $\Lambda^*(X)$ and $\Lambda^*(Y)$ respectively. Assume that $dA_X^0 = A_X^1 \cap d\Lambda^0(X)$, and $dA_Y^0 = A_Y^1 \cap d\Lambda^0(Y)$.

Theorem 3.6 If $H^1(A_X) \longrightarrow H^1(A_Y)$ is an isomorphism (resp. epimorphism) and $H^2(A_X) \longrightarrow H^2(A_Y)$ is a monomorphism, then $H^0(A'_X) \longrightarrow H^0(A'_Y)$ is an isomorphism (resp. epimorphism), where A'_X is the subcomplex of $\Lambda^*(\Omega_{x_0}(X))$ spanned by iterated integrals.

In Theorem 3.7, we shall prove that under what condition, $H(A') \cong H(B)$, where $B = \mathrm{Hom}_Z(F(\hat{C}_*), k)$, $F(\hat{C}_*)$ has a basis all elements of the type $[c_1|\cdots|c_r]$, $r \geq 0$, where each $c_i \in \hat{C}_s(X)$ for $s \geq 2$ and k is the field of all real (or complex) numbers.

Section 4 deals with the properties of formal power series connection ω.
Especially, we prove the Theorem 4.4 :

Let M be a differentiable space such that the exterior algebra $\Lambda*(M)$ is generated by $\Lambda^0(M)$ and $d\Lambda^0(M)$ and let ω be locally flat modulo a homogeneous ideal ϑ of $k[[X]]$. Then there is a ring homomorphism

$$H_*(\Omega_{x_0}(M)) \longrightarrow k[[X]]/\vartheta.$$

2. Iterated Integrals

Throughout this paper, k will denote the field of real (or complex) numbers. By a *convex n-region,* we mean a closed convex region in R^n. A convex 0-region consists of a single point.

Definition 2.1. A *differentiable space* X is a Hausdorff space equipped with a family $\Phi(X)$ of maps called *plots* which satisfy the following conditions. ([4], [6], [7])

(i) Every plot is a continuous map of the type $\phi : U \longrightarrow X$, where U is a convex region.

(ii) If U' is also a convex region (not necessarily of the same dimension as U) and if $\theta : U' \longrightarrow U$ is a C^∞-map, then $\phi\theta$ is also a plot.

(iii) Each map $\{0\} \longrightarrow X$ is a plot.

(iv) Let $\phi : U \longrightarrow X$ be a continuous map and let $\{\theta_i : U_i \longrightarrow U\}$ be a family of C^∞-maps, U, U_i being convex regions, such that a function f on U is C^∞ if and only if each $f\circ\theta_i$ is C^∞ on U_i. If each $\phi\circ\theta_i$ is a plot of X, then ϕ itself is a plot of X.
If X satisfies only (i), (ii) and (iii) of the above conditions, then it is called a *predifferentiable space.*

Definition 2.2. Let X and X' be differentiable spaces. A *differentiable map* is a continuous map $f : X \longrightarrow X'$ such that, for every plot ϕ of X, $f\phi$ is a plot of X'.

Example: (1) Every C^∞-manifold M(with or without boundary) is a differentiable space, whose family of plots consists of all C^∞-maps from a convex region to M.

(2) Let X be a differentiabe space with a family $\Phi(X)$ of plots. For each subspace $S\subset X$, if we take a family $\Phi(S)$ of plots such that $\phi\in\Phi(S) \Longleftrightarrow \phi\in\Phi(X)$ and the image of $\phi\subset S$, then S is also a differentiable space, which is called a differentiable subspace of X.

(3) Every simplicial complex K induces a natural differentiable space structure on $|K|$, whose family of plots consists of all maps of the type

$$U \xrightarrow{\theta} \triangle^n \xrightarrow{\psi} |K|$$

where U is a convex region, $\triangle^n$ is the standard n-simplex, θ is a C^∞-map and ψ is a simplicial map.

Definition 2.3. A *p-form* ω on a *differentiable space* X is a rule that assigns to each plot $\phi : U \longrightarrow X$ a p-form ω_ϕ on U satisfying the condition ([6], [7]) :

If $\theta : U' \longrightarrow U$ is given as in (ii) of Definition 2.1, then $\omega_{\phi\theta} = \theta^* \omega_\phi$, where θ^* is a homomorphism induced by θ.

In an obvious manner, we obtain the exterior algebra $\Lambda^*(X) = \Sigma \Lambda^p(X)$ whose sum, k-action, exterior product and exterior differential are respectively given by the formulas ([6]) :

$$(\omega + \omega')_\phi = \omega_\phi + \omega'_\phi, \quad (c\omega)_\phi = c\omega_\phi, \quad (\omega \Lambda \omega')_\phi = \omega_\phi \Lambda \omega'_\phi$$

and $\quad (d\omega)_\phi = d\omega_\phi$, where $c \in k$.

The de Rham cohomology will be denoted by $H^*(X)$ ([6]). If $f : X \longrightarrow X'$ is a differentiable map, we shall use f^* to denote both of the induced maps $\Lambda^*(X') \longrightarrow \Lambda^*(X)$ and $H^*(X') \longrightarrow H^*(X)$.

A *piecewise smooth path* (or simply a *path*) on a differentiable space X is a continuous map $\alpha : I \longrightarrow X$ such that, for some partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of the unit interval I, each restriction $\alpha | [t_{i-1}, t_i]$ is a plot of X. Let $P(X)$ denote the space of all paths on X with the compact open topology. Every map $\alpha : U \longrightarrow P(X)$ gives rise to a map

$$\phi_\alpha : U \times I \longrightarrow X$$

given by $(\xi, t) \longmapsto \alpha(\xi)(t)$. A plot of $P(X)$ is defined to be a continuous map $\alpha : U \longrightarrow P(X)$, U being a convex region, such that, for some partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of the unit interval, the restriction of ϕ_α to each $U \times [t_{i-1}, t_i]$ is a plot of X.

If $\omega \in \Lambda^{p+1}(X)$, then ω_{ϕ_α} denotes the piecewise defined $(p+1)$-form on $U \times I$ whose restriction on each $U \times [t_{i-1}, t_i]$ is $\omega_{\phi_\alpha | U \times [t_{i-1}, t_i]}$

Let U be a convex n-region in $\boldsymbol{R}^n$, whose coordinates are $\xi = (\xi^1, \cdots \xi^n)$. Then $U \times I$ has (ξ, t) as coordinates. By a *$\Lambda^*(U)$-valued function of t* (on I), we shall mean an element of the subalgebra of $\Lambda^*(U\times I)$ which is generated by $d\xi^1, \cdots, d\xi^n$ and $\Lambda^0(U\times I)$. In particular, a $\Lambda^p(U)$-valued function $\omega(t)$ is a p-form in $\Lambda^*(U\times I)$ of the type

$$\omega(t)=\sum a_{i_1\cdots i_p}(\xi,t)\, d\xi^{i_1}\Lambda\cdots\Lambda d\xi^{i_p}.$$

For $a,b\in I$, define

$$\int_a^b \omega(t)dt=\sum\Big(\int_a^b a_{i_1\cdots i_p}(\xi,t)dt\Big)d\xi^{i_1}\Lambda\cdots\Lambda d\xi^{i_p}.$$

Thus $\int_a^t \omega(t)dt$ is again a $\Lambda^p(U)$-valued function of t. If $\omega_1(t)$, $\omega_2(t),\cdots$ are $\Lambda^*(U)$-valued functions of t, define inductively

$$\int_a^b \omega_1(t)dt\cdots\omega_r(t)dt=\int_a^b\Big[\int_a^t \omega_1(t)dt\cdots\omega_{r-1}(t)dt\Big]\Lambda\omega_r(t)dt.$$

Lemma 2.4. If $a,b,c\in I$, then

$$\int_a^c \omega_1(t)dt\cdots\omega_r(t)dt=\int_a^b \omega_1(t)dt\cdots\omega_r(t)dt+\cdots+\int_a^b \omega_1(t)dt\cdots\omega_i(t)dt\Lambda$$
$$\int_b^c \omega_{i+1}(t)dt\cdots\omega_r(t)dt+\cdots+\int_b^c \omega_1(t)dt\cdots\omega_r(t)dt.$$

Proof. If $\omega_1(t)=\sum a_{i_1\cdots i_p}(\xi,t)d\xi^{i_1}\Lambda\cdots\Lambda d\xi^{i_p}$, then we have

$$\begin{aligned}\int_a^c \omega_1(t)dt&=\sum\Big(\int_a^c a_{i_1\cdots i_p}(\xi,t)dt\Big)\, d\xi^{i_1}\Lambda\cdots\Lambda d\xi^{i_p}\\&=\sum\Big(\int_a^b a_{i_1\cdots i_p}(\xi,t)\,dt+\int_b^c a_{i_1\cdots i_p}(\xi,t)dt\Big)d\xi^{i_1}\Lambda\cdots\Lambda d\xi^{i_p}\\&=\int_a^b \omega_1(t)dt+\int_b^c \omega_1(t)dt.\end{aligned}$$

Assume that for $r\geq 2$,

$$\int_a^c \omega_1(t)dt\cdots\omega_{r-1}(t)dt=\int_a^b \omega_1(t)dt\cdots\omega_{r-1}(t)dt+\cdots+\int_a^b \omega_1(t)dt\cdots\omega_i(t)dt\Lambda$$
$$\int_b^c \omega_{i+1}(t)dt\cdots\omega_{r-1}(t)dt+\cdots+\int_b^c \omega_1(t)dt\cdots\omega_{r-1}(t)dt.$$

Then we get

$$\begin{aligned}\int_a^c \omega_1(t)dt\cdots\omega_r(t)dt&=\int_a^c\Big[\int_a^t \omega_1(t)dt\cdots\omega_{r-1}(t)dt\Big]\Lambda\omega_r(t)dt\\&=\int_a^b\Big[\int_a^t \omega_1(t)dt\cdots\omega_{r-1}(t)dt\Big]\Lambda\omega_r(t)dt+\int_b^c\Big[\int_a^t \omega_1(t)dt\cdots\omega_{r-1}(t)dt\Big]\Lambda\omega_r(t)dt\\&=\int_a^b \omega_1(t)dt\cdots\omega_r(t)dt+\int_b^c\Big[\int_a^b \omega_1(t)dt\cdots\omega_{r-1}(t)dt+\cdots\end{aligned}$$

$$\cdots+\int_a^b \omega_1(t)dt\cdots\omega_i(t)dt\Lambda\int_b^t \omega_{i+1}(t)dt\cdots\omega_{r-1}(t)dt+\cdots$$
$$\cdots+\int_b^t \omega_1(t)dt\cdots\omega_{r-1}(t)dt]\Lambda\omega_r(t)dt$$
$$=\int_a^b \omega_1(t)dt\cdots\omega_r(t)dt+\int_a^b \omega_1(t)dt\cdots\omega_{r-1}(t)dt\Lambda\int_b^c \omega_r(t)dt+\cdots$$
$$+\int_a^b \omega_1(t)dt\cdots\omega_i(t)dt\,\Lambda\int_b^c \omega_{i+1}(t)dt\cdots\omega_r(t)dt+\cdots$$
$$+\int_b^c \omega_1(t)dt\cdots\omega_r(t)dt.\quad ///$$

Every $(p+1)$-form on $U\times I$, $p\geq 0$, can be uniquely written as $dt\Lambda\,\omega'(t)+\omega''(t)$, where $\omega'(t)$ and $\omega''(t)$ are respectively $\Lambda^p(U)$-valued and $\Lambda^{p+1}(U)$-valued functions of t on I.

Let ω be a $(p+1)$-form on a differentiable space X. If $\alpha: U\longrightarrow P(X)$ is a plot, then the $(p+1)$-form ω_{ϕ_α}, which is piecewise defined on $U\times I$, can be uniquely written as $dt\Lambda\,\omega'(t)+\omega''(t)$ where $\omega'(t)$ and $\omega''(t)$ are piecewise defined $\Lambda^*(U)$-valued functions of t. We define

$$\omega(\alpha,\dot{\alpha})=\omega'(t).$$

Put

$$\Lambda^+(X)=\sum_{p>0}\Lambda^p(X).$$

If $\omega_1, \omega_2, \cdots\in\Lambda^+(X)$, define $\int\omega_1\cdots\omega_r\in\Lambda^*(P(X))$ such that, for any plot $\alpha: U\rightarrow P(X)$

$$(\int\omega_1\cdots\omega_r)_\alpha=\int_0^1\omega_1(\alpha,\dot{\alpha})dt\cdots\omega_r(\alpha,\dot{\alpha})dt\in\Lambda^*(U) \quad\cdots\cdots\cdots\cdots(\mathrm{A})$$

For $r=0$, we put $\int\omega_1\cdots\omega_r=1\in\Lambda^0(P(X))$. We see that if there exists a C^∞-map $\theta: U'\longrightarrow U$ such that $\alpha'=\alpha\theta: U'\longrightarrow P(X)$, then

$$(\int\omega_1\cdots\omega_r)_{\alpha'}=\theta^*(\int\omega_1\cdots\omega_r)_\alpha.$$

Observe that, if each ω_i is a p_i-form on X, then $\int\omega_1\cdots\omega_r$ is a $(p_1+\cdots+p_r-r)$-form on $P(X)$.

Let $x_0, x_1\in X$. A plot $\alpha: U\longrightarrow P(X)$ is said to be *from* x_0 (resp. *to* x_1), if $\alpha(\xi)(0)=x_0$ (resp. $\alpha(\xi)(1)=x_1$) for any $\xi\in U$. Let $\alpha,\beta: U\longrightarrow P(X)$ be plots with $\alpha(\xi)(1)=\beta(\xi)(0)$ for any $\xi\in U$. Define the plot

$$\alpha\beta: U\longrightarrow P(X)$$

such that

$$\alpha\beta(\xi)(t)=\begin{cases}\alpha(\xi)(2t) & \text{for } 0\leq t\leq\frac{1}{2}\\ \beta(\xi)(2t-1) & \text{for } \frac{1}{2}\leq t\leq 1.\end{cases}$$

Define the plot $\alpha^{-1}: U\longrightarrow P(X)$ such that $(\alpha^{-1})(\xi)(t)=\alpha(\xi)(1-t)$. If $\alpha: U\longrightarrow P(X)$ is a plot to x and if $\alpha': U'\longrightarrow P(X)$ is a plot from the same point x, define the plot

$$\alpha\times\alpha': U\times U'\longrightarrow P(X)$$

such that

$$(\alpha\times\alpha')(\xi,\eta)(t)=\begin{cases}\alpha(\xi)(2t) & \text{for } 0\leq t\leq\frac{1}{2}\\ \alpha'(\eta)(2t-1) & \text{for } \frac{1}{2}\leq t\leq 1.\end{cases}$$

Observe that, if

$$p_1: U\times U'\longrightarrow U,\quad p_2: U\times U'\longrightarrow U'$$

denote projections, then

$$\alpha\times\alpha'=(\alpha p_1)(\alpha' p_2).$$

Proposition 2.5. If $\alpha\beta$ is defined, then

$$\left(\int\omega_1\cdots\omega_r\right)_{\alpha\beta}=\sum_{0\leq i\leq r}\left(\int\omega_1\cdots\omega_i\right)_\alpha\Lambda\left(\int\omega_{i+1}\cdots\omega_r\right)_\beta.$$

Moreover, if $r\geq 1$, then $\left(\int\omega_1\cdots\omega_r\right)_{\alpha\alpha^{-1}}=0$.

Proof. Put $\gamma=\alpha\beta$. Then, by Lemma 2.4,

$$\begin{aligned}\left(\int\omega_1\cdots\omega_r\right)_\gamma&=\int_0^1\omega_1(\gamma,\dot\gamma)dt\cdots\omega_r(\gamma,\dot\gamma)dt\\ &=\sum_{0\leq i\leq r}\int_0^{\frac{1}{2}}\omega_1(\gamma,\dot\gamma)dt\cdots\omega_i(\gamma,\dot\gamma)dt\Lambda\int_{\frac{1}{2}}^1\omega_{i+1}(\gamma,\dot\gamma)dt\cdots\omega_r(\gamma,\dot\gamma)dt\end{aligned}$$

which is equal to

$$\sum\left(\int\omega_1\cdots\omega_i\right)_\alpha\Lambda\left(\int\omega_{i+1}\cdots\omega_r\right)_\beta.$$

If $\beta=\alpha^{-1}$, then for $\frac{1}{2}\leq t\leq 1$,

$$\int_{\frac{1}{2}}^t\omega_{i+1}(\gamma,\dot\gamma)dt\cdots\omega_r(\gamma,\dot\gamma)dt=\int_{\frac{1}{2}}^{1-t}\omega_{i+1}(\gamma,\dot\gamma)dt\cdots\omega_r(\gamma,\dot\gamma)dt$$

Use the above formula for $t=1$ and apply Lemma 2.4 to the right hand side of proof of the first equation. Hence

$$(\int \omega_1\cdots\omega_r)_\gamma=\int_0^0 \omega_1(\gamma,\dot{\gamma})dt\cdots\omega_r(\gamma,\dot{\gamma})dt=0. \quad ///$$

Corollary 2.6. If $\alpha\times\alpha'$ is defined, then

$$(\int \omega_1\cdots\omega_r)_{\alpha\times\alpha'}=\sum_{0\leq i\leq r}(\int \omega_1\cdots\omega_i)_\alpha\times(\int \omega_{i+1}\cdots\omega_r)_{\alpha'}.$$

Proof. Since $\alpha\times\alpha'=(\alpha p_1)(\alpha p_2)$,

$$(\int \omega_1\cdots\omega_r)_{\alpha\times\alpha'}=\sum_{0\leq i\leq r} p_1^*(\int \omega_1\cdots\omega_i)_\alpha \Lambda\, p_2^*(\int \omega_{i+1}\cdots\omega_r)_{\alpha'}$$

$$=\sum_{0\leq i\leq r}(\int \omega_1\cdots\omega_i)_\alpha\times(\int \omega_{i+1}\cdots\omega_r)_{\alpha'}. \quad ///$$

By a *compact plot,* we mean a plot such that its domain is compact. Let ω_i be a p_i-form on X, $i=1,2,\cdots$. If $\alpha: U\longrightarrow P(X)$ is a compact plot, define, for $r>0$

$$\int_\alpha\int \omega_1\cdots\omega_r=\begin{cases}\int_U(\int \omega_1\cdots\omega_r)_\alpha, & \text{when } \dim U=p_1+\cdots+p_r-r\\ 0 & \text{, otherwise.}\end{cases}$$

Extending by linearity, $\int_\alpha\int u$ is defined for any iterated integral $\int u$, where u is a form sum. When $r=0$, we use the convention:

$\int_\alpha\int \omega_1\cdots\omega_r=\delta_{on}$, where $n=\dim U$ and δ_{on} is the Kroneck's notation. Observe that if $n=0$, then the plot α can be taken of the type $\{0\}\longrightarrow P(X)$. Therefore if $r=0$, then $\int \omega_1\cdots\omega_r=1\in\Lambda^0(P(X))$.

For the case of $r=1$, sbserve that

$$\int_\alpha\int \omega_1=\int_{\phi_\alpha}\omega_1.$$

If $\alpha\times\alpha'$ is defined, then

$$\int_{\alpha\times\alpha'}\int \omega_1\cdots\omega_r=\sum_{0\leq i\leq r}\int_\alpha\int \omega_1\cdots\omega_i\int_{\alpha'}\int \omega_{i+1}\cdots\omega_r. \quad \cdots\cdots\cdots\cdots\cdots\cdots(\text{B})([6]).$$

For forms $\omega_1,\cdots,\omega_r$ on X, we can given the definition of $\int \omega_1\cdots\omega_r$ as another way. Now a *$\Lambda^p(X)$-valued function u of t on I* is an element of $\Lambda^p(X\times I)$ such that, for every plot $\phi: U\longrightarrow X$, the p-form $u_{\phi\times 1}$ is of the type

$$\sum a_{i_1 \cdots i_p}(\xi, t)\, d\xi^{i_1} \Lambda \cdots \Lambda d\xi^{i_p}$$

where $1 = 1_I$ denotes the identity map of I and $\xi = (\xi^1, \cdots, \xi^n)$ denotes the coordinates of U. For $t_0 \in I$, $u(t_0)$ is the image of u under the homomorphism induced by the inclusion $X = X \times \{t_0\} \subset X \times I$.

If u is a $\Lambda^p(X)$-valued function of t on I, define

$$\int_a^b u\,dt \in \Lambda^p(X), \quad a, b, \in I.$$

such that

$$\left(\int_a^b u\,dt\right)_\phi = \sum \left(\int_a^b a_{i_1 \cdots i_p}(\xi, t)\,dt\right) d\xi^{i_1} \Lambda \cdots \Lambda d\xi^{i_p}.$$

Similary define the $\Lambda^p(X)$-valued function $\partial u/\partial t$ of t on I such that $(\partial u/\partial t)_{\phi \times 1}$ is obtained the differentiating the coefficients of $u_{\phi \times 1}$ with respect to t.

Every p-form v on $X \times I$ can be uniquely written as

$$v = dt \Lambda v' + v''$$

where v' and v'' are respectively $\Lambda^{p-1}(X)$ and $\Lambda^p(X)$-valued functions of t on I. We shall denote $v' = (\partial/\partial_t) \lrcorner\, v$, which coincides with the usual notation for an interior product in the case where X is a manifold.

Lemma 2.7. If $f_0, f_1 : X' \longrightarrow X$ is differentiable maps and $F : X' \times I \longrightarrow X$ is a homotopy from f_0 to f_1, then F induces a chain homotopy

$$\int_F : \Lambda(X) \longrightarrow \Lambda(X')$$

given by $\omega \longmapsto \int_0^1 ((\partial/\partial t) \lrcorner\, F^*\omega)\; dt$ such that

$$d\int_F + \int_F d = f_1^* - f_0^*.$$

Proof. Let d' denote the exterior differential in X'. If $\omega \in \Lambda^p(X)$ and $F^*\omega = v = dt \Lambda v' + v''$, then

$$d\int_F \omega + \int_F d\omega = \int_0^1 d'v'dt + \int_0^1 ((\partial/\partial t) \lrcorner\, F^*d\omega)dt.$$

Since $F^*\omega = v = dt\Lambda v' + v''$ and $v' = \sum a'_{i_1, \cdots, i_{p-1}}(\xi_1, t)\, d\xi_1^{i_1} \Lambda,\ \Lambda d\xi_1^{i_{p-1}}$ and

For any element u in a graded vector space $\Lambda^*(X)$, put $Ju=(-1)^{\deg u}u$.

Definition 2.9. For forms $\omega_1,\cdots,\omega_r$ on X, define $\int\omega_1=\int' p_1^*\,\omega_1$ and for $r>1$,

$$\int\omega_1\cdots\omega_r=\int'(J\int\omega_1\cdots\omega_{r-1})\Lambda\, p_1^*\,\omega_r.$$

Set

$$\int\omega_1\cdots\omega_r=\begin{cases}1 \text{ if } r=0\\ 0 \text{ if } r<0.\end{cases}$$

Elements of the graded subspace of $\Lambda^*(P(X))$ spanned by all $\int\omega_1\cdots\omega_r$ $r\geq 0$, will be called *iterated integrals.* Now, the definition of $\int\omega_1\cdots\omega_r$ was given as two different ways. In the proposition 2.10, we are going to verify that they are equal.

Proposition 2.10. In the above situation, the present definition agrees with the definition of (A).

Proof. Put $\omega_i'=(\partial/\partial t)\lrcorner\, F^*p_1{}^*\omega_i$ and $u_r=F^*\int\omega_1\cdots\omega_r$, both of which are $\Lambda^*P(X)$-valued functions of t on I. Since F is a homotopy from $\eta\circ p_0$ to $1_{P(X)}$, we have

$$u_r(1)=\int\omega_1\cdots\omega_r.$$

If $\alpha: U\longrightarrow P(X)$ is a plot, then

$$(\omega_i')_{\alpha\times 1}=(\partial/\partial t)\lrcorner\ (F^*p_1{}^*\omega_i)_{\alpha\times 1}=(\partial/\partial t)\lrcorner\ (\omega_i)_{\phi_\alpha}$$

which is $\omega_i\ (\alpha,\dot\alpha)$ as defined in (A). Therefore, in order that the two definitions agree, it suffices to verify that

$$u_r(s)=\int_0^s(\int_0^{t_r}(\cdots\int_0^{t_2}\omega_1'\,(t_1)dt_1\cdots)\Lambda\,\omega'_{r-1}(t_{r-1})dt_{r-1})\Lambda\,\omega_r'\,(t_r)dt_r.$$

For $r>1$, u_r is a $\Lambda^*(P(X))$-valued function of t on I such that

$$u_r(s)=\int_0^s v'(t)dt$$

where since if $u_r=F^*\int' v$, then $u_r(s)=\int_0^s((\partial/\partial t)\lrcorner\ F^*v)dt$, $v'(t)=(\partial/\partial t)\lrcorner\ F^*(J\int\omega_1\cdots\omega_{r-1}\Lambda\ p_1{}^*\omega_r)=(\partial/\partial t)\lrcorner\ (Ju_{r-1}(t)\Lambda\ F^*p_1{}^*\omega_r)$.

According to Lemma 2.8, u_{r-1} is also a $\Lambda^*(P(X))$-valued function of t on I so that

$v''=\sum a''_{i_1\cdots i_p}(\xi_1,t)d\xi_1^{i_1}\Lambda\cdots\Lambda d\xi_1^{i_p}$, where $\xi_1=(\xi_1^1,\cdots,\xi_1^n)$ is a coordinate of X', $(\partial/\partial t)\lrcorner F^*\omega=v'$ and $dF^*\omega=dt\Lambda(-d'v'+\frac{\partial}{\partial t}v'')+d'v''$.

Hence

$$d\int_F\omega+\int_F d\omega=\int_0^1 d'v'dt+\int_0^1(-d'v'+\frac{\partial}{\partial t}v'')\,dt$$
$$=v''(1)-v''(0)=f_1^*\omega-f_0^*\omega. \quad ///$$

In the special case where $X=X'$ is a convex set in $\boldsymbol{R}^n$ and F representing a contraction, the above lemma is the usual Poincaré lemma. Therefore we call $\int_F$ the *Poincaré operator* of the homotopy F.

For every $t\in I$, there is a differentiable map $p_t: P(X)\longrightarrow X$ given by $\gamma\longrightarrow\gamma(t)$. Let γ^t be the path such that $\gamma^t(\tau)=\gamma(t\tau)$. Denote by $\tilde{p}_t: P(X)\longrightarrow P(X)$ the differentiable maps given by $\gamma\longmapsto\gamma^t$. Denote by η_x the constant path at x in X. Denote by $\eta: X\longrightarrow P(X)$ the canonical differentiable map given by $x\longmapsto\eta_x$. Then there is a homotopy $P(X)\times I\longrightarrow P(X)$ from $\eta\circ p_0$ to the identity map $1_{P(X)}$ given by $(\gamma,t)\longmapsto\gamma^t$.

This is the homotopy obtained by contracting each path along itself. Denote by

$$\int' : \Lambda^*(P(X))\longrightarrow\Lambda^*(P(X))$$

the resulting Poincaré operator. Let $F: P(X)\times I\longrightarrow P(X)$ be a homotopy from $\eta\circ p_0$ to the identity map $1_{P(X)}$ given by $(\gamma,t)\longmapsto\gamma^t$.

Lemma 2.8. If $v\in\Lambda^p(P(X))$, then $u=F^*\int' v$ is a $\Lambda^{p-1}(P(X))$-valued function of t on I, and

$$u(s)=\int_0^s((\partial/\partial t)\lrcorner F^*v)dt, \qquad s\in I.$$

Proof. Put $\alpha'=F\circ(\alpha\times 1_I)$ and $\alpha''=F\circ(\alpha'\times 1_I)$ for any plot $\alpha: U\longrightarrow P(X)$. Then $\alpha''(\xi,t,s)(\tau)=\alpha(\xi)^{ts}(\tau)=\alpha(\xi)(ts\tau)$ and can be factorized as

$$U\times I\times I\xrightarrow{1_U\times\mu}U\times I\xrightarrow{\alpha'}P(X)$$

where $\mu: I\times I\longrightarrow I$ is such that $(t,s)\longmapsto ts$.

We get $v_{\alpha'}=dt\Lambda v'+v''$. Then $v_{\alpha''}=d(st)\Lambda v'(st)+v''(st)=dt\Lambda sv'(st)+(ds\Lambda tv'(st)+v''(st))$.

Hence

$$(F^*\int' v)_{\alpha\times 1}=(\int' v)_{\alpha'}=\int_0^1 sv'(st)dt=\int_0^s v'(t)dt=\int_0^s(\partial/\partial_t)\lrcorner F^*v)dt. \quad ///$$

$$v'(t)=u_{r-1}(t)\,\Lambda((\partial/\partial t)\lrcorner\, F^*p_1^*\omega_r)=u_{r-1}(t)\Lambda\,\omega_r'(t).$$

Finally, it follows by induction. ///

3. A de Rham Theorem for the Loop Space

Throughout this section, $\omega, \omega_1, \omega_2, \cdots$ will denote forms on X of positive degrees $p, p_1, p_2, \cdots$ respectively. Denote by $P(X; x_0, *)$ the differentiable subspace of $P(X)$ consisting of those paths which initiate from x_0. Let τ be a permutation of n letters. Then $\omega_1\Lambda\cdots\Lambda\omega_n$ and $\omega_{\tau(1)}\Lambda\cdots\Lambda\omega_{\tau(n)}$ differ only by a sign, which depends on the permutation τ and the degrees $p_1, \cdots, p_n$ and will be denoted by $\varepsilon(\tau; p_1, \cdots, p_n)$.

Recall that an (r,s)-shuffle σ is a permutation of $r+s$ letters such that

$$\sigma^{-1}(1)<\cdots<\sigma^{-1}(r),\quad \sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s).$$

If $f_1(t), f_2(t)\cdots$ are piecewise continuous functions, define for $r>1$

$$\int_a^b f_1dt\cdots f_rdt=\int_a^b\left(\int_a^t f_1dt\cdots f_{r-1}dt\right) f_r(t)dt.$$

Then

$$\left(\int_a^b f_1dt\cdots f_rdt\right)\left(\int_a^b f_{r+1}dt\cdots f_{r+s}dt\right)=\sum\int_a^b f_{\sigma(1)}dt\cdots f_{\sigma(r+s)}dt$$

summing over all (r,s)-shuffle σ ([4]). Its verification can be illustrated by the following particular cases :

$$\begin{aligned}\int_a^b f_1dt\int_a^b f_2dt&=\int_a^b\left[\left(\int_a^t f_1dt\right)f_2+\left(\int_a^t f_2dt\right)f_1\right]dt\\&=\int_a^b f_1dt\, f_2dt+\int_a^b f_2dt\, f_1dt,\end{aligned}$$

$$\begin{aligned}\left(\int_a^b f_1dt\right)\left(\int_a^b f_2dt\, f_3dt\right)&=\int_a^b\left[\left(\int_a^t f_1dt\int_a^t f_2dt\right) f_3+\left(\int_a^t f_2dt\, f_3dt\right)f_1\right]dt\\&=\int_a^b f_1dt\, f_2dt\, f_3dt+\int_a^b f_2dt\, f_1dt\, f_3dt+\int_a^b f_2dt\, f_3dt\, f_1dt,\end{aligned}$$

$$\begin{aligned}\left(\int_a^b f_1dt\, f_2dt\right)\left(\int_a^b f_3dt\, f_4dt\right)=\cdots&=\int_a^b f_1dt\, f_2dt\, f_3dt\, f_4dt+\int_a^b f_1dt\, f_3dt\, f_2dt\, f_4dt\\&+\int_a^b f_1dt\, f_3dt\, f_4dt\, f_2dt+\int_a^b f_3dt\, f_1dt\, f_2dt\, f_4dt+\int_a^b f_3dt\, f_1dt\, f_4dt\, f_2dt\\&+\int_a^b f_3dt\, f_4dt\, f_1dt\, f_2dt.\end{aligned}$$

It follows from the definition of iterated integrals that, in $\Lambda^*(P(X))$,

$$\int \omega_1 \cdots \omega_r \Lambda \int \omega_{r+1} \cdots \omega_{r+s} = \sum \varepsilon(\sigma;\ p_1-1, \cdots, p_{r+s}-1) \int \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)}.$$

Proposition 3.1. If $f \in \Lambda^0(X)$, then in $\Lambda^*(P(X))$,

$$\int \omega_1 \cdots \omega_{i-1}(f\omega_{i+1})\omega_{i+2} \cdots \omega_r = \int \omega_1 \cdots \omega_{i-2}(f\omega_{i-1})\omega_{i+1} \cdots \omega_r$$
$$+ \int \omega_1 \cdots \omega_{i-1} df\ \omega_{i+1} \cdots \omega_r \qquad 1 < i < r.$$

Proof. For any plot $\alpha : U \longrightarrow P(X)$, if we put $\omega_i = df$, then

$$\omega_i(\alpha, \dot{\alpha}) = \frac{\partial}{\partial t} f \circ \phi_\alpha,$$

since $\omega_{i\phi_\alpha} = (\sum \frac{\partial}{\partial x^i} f \circ \phi_\alpha) + \frac{\partial}{\partial t} f \circ \phi_\alpha\ dt$, where x^i is the i-th coordinate of U. Let $\alpha^t : U \longrightarrow P(X)$ be the plot given by $\alpha^t(\xi)(\tau) = \alpha(\xi)(t\tau)$, $0 \leq \tau \leq 1$. Since

$$(\int \omega_1 \cdots \omega_r)_\alpha = \int_0^1 (\int \omega_1 \cdots \omega_{r-1})_{\alpha^t} \Lambda\ \omega_r(\alpha, \dot{\alpha}) dt,$$

it suffices to verify proposition for the case of $i = r-1$. Then we have for $r > 2$

$$(\int \omega_1 \cdots \omega_{r-1})_{\alpha^s} = \int_0^S (\int \omega_1 \cdots \omega_{r-2})_{\alpha^t} \left(\frac{\partial}{\partial t} f \circ \phi_\alpha\right) dt$$
$$= [(\int \omega_1 \cdots \omega_{r-2})_{\alpha^t} f \circ \phi_\alpha]_{t=s}$$
$$- \int_0^S (\int \omega_1 \cdots \omega_{r-3})(f\omega_{r-2})(\alpha, \dot{\alpha}) dt.$$

Hence the proposition is proved. ///

Verify also that, in $\Lambda^*(P(X; x_0, *))$,

$$\int (df)\omega_2 \cdots \omega_r + f(x_0) \int \omega_2 \cdots \omega_r = \int (f\omega_2)\omega_3 \cdots \omega_r \qquad ([6]) \cdots\cdots\cdots (C)$$

and that, in $\Lambda^*(P(X; x_0, x_1))$,

$$f(x_1) \int \omega_1 \cdots \omega_{r-1} - \int \omega_1 \cdots \omega_{r-1} df = \int \omega_1 \cdots \omega_{r-2}(f\omega_{r-1}) \qquad ([6]) \cdots (D)$$

Proposition 3.2. For forms $\omega_1, \cdots, \omega_r$ on X,

$$d\int \omega_1 \cdots \omega_r = \sum_{1 \leq i \leq r} (-1)^i \int J\omega_1 \cdots J\omega_{i-1}(d\omega_i)\omega_{i+1} \cdots \omega_r$$
$$- \sum_{1 \leq i \leq r} (-1)^i \int J\omega_1 \cdots J\omega_{i-1}(J\omega_i \Lambda\ \omega_{i+1})\omega_{i+2} \cdots \omega_r$$
$$- p_0^* \omega_1 \Lambda \int \omega_2 \cdots \omega_r + (J \int \omega_1 \cdots \omega_{r-1}) \Lambda p_1^* \omega_r.$$

Proof. This proposition holds when $r=0$. For $r>0$,

$$d\int\omega_1\cdots\omega_r=d\int'(J\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r$$

$$=-\int'd((J\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r)+(J\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r \text{ (By Lemma 2.7)}$$

$$=-\int'(dJ\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r-\int'(\int\omega_1\cdots\omega_{r-1})\Lambda dp_1^*\omega_r$$

$$+(J\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r$$

$$=\int'((Jd\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*\omega_r)+(-1)^r\int'(J\int J\omega_1\cdots J\omega_{r-1})\Lambda p_1^*d\omega_r$$

$$+(J\int\omega_1\cdots\omega_{r-1})\Lambda p_1^*d\omega_r.$$

Hence the proposition follows by induction. ///

A differentiable space is said to be *path connected* if any two points can be connected by a piecewise smooth path in X.

Let A be a differential graded subalgebra of the exterior algebra $\Lambda^*(X)$ such that $dA^0=A^1\cap d\Lambda^0(X)$. The reader may keep in mind of the special case of $A=\Lambda^*(X)$. Denote by $\bar{A}$ the cochain complex obtained from A by setting $\bar{A}^p=0$ for $p<0$, $\bar{A}^0=A^1/dA^0$, and $\bar{A}^p=A^{p+1}$ for $p<0$. Then

$$H^p(\bar{A})=\begin{cases}H^{p+1}(A) & \text{for } p\geq 0\\ 0 & \text{for } p<0.\end{cases}$$

Let $\Omega_{x_0}X$ is a loop space at $x_0\in X$ and let $C^*(\Omega_{x_0}X;k)$ be a cochain complex of smooth singular cochain of $\Omega_{x_0}X$. Then there is a cochain map

$$\Psi:\bar{A}\longrightarrow C^*(\Omega_{x_0}X;k)$$

which assigns to each form $\omega\in\bar{A}$ the integral of the form $\int\omega$ on $\Omega_{x_0}X$. We know that the map Ψ is well-defined on $\bar{A}^\circ$.

Lemma 3.3. If X is connected, then the map Ψ is an injection.

Proof. Let ω be an element of A^p. If $p=1$, we use the usual argument to show that, if ω vanishes on $\Omega_{x_0}X$, then ω must be exact on X ([7]). Assume that $p>1$ and $\omega\neq 0$. There exists a plot $\phi:U\longrightarrow X$ such that ω_ϕ vanishes nowhere on U. Construct a plot

$$\theta:\triangle^{p-1}\longrightarrow P(U;\xi_0,\xi_1)$$

for some $\xi_0, \xi_1 \in U$ such that $\phi_\theta : \triangle^{p-1}\times I \longrightarrow U$ maps the interior of $\triangle^{p-1}\times I$ homeomorphically into U. Put $x_i' = \phi(\xi_i)$, $i=0, 1$, and let

$$\sigma : \triangle^{p-1} \longrightarrow P(X; x_0', x_1')$$

be given by $\sigma(\xi)(t) = \phi(\phi_\theta(\xi, t))$. Then $\int_\sigma \int \omega \neq 0$. Choose any $\alpha_0 \in P(X; x_0, x_0')$ and $\alpha_1 \in P(X; x_1', x_0)$. Then $\alpha_0 \times \sigma \times \alpha_1$ is a smooth $(p-1)$-simplex of $\Omega_{x_0}X$, and

$$\int_{\alpha_0 \times \sigma \times \alpha_1} \int \omega = \int_\sigma \int \omega \neq 0.$$

Hence the Ψ is an injection. ///

For $s \geq 0$, let $A'(s)$ be the subcomplex of $\Lambda^*(\Omega_{x_0}X)$ spanned by iterated integrals of the type

$$\int \omega_1 \cdots \omega_r$$

$0 \leq r \leq s$, $\omega_1, \cdots, \omega_r \in A^+ = \sum_{p>0} A^p$. Then $A' = \bigcup_s A'(s)$ is a differential graded subalgebra of $\Lambda^*(\Omega_{x_0}X)$. Then

$$k = A'(0) \subset A'(1) \subset \cdots \subset A'(s) \subset \cdots\cdots$$

is an ascending filtration of the differential graded algebra A'. Put $A'(s)=0$ for $s<0$.

Consider the map

$$\Phi : \otimes^s \bar{A} \longrightarrow A'(s)/A'(s-1) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{(E)}$$

given by $\omega_1 \otimes \cdots \otimes \omega_s \longmapsto \int \omega_1 \cdots \omega_s + A'(s-1)$. If $df = \omega_i$ for some $f \in A^0$, one can show that $\int \omega_1 \cdots \omega_i \cdots \omega_s \in A'(s-1)$ (by Proposition 3.1, (C) and (D)). Therefore the map Φ is well-defined.

Lemma 3.4. For any path connected space X, the map Φ is a bijection.

Proof. Put $L_r(C_*(\Omega_{x_0}X), k)$ = the vector space over k of r-linear functions on $C_*(\Omega_{x_0}X)$. Let $b_{\omega_1 \cdots \omega_r}$ denote the element of $L_r(C_*(\Omega_{x_0}X), k)$ such that, for any simplices $c_1, \cdots, c_r$ belonging to $C_*(\Omega_{x_0}X)$.

$$b_{\omega_1 \cdots \omega_r}(c_1, \cdots, c_r) = \left(\int_{c_1}\int \omega_1\right) \cdots \left(\int_{c_r}\int \omega_r\right).$$

Then the map Ψ of Lemma 3.3 induces a monomorphism

$$\otimes^s \bar{A} \longrightarrow L_s(C_*(\Omega_{x_0}X), k)$$

given by $\omega_1 \otimes \cdots \otimes \omega_s \longmapsto b_{\omega_1 \cdots \omega_s}$.

For every p-simplex $\sigma : \triangle^p \longrightarrow \Omega_{x_0}X$, definee the reduced simplex σ' to be σ for $p>0$ and $\sigma-\varepsilon$ for $p=0$, where ε denotes the 0-simplex at the base point (i. e. the null loop) of $\Omega_{x_0}X$. Observe that $\int_\varepsilon \int \omega_1 \cdots \omega_r = 0$ when $r=0$.

Define the linear map

$$A'(s)/A'(s-1) \longrightarrow L_s(C_*(\Omega_{x_0}X); k)$$

such that $\int \omega_1 \cdots \omega_s + A'(s-1)$ is sent to the s-linear function given by

$$(c_1, \cdots, c_s) \longmapsto \int_{c_1' \times \cdots \times c_s'} \int \omega_1 \cdots \omega_s$$

where the integral over $c_1' \times \cdots \times c_s'$ is defined by linearity. By using successively (B), we obtain

$$\int_{c_1' \times \cdots \times c_s'} \int \omega_1 \cdots \omega_s = \left(\int_{c_1'} \int \omega_1\right) \cdots \left(\int_{c_s'} \int \omega_s\right) = \left(\int_{c_1} \int \omega_1\right) \cdots \left(\int_{c_s} \int \omega_s\right)$$

This means that the map $A'(s)/A'(s-1) \longrightarrow L_s(C_*(\Omega_{x_0}X), k)$ is such that

$$\int \omega_1 \cdots \omega_s + A'(s) \longmapsto b_{\omega_1 \cdots \omega_s}.$$

Comparing with $\otimes^s \bar{A} \longrightarrow L_s(C_*(\Omega_{x_0}X), k)$, we conclude that the map Φ is injective. On the other hand, it is obviously surjective. ///

Now, we can get the next Lemma 3.5 regarding the spectral sequence of A'.

Lemma 3.5. The cohomology spectral sequence $\{E_r(A')\}$ associated to the filtered cochain complex A' converges to $H(A')$, and

$$E_1^{s,t} = H^{s+t}(A'(s)/A'(s-1)) \cong H^{s+t}(\otimes^s \bar{A}) \ ([6]). \quad ///$$

For any $r, s \geq 0$, define

$$A'(s, s-r) = \{u \in A'(s) \mid du \in A'(s-r)\}$$
$$L'^s_r = A'^0(s, s-r)/A'^0(s-1, s-r)$$
$$M'^s_r = A'^1(s, s-r)/[A'^1(s-1, s-r) + dA'^0(s+r-1, s)]$$

For $r>s$, $A'(s-r)=0$ since $s-r<0$. Now consider the following cochain complex

$$0 \longrightarrow A'^0(s) \xrightarrow{d^0} A'^1(s) \longrightarrow A'^2(s) \longrightarrow \cdots\cdots.$$

Then we get that $\mathrm{Ker}\, d^0 = \{u \in A'^0(s) \mid d^0u = 0 = du\} = A'^0(s, s-r) = H^0(A'(s))$ and $H^0(A'(s-1)) = A'^0(s-1, s-r)$. Therefore we have the following result:

$$L'^s_r = L'^s_s = H^0(A'(s)/H^0(A'(s-1)) \text{ for } r > s.$$

Next, we can see that

$$0 \longrightarrow A'^0(s)/A'^0(s-1) \xrightarrow{d^0} A'^1(s)/A'^1(s-1) \xrightarrow{d^1} A'^2(s)/A'^2(s-1) \longrightarrow \cdots\cdots$$

is a cochain complex. For any $u + A'^0(s-1) \in \mathrm{Ker}\, d^0$, $u \in A'^0(s)$, $d^0u = du \in A'^1(s-1)$ and $u \in A'^0(s, s-1)$. Also, for any $u + A'^0(s-1, s-1) \in L_1'^s$, we have $du \in A'^1(s-1)$. Therefore we get

$$L'^s_1 \cong H^0(A'(s)/A'(s-1)).$$

Similary, we obtain the following result:

$$M'^s_1 \cong H^1(A'(s)/A'(s-1)).$$

Since $A'^0(s, s-r-1) \subset A'^0(s, s-r)$, $A'^0(s-1, s-r-1) \subset A'^0(s-1, s-r)$ and if $u, u' \in A'^0(s, s-r-1)$ and $u - u' \notin A'^0(s-1, s-r-1)$, then $u - u' \notin A'^0(s-1)$, we have an injection

$$\begin{array}{ccc} L'^s_{r+1} & \xrightarrow{\quad i \quad} & L'^s_r \\ \Cup & & \Cup \\ u + A'^0(s-1, s-r-1) & \longmapsto & u + A'^0(s-1, s-r). \end{array}$$

The exterior differentiation induces a homomorphism d_r such that

$$\begin{array}{ccc} d_r : L'^s_r & \longrightarrow & M'^{s-r}_r \\ \Cup & & \Cup \\ u + A'^0(s-1, s-r) & \longmapsto & du + [A'^1(s-r-1, s-2r) + dA'^0(s-1, s-r)]. \end{array}$$

For any $u \in A'^0(s, s-r-1)$,

$$\begin{aligned} &d_r i(u + A'^0(s-1, s-r-1)) = d_r(u + A'^0(s-1, s-r)) \\ &= du + [A'^1(s-r-1, s-2r) + dA'^0(s-1, s-r)] = 0. \end{aligned}$$

If $d_r(u + A'^0(s-1, s-r)) = 0$ for $u \in A'^0(s, s-r)$, then $du \in A'^1(s-r-1, s-2r)$ or $du \in dA'^0(s-1, s-r)$. We can see that $u \in A'^0(s, s-r-1)$ or $u \in A'^0(s, s-1)$, and hence $u + A'^0(s-1, s-r-1) \in L'^s_{r+1}$ or $u + A'^0(s-1, s-r) = 0 \in L'^s_r$.

Hence we obtain the following exact sequence

$$0 \longrightarrow L'^{s}_{r+1} \xrightarrow{i} L'^{s}_{r} \xrightarrow{d_r} M'^{s-r}_{r}.$$

By Lemma 3.4, $\Phi : \otimes^s \bar{A} \longrightarrow A'(s)/A'(s-1)$ is an isomorphism provided the differentiable space is path connected. Hence if X is path connected, then it follows that there is an induced isomorphism

$$H(\otimes^s \bar{A}) \cong H(A'(s)/A'(s-1)).$$

Since $H^0(A'(s)/A'(s-1)) = L'^{s}_{1}$ and $H^1(A'(s)/A'(s-1)) = M'^{s}_{1}$, we obtain

$$L'^{s}_{1} \cong H^0(\otimes^s \bar{A}) \text{ and } M'^{s}_{1} \cong H^1(\otimes^s \bar{A}).$$

Let X and Y be differentiable spaces and $h : Y \longrightarrow X$ differentiable map. Let A_X and A_Y be differential graded subalgebras of $\Lambda^*(X)$ and $\Lambda^*(Y)$ respectively. Then the differentiable map h induces a homomorphism $h^* : A_X \longrightarrow A_Y$ and a cochain map $h' : A'_X \longrightarrow A'_Y$.

We assume that both A_X and A_Y satisfy the following conditions:

$$dA^0_X = A^1_X \cap d\Lambda^0(X) \text{ and } dA^0_Y = A^1_Y \cap d\Lambda^0(Y).$$

Let $h^{*1} : H^1(A_X) \longrightarrow H^1(A_Y)$ and $h^{*2} : H^2(A_X) \longrightarrow H^2(A_Y)$ be homomorphisms induced by h^*, respectively and let $h'^0 : H^0(A'_X) \longrightarrow H^0(A'_Y)$ be a homomorphism induced by h'.

Theorem 3.6. Let h^{*2} be a monomorphism.

(1) If h^{*1} is an epimorphism, then h'^0 is an epimorphism.

(2) If h^{*1} is an isomorphism, then h'^0 is an isomorphism.

Proof. For A'_X and A'_Y, we have $(L'_X)^s_r$ and $(L'_Y)^s_r$. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & (L_X)^s_{r+1} & \longrightarrow & (L'_X)^s_r & \longrightarrow & (M'_X)^{s-r}_r \\ & \downarrow h^s_{r+1} & & \downarrow h^s_r & & \downarrow m^{s-r}_r \\ 0 \longrightarrow & (L'_Y)^s_{r+1} & \longrightarrow & (L'_Y)^s_r & \longrightarrow & (M'_Y)^{s-r}_r \end{array}$$

where the rows are exact, and the vertical arrows are induced by h'. If h^s_r is an isomorphism (resp. epimorphism) and if m^{s-r}_r is a monomorphism, then by the five lemma, h^s_{r+1} is an isomorphism (resp. epimorphism). Since $A'^0(s+r-1, s) \subset A'^0(s+r, s)$ and if $u \in A'^1(s-1, s-r) - A'^1(s-1, s-r-1)$ then $u \notin A'^1(s, s-r-1)$ we can see that

$M_{r+1}^{\prime s}$ can be taken as a subgroup of $M'^{s}_{r}/d_r L'^{s+r}_{r}$. It also follows that if m_r^{s-r} is a monomorphism, then m_r^{s-r} induces a monomorphism

$$m'^{s-r}_{r} : (M_X')_r^{s-r}/d_r(L_X')_r^s \longrightarrow (M_Y')_r^{s-r}/d_r(L_Y')_r^s.$$

Therefore m_{r+1}^{s-r} can be taken as a restriction of m'^{s-r}_{r} and is a monorphism.

Starting from the fact that

$$h_1^s : (L_X')_1^s \cong H^1(\otimes^s A_X) \longrightarrow H^1(\otimes^s A_Y) \cong (L_Y')_1^s$$

is an isomorphism (resp. epimorphism) if h^{*1} is an isomorphism (resp. epimorphism) and the fact that

$$m_1^{s-1} : (M_X')_1^{s-1} \cong H^2(\otimes^{s-1} A_X) \longrightarrow H^2(\otimes^{s-1} A_Y) \cong (M_Y')_1^{s-1}$$

is a monomorphism since h^{*2} is a monomorphism, we now conclude that for $r \geq 1$, every h_r^s is an isomorphism (resp. epimorphism). In particular

$$h_s^s : H^0(A_X'(s))/H^0(A_X'(s-1)) \cong H^0(A_Y'(s))/H^0(A_Y'(s-1))$$

is an isomorphism (resp. epimorphism). Hence h induces an isomorphism (resp. epimorphism)

$$H^0(A_X'(s)) \cong H^0(A_Y'(s))$$

and

$$H^0(A_X') = \varinjlim H^0(A_X'(s)) \cong \varinjlim H^0(A_Y'(s)) = H^0(A_Y'). \quad ///$$

Let X be simply connected as a topological space. Let $\hat{C}_*(X)$ be the chain complex of those smooth singular simplices of X(i.e., $\sigma : \triangle^q \longrightarrow X$ is a plot of X) that map the 1-skeleton of the standard simplex to the base point x_0 of X. Then we see that $\hat{C}_1(X)=0$. Assume that the canonical map of $\hat{C}_*(X)$ into the normalized total singular chain complex of X is a chain equivalence. According to a theorem of Adams (p. 83, [1]), the cobar construction $F(\hat{C}_*)$ of chain complex $\hat{C}_*(X)$ yields the correct homology for the loop space ΩX.

Recall that $F(\hat{C}_*)$ has as a basis all elements of the type $[c_1|\cdots|c_r]$ $r\geq 0$, where each c_i is an element of $\hat{C}_s(X)$ for $s\geq 2$. The dimension of $[c_1|\cdots|c_r]$ is defined to be $\sum_{1\leq i\leq r}$ (dim c_i-1). Now we can define the multiplication and the differential d_F of $F(\hat{C}_*)$ as follows;

$$[c_1|\cdots|c_r][c_{r+1}|\cdots|_{r+s}]=[c_1|\cdots|c_{r+s}]$$
$$d_F[\sigma]=[\partial\sigma]-\sum_{1<i<n-1}(-1)^i[{}_i\sigma|\sigma_{n-i}]$$

where σ is an n-simplex in $\hat{C}_*(X)$, and ${}_i\sigma$ and σ_j are the first i-face and the last j-face of σ, respectively. $F(\hat{C}_*)$ becomes a differential graded algebra.

Let $F_r(\hat{C}_*)$ be the subcomplex of $F(\hat{C}_*)$ spanned by elements of the type $[c_1|\cdots|c_s]$, $s\geq r$. Now we get the following descending filtration

$$F(\hat{C}_*)=F_0(\hat{C}_*)\supset\cdots\supset F_r(\hat{C}_*)\supset\cdots\cdots.$$

Note that $(F_r(\hat{C}_*))_q=0$ for r sufficiently large and filtration is complete. Also, note that

$$d_F[c_1|\cdots|c_r]$$
$$=\Sigma(-1)^{\dim[c_1|\cdots|c_{i-1}]}[c_1|\cdots|c_{i-1}|\partial c_i|c_{i+1}|\cdots|c_r] \bmod F_{r+1}(\hat{C}_*).$$

Let $\hat{C}_{*-1}$ be the chain complex obtained from the reduced chain complex of $\hat{C}_*$ through lowering the degree by 1(*i.e.*, $(\hat{C}_{*-1})_q=\hat{C}_{q+1}$, $q\geq 0$).
There is a homomorpnism

$$\otimes^s\hat{C}_{*-1}\longrightarrow F_s(\hat{C}_*)/F_{s+1}(\hat{C}_*)$$

given by $c_1\otimes\cdots\otimes c_s\longmapsto[c_1|\cdots|c_s] \bmod F_{s+1}(\hat{C}_*)$. Then this chain complex homomorphism is an isomorphism. The homology spectral sequence $\{E^r\}$ associated to the filtered chain complex $F(\hat{C}_*)$ converges to $H(F(\hat{C}_*))\cong H_*(\Omega X)$, and

$$E^1_{s,t}\cong H_{s+t}(\otimes^s\hat{C}_{*-1})$$

If $H_*(X)$ is of finite type, so are E^1 and, consequently, $H_*(\Omega X)$.

The cochain complex $B=\mathrm{Hom}_Z(F(\hat{C}_*),k)$ has a descending filtration

$$k=B(0)\subset\cdots\subset B(s)\subset\cdots\cdots$$

where $B(s)$ is the subcomplex of B orthogonal to $F_{s+1}(\hat{C}_*)$. Since $(B(s))^q=B^q$ for s sufficiently large, the filtration is complete. The associated cohomology spectral sequence $\{E_r(B)\}$ converges to $H(B)=H^*(F(\hat{C}_*);\ k)$ and

$$E_1^{s,t}(B)=H^{s+t}(B(s)/B(s-1)).$$

Observe that

$$B(s)/B(s-1)\cong \mathrm{Hom}_Z(F_s(\hat{C}_*)/F_{s+1}(\hat{C}_*);k)\ \mathrm{Hom}_Z(\otimes^s\hat{C}_{*-1},k).$$

By assuming $H_*(X)$ to be of the finite type, we conclude that $\{E_r(B)\}$ converges to $H^*(\Omega X;k)$, and shat

$$E_1^{s,t}(B)\cong(\otimes^s H^{*-1}(X;k))^{s+t}. \quad\cdots\cdots(F)$$

Let a denote a partition of the unit interval

$$0=a_0<a_1<\cdots<a_n=1.$$

Let $H=\{h_1,\cdots,h_r\}$ be an ordered subeset of $\{1,\cdots,n-1\}$. Set $h_0=0$ and $h_{r+1}=n$. Let $\gamma_{n,H,a}:I\longrightarrow\triangle^n$ be the edge path $<v_{h_0},\cdots,v_{h_{r+1}}>$such that $\gamma_{n,H,a}(a_{h_i})=v_{h_i}$, $0\leq i\leq r+1$.

Define $\theta_{n,a}:I^{n-1}\longrightarrow P(\triangle^n;v_0,v_1)$ such that

$$\theta_{n,a}(\xi)(t)=\Sigma_H\Pi_{j\in H}\xi^j\Pi_{j\notin H}(1-\xi^j)\gamma_{n,H,a}(t)\ \text{(See, [6])}$$

summing over the 2^{n-1} ordered subsets H of $\{1,\cdots,n-1\}$, where ξ^i is the i-th coordinate of ξ.

For an n-simplex σ in $\hat{C}_*(X)$, define $\tilde{\sigma}$ to be the $(n-1)$-cube

$$I^{n-1}\xrightarrow{\theta_{n,a}}P(\triangle^n;v_0,v_1)\xrightarrow{p(\sigma)}\Omega_{X_0}X,$$

where $p(\sigma)$ is the induced map by σ. We shall consider the pairing

$$<,>:A'\times F(\hat{C}_*)\longrightarrow k$$

given by

$$<\int\omega_1\cdots\omega_r,[c_1|\cdots|c_r]>=\int_{\bar{c}_1\times\cdots\times\bar{c}_r}\int\omega_1\cdots\omega_r$$

From (B), we obtain

$$<\int\omega_1\cdots\omega_r,[c_1|\cdots|c_r]>=\int_{\bar{c}_1}\int\omega_1\cdots\int_{\bar{c}_r}\int\omega_r$$
$$=\int_{c_1}\omega_1\cdots\int_{c_r}\omega_r,$$

and, for $r<s$

$$<\int\omega_1\cdots\omega_r,\ [c_1|\cdots|c_r]>=0. \quad\cdots\cdots(G)$$

Let X be a path connected differentiable space and let A be a differential graded subalgebra of $A^*(X)$ such thet $dA^0 = A^1 \cap dA^0(X)$.

Theorem 3.7. If the following conditions hold;

(i) X is simply connected as a topological space, and its integral singular homology $H_*(X)$ is of finite type.

(ii) The canonical map from $\hat{C}_*(X)$ into the normalized singular simplicial chain complex of X is a chain equivalence.

(iii) $H(A) \cong H^*(X;k)$ via $\hat{C}_*(X)$,

then $H(A') \cong H(B)$.

Proof. The pairing $\langle , \rangle : A' \times F(\hat{C}_*) \longrightarrow k$ is that of a cochain complex and a chain complex. Hence the pairing gives rise to a cochain map

$$A' \longrightarrow B = \mathrm{Hom}_Z(F(\hat{C}_*), k)$$

which, by (G), preserves the filtration. Consequently there is an induced map of spectral sequences

$$E_r(A') \longrightarrow E_r(B)$$

Using Lemma 3.5 and (F), we verify that, on the E_1-level, the map can be composed as follows :

$$E_1^{s,t}(A') \cong H^{s,t}(\otimes^s \bar{A}) \cong (\otimes^s H^{*-1}(A))^{s+t} \cong (\otimes^s H^{*-1}(X;k)^{s+t} \cong E_1^{s,t}(B).$$

Hence $H(A') \cong H(B) \cong H^*(\Omega X;k)$. ///

4. Loop Space Homology

Let M be a differentiable space. Recall that $k[[X]]$ is the formal power series algebra in the noncommutative indeterminates $X_1, \cdots, X_m$. A *fomal power series connection* on a differentiable space M is an element of $A^*(M)[[X]]$ of the type

$$\omega = \sum \omega_i X_i + \sum \omega_{ij} X_i X_j + \cdots + \sum \omega_{i_1 \ldots i_r} X_{i_1} \ldots X_{i_r} + \cdots$$

where the coefficients are forms of positive degree on M. The *curvature* of ω is the element $\mathscr{K} = d\omega - J\omega \Lambda \omega$ of $A^*(M)[[X]]$.

Definition 4.1. The *transport* of a formal power series connection ω is the element

$$T=1+\int\omega+\int\omega^2+\cdots+\int\omega^r+\cdots\cdots$$

of $\Lambda(P(M))$ $[[X]]$.

Write the transport T in the form of

$$T=1+\sum T_iX_i+\sum T_{ij}X_iX_j+\cdots\cdots$$

verify that

$$T_i=\int\omega_i,\quad T_{ij}=\int(\omega_{ij}+\omega_i\omega_j),$$

$$T_{ijk}=\int(\omega_{ijk}+\omega_i\omega_{jk}+\omega_{ij}\omega_k+\omega_i\omega_j\omega_k),\cdots\cdots \text{ etc}$$

In general,

$$T_{i_1,\cdots i_r}=\sum\int\omega_{i_{\lambda_1},\cdots i_{\lambda_2-1}}\omega_{i_{\lambda_2},\cdots i_{\lambda_3-1}}\omega_{i_{\lambda_l},\cdots i_r}$$

summing over all partitions

$$1=\lambda_1<\lambda_2<\cdots<\lambda_l\leq r,\quad 1\leq l\leq r,$$

of the ordered set $\{1,\cdots,r\}$.

If $\alpha : U\longrightarrow P(M)$ is a plot, define

$$T_\alpha=1+\sum(T_i)_\alpha X_i+\sum(T_{ij})_\alpha X_iX_j+\cdots\cdots$$

If α is a compact plot with dim $U=n$, define

$$T(\alpha)=<T,\alpha>=\delta_{on}+\sum<T_i,\alpha>X_i+\sum<T_{ij},\alpha>X_iX_j+\cdots\cdots$$

where $<T_i,\alpha>=\int_\alpha T_i=\int_\alpha\int\omega_i$.

It follows from B and its extension to $\Lambda^*(P(M))$ $[[X]]$ that if $\alpha' : U'\longrightarrow P(M)$ is another compact plot such that the product plot $\alpha\times\alpha'$ is well-defined, then

$$T(\alpha\times\alpha')=T(\alpha)\ T(\alpha').$$

Also, we can see that if α,β are plots of $P(M)$ such that $\alpha\beta$ is defined, then

$$T_{\alpha\beta}=T_\alpha\Lambda T_\beta \text{ and } T_\alpha\Lambda T_\alpha^{-1}=1.$$

Lemma 4.2. Let $\omega=fdx^1\Lambda\cdots\Lambda dx^p$ where $f,x^1,\cdots,x^p\in\Lambda^0(M)$.

Let $\alpha : U \longrightarrow P(M;x_0,x_1)$ be a plot. Then for any $\Lambda^*(U)$-valued functions g and h of t,

$$\int_0^1 g\Lambda d_\xi\omega(\alpha,\dot\alpha)\Lambda\ hdt = -\int_0^1 g\Lambda(d\omega)(\alpha,\dot\alpha)\Lambda\ hdt - \int_0^1(\dot g\Lambda\omega''\Lambda h + g\Lambda\omega''\Lambda\dot h)dt,$$

where $\omega_{\phi_\alpha} = dt\Lambda\omega\ (\alpha,\dot\alpha)+\omega''$, ξ is a coordinate of U, $d_\xi f = \sum\frac{\partial f}{\partial\xi^i}d\xi^i$ and $\dot g = \frac{\partial g(\xi,t)}{\partial_t}$

Proof. For simplicity, we write $f, x^1,\cdots,x^p$ instead of $f\circ\phi_\alpha,\ x^1\circ\phi_\alpha,\cdots,x^p\circ\phi_\alpha$. Put

$$u = d_\xi x^1\Lambda\cdots\Lambda d_\xi x^p,$$
$$u_\lambda = d_\xi x^1\Lambda\cdots\Lambda\ \widehat{d_\xi x^\lambda}\Lambda\cdots\Lambda\ d_\xi x^p$$

where $\widehat{d_\xi x^\lambda}$ is the notation of ommition of $d_\xi x^\lambda$. Then since $\phi_\alpha^*\ (dx^i) = d_\xi x^i + \dot x^i dt$ and $\omega_{\phi_\alpha} = f\,(d_\xi x^1 + \dot x^1 dt)\Lambda\cdots\Lambda(d_\xi x^p + \dot x^p dt)$ so that

$$\omega(\alpha,\dot\alpha) = -\sum(-1)^\lambda\dot x^\lambda f u_\lambda,$$
$$\omega'' = fu.$$

we have

$$(d\omega)_{\phi_\alpha} = (d_\xi f + \dot f dt)\Lambda\ (d_\xi x^1 + \dot x^1 dt)\Lambda\cdots\Lambda(d_\xi x^p + \dot x^p dt)$$
$$= dt\Lambda\ (\dot f u + \sum(-1)^\lambda\ \dot x^\lambda d_\xi(fu_\lambda)) + d_\xi f\Lambda u$$

and hence $\quad (d\omega)(\alpha,\dot\alpha) = \dot f u + \sum(-1)^\lambda\ \dot x^\lambda d_\xi(fu_\lambda).$

Now

$$\sum(-1)^\lambda d_\xi\dot x^\lambda\Lambda u_\lambda = -\sum d_\xi x^1\Lambda\cdots\Lambda d_\xi\dot x^\lambda\Lambda\cdots\Lambda d_\xi x^p = -\dot u$$

and

$$d_\xi\omega(\alpha,\dot\alpha) = f\dot u - \sum(-1)^\lambda\dot x^\lambda d\xi(fu_\lambda)$$
$$= f\dot u + \dot f u - (d\omega)(\alpha,\dot\alpha) = \dot\omega'' - (d\omega)(\alpha,\dot\alpha).$$

An integration by parts gives

$$\int_0^1 g\Lambda\dot\omega''\Lambda hdt = [g\Lambda\omega''\Lambda h]_0^1 - \int_0^1(\dot g\Lambda\omega''\Lambda h + g\Lambda\omega''\Lambda\dot h)dt$$
$$= -\int_0^1(\dot g\Lambda\omega''\Lambda h + g\Lambda\omega''\Lambda\dot h)dt \quad \text{(owing to } d_\xi x^\lambda = 0 \text{ when } t = 0, 1\text{)}$$

Hence

$$\int_0^1 g\Lambda d_t\omega(\alpha,\dot{\alpha})\Lambda\, h dt = -\int_0^1 g\Lambda(d\omega)(\alpha,\dot{\alpha})\Lambda\, h dt$$
$$-\int_0^1 (\dot{g}\Lambda\omega''\Lambda\, h + g\Lambda\omega''\Lambda\dot{h})dt. \quad ///$$

Put

$$\omega(\alpha,\dot{\alpha}) = \sum\omega_i(\alpha,\dot{\alpha})X_i + \sum\omega_{ij}(\alpha,\dot{\alpha})X_iX_j + \cdots\cdots \text{ for any plot } \alpha.$$

For $0 \leq t \leq 1$

$$(\int\omega_i\omega_j)_{\alpha^t} = \int_0^t \omega_i(\alpha,\dot{\alpha})dt \;\; \omega_j(\alpha,\dot{\alpha})dt \cdots\cdots\cdots\cdots \text{ etc.}$$

and

$$T_{\alpha^t} = 1 + \sum(\int\omega_i)_{\alpha^t}X_i + \sum(\int(\omega_{ij}+\omega_i\omega_j))_{\alpha^t} X_iX_j + \cdots\cdots$$

Since $\frac{d}{dt}\int_0^t \omega_i(\alpha,\dot{\alpha})dt\omega_j(\alpha,\dot{\alpha})dt = \left(\int_0^t \omega_i(\alpha,\dot{\alpha})dt\right)\Lambda\, \omega_j(\alpha,\dot{\alpha})$, we get

$$\begin{aligned} dT_{\alpha^t}/dt &= \sum\omega_i(\alpha,\dot{\alpha})X_i + \sum[\omega_{ij}(\alpha,\dot{\alpha}) + (\int\omega_i)_{\alpha^t}\Lambda\,\omega_j(\alpha,\dot{\alpha})]X_iX_j \\ &+ \sum[\omega_{ijk}(\alpha,\dot{\alpha}) + (\int\omega_i)_{\alpha^t}\Lambda\,\omega_{jk}(\alpha,\dot{\alpha}) + (\int(\omega_{ij}+\omega_i\omega_j)_{\alpha^t} \\ &\Lambda\omega_k(\alpha,\dot{\alpha})]X_iX_jX_k + \cdots\cdots \\ &= T_{\alpha^t}\Lambda\,\omega(\alpha,\dot{\alpha}). \cdots\cdots\cdots\cdots\cdots\cdots \text{(H)} \end{aligned}$$

Also since $T\Lambda T^{-1}=1, d(T_{\alpha^t})^{-1}/dt = -\omega(\alpha,\dot{\alpha})\Lambda\,(T_{\alpha^t})^{-1}$.

By making use of Proposition 3.2 and its extension to $\Lambda^*(P(M))$ $[[X]]$, one may verify that

$$\begin{aligned} dT &= -\int\mathscr{X} + (-\int\mathscr{X}\omega + \int J\omega\mathscr{X}) + \cdots\cdots \\ &+ \sum_{i+j=r-1}\int(J\omega)^i\mathscr{X}\omega^j + \cdots\cdots - p_0^*\,\omega\Lambda T + JT\Lambda p_1^*\,\omega, \cdots \text{(I)} \end{aligned}$$

where $\omega^i = \omega\Lambda\omega\Lambda\cdots\Lambda\omega$ (i-times). Recall that every element of $k[[X]]$ can be written as

$$a = a_0 + \sum a_iX_i + \cdots\cdots + \sum a_{i_1,\ldots,i_r}X_{i_1}\cdots X_{i_r} + \cdots\cdots\cdots\cdots \text{(J)}$$

Let $\mathscr{I}$ be the augmentation ideal of $k[[X]]$, which consists of all elements a such that $a_0=0$. the s-th power $\mathscr{I}^s$ of $\mathscr{I}$ consists of all a such that $a_{i_1,\ldots,i_r}=0$ for $r<s$. A derivation ∂ of $k[[X]]$ is a linear endomorphism of $k[[X]]$ satisfying of the following conditions ([6]):

(a) If u and v are homogeneous elements of $k[[X]]$, then

$$\partial(uv)=(\partial u)v+(-1)^{\deg u}u(\partial v).$$

(b) For $1\leq i\leq m$, $\partial X_i\in\mathcal{J}^2$.

(c) For any $a\in k[[X]]$ as given in (J),

$$\partial a=\sum a_i\partial X_i+\cdots\cdots+\sum a_{i_1\cdots i_r}\partial(X_{i_1}\cdots X_{i_r})+\cdots\cdots.$$

Every derivation of $k[[X]]$ can be extended to a derivation of $\Lambda^*(M)$ $[[X]]$ in the obvious sense.

Theorem 4.3. Let $\mathscr{K}$ be a curvature of a formal power series connection ω on M. If $\partial\omega+\mathscr{K}=0$, then

$$dT=\partial T-p_0^*\omega\Lambda T+JT\Lambda p_1^*\omega.$$

Proof. Since

$$\partial T=\sum\left(\int\omega_i\right)\partial X_i+\sum\left(\int(\omega_{ij}+\omega_i\omega_j)X_iX_j+\cdots\cdots\right.$$

$$\partial\omega=\sum\omega_i\partial X_i+\sum\omega_{ij}\partial(X_iX_j)+\cdots\cdots$$

$$(\partial\omega)\omega=\sum\omega_i\omega_j(\partial X_i)X_j+\sum\omega_k\omega_{ij}X_k\,\partial(X_iX_j)+\sum\omega_i\omega_{jk}(\partial X_i)X_jX_k\cdots\text{etc.},$$

(I) becomes $dT=\partial T-p_0^*\omega\Lambda T+JT\Lambda p_1^*\omega$. ///

From now on, we assume that M is a differentiable space with $H_0(M)=Z$ and $H_*(M;k)$ of finite type. Let $1,\hat{Z}_1,\cdots,\hat{Z}_m$ be a basis of $H_*(M;k)$ with $\hat{Z}_i\in H_{p_i}(M;k)$. For simplicity assume that m is finite. The corresponding indeterminates $X_1,\cdots,X_m$ are such that deg $X_i=p_i-1$.

A *formal homology connection* on M is a pair (ω,∂) consisting of a formal power series connection ω and a derivation ∂ of $k[[X]]$ satisfying the condition $\partial\omega+\mathscr{K}=0$ such that

(a) $\omega_1,\cdots,\omega_m$ are closed forms dual to $\hat{z}_1,\cdots,\hat{z}_m$;

(b) deg $\omega_{i_1\cdots i_r}=p_1+\cdots+p_r-r+1$, $r\geq1$.

sincce $T(\alpha\times\alpha')=T(\alpha)T(\alpha')$ for any compact plot such that $\alpha\times\alpha'$ is well-defined, it gives rise to a multiplication preserving map

$$\theta:C_*(\Omega_{x_0}(M))\longrightarrow k[[X]]$$

given by $c\longmapsto\langle T,c\rangle=T(c)$, where $C_*(\Omega_{x_0}(M))$ is the normalized cubical chain complex of M. If (ω, ∂) is a formal homology connection on M, then $\partial\partial=0$ follows the fact that the map θ has a dense image in $k[[X]]$ and the fact that $\partial\partial<T,c>=<T,\partial\partial c>=0$, $c\in C_*(\Omega_{x_0}(M))$.

Let (ω, ∂) be a formal homology connection on M. Make $k[[X]]$ graded by assigning $\deg X_i = p_i - 1$, $1\leq i\leq m$. We have

$$\deg \omega_{i_1\cdots i_r} = 1 + \deg X_{i_1}\cdots X_{i_r}$$

where $\omega_{i_1\cdots i_r}$ is the coefficient of $X_{i_1}\cdots X_{i_r}$ in formal power series connection ω. The condition $\partial\omega + \mathscr{K} = 0$ forces the derivation ∂ to be a graded map of degree 1.

Equip $k[[X]]$ with an descending filtration by powers of the augmentation ideal $\mathscr{T}$. Then $(k[[X]], \partial)$ is a filtered chain complex, whose spectral sequence $\{C^r, \delta^r\}_{r\geq 0}$ is such that $C^0_s = \mathscr{T}^s/\mathscr{T}^{s+1}$, $s\geq 1$, $C^0_0 = k[[X]]/\mathscr{T} = k$, and $C^0_s = 0$ for $s<0$. since $\partial X_i \in \mathscr{T}^2$(note that $\partial X_i = \sum c^i_{jk} X_j X_k + \cdots$ where c^i_{jk} are determiend by cup products $[J\omega_j \Lambda \omega_k] = \sum c^i_{jk}[\omega_i]$ and $[\omega_i]$ is the cohomology class of the closed form ω_i.), we have

$$C^1_s = \mathscr{T}^s/\mathscr{T}^{s+1} \cong \otimes^s \mathscr{T}/\mathscr{T}^2 \cong \otimes^s H_{*-1}(M;k), \quad s\geq 1.$$

A formal power series connection will rewritten as

$$\omega = \omega_1 + \omega_2 + \cdots$$

where $\omega_1 = \Sigma \omega_i X_i$, $\omega_2 = \Sigma \omega_{ij} X_i X_j \cdots$ etc. The curvature is

$$\mathscr{K} = \mathscr{K}_1 + \cdots + \mathscr{K}_r + \cdots$$

where

$$\mathscr{K}_r = d\omega_r - \sum_{1\leq i\leq r} J\omega_i \Lambda \omega_{r-i}.$$

For a formal power series connection ω, if each X_i is of degree >0, then $T(\alpha)$ is a finite sum. Hence the free algebra $k[X]$, as a subalgebra of $k[[X]]$, is stable under ∂ and is therefore also a filtered chain complex. Therefore if $\deg X_i > 0$,

$\theta : C_*(\Omega_{x_0}(M)) \longrightarrow k[[X]]$ may be replaced by
$C_*(\Omega_{x_0}(M)) \longrightarrow k[X]$.

For any ideal ϑ(two sided) of $k[[X]]$, denote by ϑ_r the subspace of ϑ consisting of those elements which are homogeneous of degree r. We say that ϑ is a *homogeneous ideal* if ϑ is the totality of elements of type $u_0 + \cdots + u_r + \cdots$, $u_r \in \vartheta_r$. A formal power series connection ω is said to be *locally flat modulo a homogeneous ideal* ϑ of $k[[X]]$, if for the curvature $\mathscr{K}$, each $\mathscr{K}_r \in \Lambda^*(M) \otimes \vartheta_r$.

If $\eta = \eta_1 + \cdots + \eta_r + \cdots$ and $\zeta = \zeta_1 + \cdots + \zeta_r + \cdots$ are two formal power series in $X_1, \cdots, X_m$ with coefficients in $\Lambda^*(M)$ such that $\zeta_r - \eta_r \in \Lambda^*(M) \otimes \vartheta_r$, we shall write $\eta \equiv \zeta \bmod \vartheta$.

Theorem 4.4. Let M be a differentiable space such that the exterior algebra $\Lambda^*(M)$ is generated by $\Lambda^0(M)$ and $d\Lambda^0(M)$ and let a formal power series connection ω is locally flat modulo a homogeneous ideal ϑ of $k[[X]]$, then there is a ring homomorphism

$$\begin{array}{ccc} \theta : H_*(\Omega_{x_0}(M)) & \longrightarrow & k[[X]]/\vartheta \\ \Cup & & \Cup \\ [z] & \longmapsto & T(z)+\vartheta. \end{array}$$

In particular, if ω is such that each of $\omega_i, \omega_{ij}, \cdots$ is of degree >1, then there is a ring homomorphism

$$\theta : H_*(\Omega_{x_0}(M)) \longrightarrow k[X]/\vartheta'_\omega$$

where ϑ_ω is the intersection of all homogeneous ideals of $k[[X]]$ such that ω is locally flat modulo ϑ and $\vartheta'_\omega = \vartheta_\omega \cap k[X]$.

Proof. Write $d_\xi T_{\alpha^t} = u \Lambda T_{\alpha^t}$. Then since $d_\xi(dT_{\alpha^t}/dt) = \dfrac{d}{dt}(d_\xi T_{\alpha^t})$,

$$d_\xi(dT_{\alpha^t}/dt) = \dot{u} \Lambda T_{\alpha^t} + u \Lambda T_{\alpha^t} \Lambda\, \omega(\alpha, \dot{\alpha}) \quad (\text{by } H)$$

On the other hand

$$\begin{aligned} d_\xi(dT_{\alpha^t}/dt) &= d_\xi(T_{\alpha^t} \Lambda \omega(\alpha, \dot{\alpha})) \\ &= u \Lambda T_{\alpha^t} \Lambda \omega(\alpha, \dot{\alpha}) + J T_{\alpha^t} \Lambda d_\xi \omega(\alpha, \dot{\alpha}) \end{aligned}$$

so that

$$\dot{u} = J T_{\alpha^t} \Lambda d_\xi \omega(\alpha, \dot{\alpha}) \Lambda\, T_{\alpha^t}^{-1}.$$

When $t=0$, $T_{\alpha^t}=1$ and $d_\xi T_{\alpha^t}=0$. This makes $u(0)=0$ and

$$U(1) = \int_0^1 J T_{\alpha^t} \Lambda\; d_\xi \omega(\alpha, \dot{\alpha}) \Lambda T_{\alpha^t}^{-1} dt$$

which, according to Lemma 4.2, is equal to

$$\begin{aligned} &-\int_0^1 J T_{\alpha^t} \Lambda (d\omega)(\alpha, \dot{\alpha}) \Lambda T_{\alpha^t}^{-1} dt \\ &-\int_0^1 J T_{\alpha^t} \Lambda [J(\omega(\alpha, \dot{\alpha})) \Lambda \omega'' - \omega'' \Lambda \omega(\alpha, \dot{\alpha})] \Lambda T_{\alpha^t}^{-1} dt \end{aligned}$$

where ω'' is given by

$$\omega_{\phi_\alpha}=dt\Lambda\omega(\alpha,\dot{\alpha})+\omega''$$

Observe that $J(\omega(\alpha,\dot{\alpha}))=-(J\omega)(\alpha,\dot{\alpha})$ and

$$(J\omega\Lambda\omega)(\alpha,\dot{\alpha})=(J\omega)(\alpha,\dot{\alpha})\Lambda\ \omega''+\omega''\Lambda\omega(\alpha,\dot{\alpha}).$$

Hence $(dT\times T^{-1})_\alpha=-\int_0^1 JT_{\alpha t}\Lambda\mathscr{X}(\alpha,\dot{\alpha})\Lambda T_{\alpha t}^{-1}dt.$

Since $\mathscr{X}\equiv0 \bmod \vartheta$, we have $\mathscr{X}(\alpha,\dot{\alpha})\equiv0 \bmod \vartheta$ so that $(dT)_\alpha\equiv0 \bmod \vartheta$. Hence, for every smooth singular simplex σ of $\Omega_{x_0}(M)$,

$$\theta(\partial\sigma)\equiv0 \bmod \vartheta.$$

Therefore there is a homomorphism

$$\theta : H_*(\Omega_{x_0}(M))\longrightarrow k[[X]]/\vartheta$$

If ω_i,ω_{ij} is of degree >1, then this homomorphism may be replaced by

$$H_*(\Omega_{x_0}(M))\longrightarrow k[X]/\vartheta'\omega. \quad ///$$

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