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Some Remarks on Covariant Derivatives and Geodesics in Riemannian Surfaces

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1. Introduction

One can find the latest Tnformations about Riemannian gemetry in the paper [19]. The important problems arising in Riemannian geometry usually resolve into the investigation on the properties of curvatures and geodesics ([5], [14], [18]).

In this paper, we present the results obtained through the seminar on covariant derivatives and geodesics in 2-dimensional Riemannian manifold. We briefly outline the scheme of this paper as follows.

In Section 2, we define terminologies which will be used in Section 3 and 4. Also we prove four propositions to derive some properties concerned with the terminologies (Proposition 2.3, 2.6~2.8).

In Section 3, we prove two theorems for covariant derivatives ; that is, we present the relation between the Euclidean covariant derivative and the covariant derivative for two vectors on a surface (Theorem 3.2) and prove

$$\boldsymbol{X}_{uv}=\boldsymbol{X}_{vu}$$

for orthogonal patch $\boldsymbol{X}$ (Theorem 3.3).

In Section 4, we prove two theorems on geodesics ; that is, in Theorem 4.6, we show the condition for a unit speed curve to be geodesic, and in Theorem 4.7 we present the relation between a curve to be a unit speed geodesic and a curve to be a barrier curve.

2. Preliminaries

Let M be a n-dimensional (real) C^∞ manifold with coordinate neighborhoods $U=\{(U_\alpha,\varphi_\alpha)\}$. Then there exists an open subset V_α of $\boldsymbol{R}^n$ such that $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$ is a homeomorphism and we have a commutative diagram

$$\begin{array}{ccc} T(U_\alpha) & \xrightarrow{\varphi_{\alpha*}} & T(V_\alpha) \\ \downarrow & & \downarrow \\ U_\alpha & \xrightarrow{\varphi_\alpha} & V_\alpha \end{array} ,$$

where $T(U_\alpha)$ is the tangent space over U_α and $\varphi_{\alpha*}$ is the natural map induced by φ_α. For each $q \in \varphi_\alpha(U_\alpha) = V_\alpha$, $T_q(V_\alpha)$ has the natural basis

$$\frac{\partial}{\partial x_1}, \cdots\cdots\cdots\cdots, \frac{\partial}{\partial x^n},$$

and if we put

$$E_j = \varphi_{\alpha*}^{-1}\left(\frac{\partial}{\partial x^j}\right)$$

then $E_1, \cdots\cdots, E_n$ are C^∞ vector fields over U. That is, if $\varphi_\alpha(p) = q$, then

$$E_1(p) = E_{1p}, \cdots\cdots, E_n(p) = E_{np}$$

is a basis of $T_p(U)$ which is called the coordinate frame.

Let $\Phi : T(M) \times T(M) \longrightarrow \boldsymbol{R}$ be a bilinear form, i.e., for each $p \in M$ $\Phi | T_p(M) = \Phi_p : T_p(M) \times T_p(M) \longrightarrow \boldsymbol{R}$ is a bilinear map. For each coordinate neighborhood (U, φ) of M a C^∞ function $X : U \longrightarrow T(U)$ is called a vector field over U. We put $X(p) = X_p$ for each $p \in U$.

Now we consider two vector fields X and Y over U and put $\Phi(X_p, Y_p) = \Phi_p(X_p, Y_p)$. Then we have a function

$$\begin{array}{l} \Phi(X,Y) : U \longrightarrow \boldsymbol{R} \\ \qquad\qquad\quad \Cup \qquad \Cup \\ \qquad\qquad\quad p \longmapsto \Phi(X,Y)(p) = \Phi_p(X_p, Y_p). \end{array}$$

If for all pair of vector fields and all coordinate neighborhoods Φ is of class C^∞, then we say that Φ is a C^∞-function. If a bilinear function $\Phi : T(M) \times T(M) \longrightarrow \boldsymbol{R}$ is symmetric, positive definite and of C^∞-class then Φ is called a Riemannian metric or an inner product of M ([3]). An n-dimensional manifold with Riemannian metric is called an n-dimensional Riemannian manifold.

Definition 2.1. By a surface we mean a 2-dimensional Riemannian manifold. In this case we shall put Riemannian metric $= \cdot$. A frame field on a surface M consists of two orthogonal unit fields E_1, E_2 defined on some coordinate neighborhood of M. That is, $E_i \cdot E_j = \delta_{ij} (1 \leq i, j \leq 2)$, where δ_{ij} is the Kronecker delta.

Let E_1 and E_2 be a frame field on a surface M. We shall define the dual 1-forms θ_1 and θ_2 by

$$\theta_i(E_j) = \delta_{ij} \ (1 \leq i, j \leq 2).$$

Then we have the first structural equations

$$d\theta_1 = w_{12} \wedge \theta_2, \ d\theta_2 = w_{21} \wedge \theta_1,$$

where w_{ij} $(1 \leq i, j \leq 2)$ is the connection form. Moreover we have second structural equation

$$dw_{12} = -K \ \theta_1 \wedge \theta_2,$$

where K is the Gaussian curvature ([16]).

Definition 2.2. Let M be a surface. A covariant derivative ∇ on M assigns to each pair of vector fields V and W on M a new vector field $\nabla_V W$ satisfying the following properties : For vector fields V, W, Y, Z, differentiable functions $f, g : M \longrightarrow \boldsymbol{R}$ and the connection form $w_{12} = -w_{21}$ of a frame field E_1, E_2

(i) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ $(a, b$: constants).

(ii) $\nabla_{fv+hw}(Y) = f\nabla_V Y + g\nabla_W Y$.

(iii) $\nabla_V(fY) = V[f]Y + f\nabla_V Y$, where for $p \in M$ and a coordinate neighborhood (U, φ) of p

$$V_p[f] = \frac{d}{dt}(f(\varphi^{-1}(\varphi(p) + tV(p))))|_{t=0}.$$

(iv) $V[Y \cdot Z] = (\nabla_V Y) \cdot Z + Y \cdot \nabla_V Z$.

(v) $w_{12}(V) = \nabla_V E_1 \cdot E_2$.

Proposition 2.3. Let ∇ be a covariant derivative on a surface M, and let E_1, E_2 be a frame field of M. Then

$$\nabla_V E_1 = w_{12}(V) E_2,$$
$$\nabla_V E_2 = w_{21}(V) E_1.$$

Moreover, if $W = f_1 E_1 + f_2 E_2$ where $f_1, f_2 : M \longrightarrow \boldsymbol{R}$ are differentiable, then we have the covariant derivative formula

$$\nabla_V W = \{V[f_1] + f_2 w_{21}(V)\} E_1 + \{V[f_2] + f_1 w_{12}(V)\} E_2.$$

Proof. Since $E_1 \cdot E_2 = 0$ by (iv) of definition 2.2

$$0 = V[E_1 \cdot E_2] = (\nabla_V E_1) \cdot E_2 + E_1 \cdot \nabla_V E_2$$

and thus by (V) of definition 2.2.

$$\nabla_V E_2 \cdot E_1 = -\nabla_V E_1 \cdot E_2 = -w_{12}(V) = w_{21}(V). \quad \cdots\cdots\cdots\cdots (2-1)$$

Since $w_{ii}(V) = 0$ for $i = 1.2$ by (V) of definition 2.2

$$\nabla_V E_i \cdot E_i = 0 \quad (i = 1, 2). \quad \cdots\cdots\cdots\cdots (2-2)$$

As $\nabla_V E_i (1 \leq i \leq 2)$ is a vector field, we can write

$$\nabla_V E_i = a_{i1} E_1 + a_{i2} E_2.$$

That is,

$$\nabla_V E_1 = a_{11} E_1 + a_{12} E_2, \quad \triangle_V E_2 = a_{21} E_1 + a_{22} E_2.$$

By (2-1) and (2-2) above we get

$$a_{11} = 0, \quad a_{12} = w_{12}(V), \quad a_{21} = w_{21}(V), \quad a_{22} = 0.$$

Thus,

$$\triangle_V E_1 = w_{12}(V) E_2, \quad \nabla_V E_2 = w_{21}(V) E_1.$$

Next,

$$\nabla_V W = \nabla_V (f_1 E_1 + f_2 E_2) = \nabla_V (f_1 E_1) + \nabla_V (f_2 E_2)$$
$$= V[f_1] E_1 + f_1 \nabla_V E_1 + V[f_2] E_2 + f_2 \nabla_V E_2$$

by (iii) of definition 2.2. Since

$$\nabla_V E_1 = w_{12}(V) E_2 \text{ and } \nabla_V E_2 = w_{21}(V) E_1$$

as before, we have

$$\nabla_V W = \{V[f_1] + f_2 w_{21}(V)\} E_1 + \{V[f_2] + f_1 w_{12}(V)\} E_2.$$

Let M be a surface with coordinate neighborhoods (U_φ, φ). Then there exists an open subset V_φ in E^2, 2-dimensional Euclidean space such that

$$\varphi : U_\varphi \xrightarrow{\cong} V_\varphi$$

is a homeomorphism. Therefore

$$\varphi^{-1} : V_\varphi \longrightarrow M$$

is also a one-to-one regular mapping into M for some coordinate neighborhood (U_ψ, ψ) of M ($\psi \circ \varphi^{-1}$ is regular : A mapping $f : \boldsymbol{E}^2 \longrightarrow \boldsymbol{E}^2$ is regular provided that for each point $p \in \boldsymbol{E}^2$ the derivative map f_{*p} is one-to-one).

Thus, for each surface we can define a coordinate patch as follows :

Definition 2.4. For each surface M a coordinate patch

$$\boldsymbol{X} : D \longrightarrow M$$

is a one-to-one regular mapping of an open subset D of $\boldsymbol{E}^2$ into M.

Let us suppose a coordinate patch

$$\begin{array}{ccc} \boldsymbol{X} : D & \longrightarrow & M \\ \Uparrow & & \Uparrow \\ (u, v) & \longmapsto & \boldsymbol{X}(u, v) \end{array}$$

where D is open in $\boldsymbol{E}^2$ and M is a surface. We put

$$\boldsymbol{X}_u = \frac{\partial \boldsymbol{X}}{\partial u} \text{ and } \boldsymbol{X}_v = \frac{\partial \boldsymbol{X}}{\partial v}.$$

Then, for each $P = \boldsymbol{X}(u_0, v_0) \in \boldsymbol{X}(D)$ $\boldsymbol{X}_u(u_0, v_0)$, $\boldsymbol{X}_v(u_0, v_0) \in T_p(M)$ and thus we can define $\boldsymbol{X}_u \boldsymbol{X}_u = E$, $\boldsymbol{X}_v \boldsymbol{X}_v = G$ and $\boldsymbol{X}_u \boldsymbol{X}_v = F$.

If $\boldsymbol{X}_u \boldsymbol{X}_v = F = 0$, $\boldsymbol{X}$ is called an orthogonal coordinate patch.

Definition 2.5. The associated frame field E_1, E_2 of an orthogonal coordinate patch $\boldsymbol{X} : D \longrightarrow M$ consists of the orthogonal unit vector fields E_1 and E_2 such that

$$E_1 = \boldsymbol{X}_u(u, v) / \sqrt{E(u, v)}, \quad E_2 = \boldsymbol{X}_v(u, v) / \sqrt{G(u, v)}.$$

In the above situation the dual forms θ_1 and θ_2 are characterized by $\theta_i(E_j) = \delta_{ij}$ $(1 \leq i, j \leq 2)$. Moreover, we have the following :

$$\theta_1 = \sqrt{E}\, du, \quad \theta_2 = \sqrt{G}\, dv \cdots\cdots\cdots\cdots (2\text{—}3)$$

Proposition 2.6. In the above situation, we have

$$d\theta_1=(-(\sqrt{E})_v/\sqrt{G})\ du\wedge\theta_2$$

$$d\theta_2=(-(\sqrt{G})_u/\sqrt{E})\ dv\wedge\theta_1$$

and

$$w_{12}=\frac{-(\sqrt{E})_v}{\sqrt{G}}\ du+\frac{(\sqrt{G})_u}{\sqrt{E}}\ dv$$

Proof. From (2—3) above

$$d\theta_1=d\sqrt{E})\wedge du+\sqrt{E}\wedge d^2u=(\sqrt{E})_u\ dv\wedge du$$
$$(=\frac{\partial}{\partial v}(X_u\cdot X_u)^{\frac{1}{2}}\ dvdu)$$
$$=(\sqrt{E})_v\cdot\frac{\theta_2}{\sqrt{G}}\wedge du=-\frac{(\sqrt{E})_v}{\sqrt{G}}\ du\wedge\theta_2$$

and similarly,

$$d\theta_2=(\sqrt{G})_u\ du\wedge dv=\frac{(\sqrt{G})_u}{\sqrt{E}}\theta_1\wedge dv$$

$$=\frac{-(\sqrt{G})_u}{\sqrt{E}}\ dv\wedge\theta_1.$$

By the first structural equations,

$$d\theta_1=w_{12}\wedge\theta_2,\ \ d\theta_2=w_{21}<\theta_1.$$

We note that there exists only one w_{12} satisfying the above equations.

Now

$$d\theta_1+d\theta_2=-\frac{(\sqrt{E})_v}{\sqrt{G}}\ \mathrm{du}\wedge\theta_2-\frac{(\sqrt{G})_u}{\sqrt{E}}\ dv\wedge\theta_1$$
$$=w_{12}\wedge(\theta_2-\theta_1). \quad\cdots\cdots(2—4)$$

On the other hand, since

$$\theta_1=\sqrt{E}\ du,\ \ \theta_2=\sqrt{G}\ dv,$$

we have a solution of (2—4) such that

$$w_{12}=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

By the uniqueness of w_{12} our proof is completed.

Proposition 2.7. In the above situation

$$dw_{12}=\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u+\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right\}\theta_1\wedge\theta_2$$

and

$$K=-\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u+\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right\},$$

where K is the Gaussian curvature ([11], [17], [20]).

Proof. By Proposition 2.6.

$$dw_{12}=\left(-\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v dv\wedge du+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u du\wedge dv.$$

By (2—3)

$$\theta_1\wedge\theta_2=\sqrt{EG}\ du\wedge dv.$$

Hence

$$dw_{12}=\frac{1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}\theta_1\wedge\theta_2.$$

Moreover, for the Gaussian curvature K, since we have

$$dw_{12}=-K\theta_1\wedge\theta_2,$$

we get

$$K=\frac{-1}{\sqrt{EG}}\left\{\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right\}.$$

Let M be a surface with coordinate neighborhoods (U,φ). A curve is defined by a differentiable function

$$\alpha:I\longrightarrow M$$

where I is an open subset of $\boldsymbol{E}^1$, 1-dimensional Euclidean space.

Therefore, if $\alpha(\mathrm{I})\cap U\neq\phi$, then for $\alpha^{-1}(\alpha(\mathrm{I})\cap U)=I'$, $\varphi\circ\alpha|I':I'\longrightarrow V\subset\boldsymbol{E}^2$ is differentiable. If each $\varphi\circ\alpha|I'$ is regular, then the curve α is said to be regular.

Let $\boldsymbol{X}:D\longrightarrow M$ be a coordinate patch and assume that $\alpha(\mathrm{I})\subset\boldsymbol{X}(D)$, where $\alpha:I$

$\longrightarrow M$ is a curve in M. Then there exists a unique differentiable functions α_1, α_2 on I such that

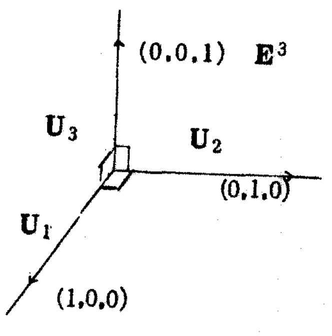
$$\alpha(t) = \boldsymbol{X}(\alpha_1(t),\ \alpha_2(t)) \in M$$

for all $t \in I$.

We assume that a surface M is in E^3, 3-dimensional Euclidean space. Let U be a unit normal vector field on a neighborhood of $p \in M$. For each $\vec{v} \in T_p(M)$ we define

$$S_p : T_p(M) \longrightarrow T_p(M)$$

by $S_p(\vec{v}) = -V_{\vec{v}}U$, where $\triangle$ is the covariant derivative for the natural frame field U_1, U_2 and U_3 as in the figure :



For each $P \in M \subset E^3$, S_p is called the shape operator, which is a linear map ([8], [9], [13]).

Proposition 2.8. In the above situation, $\vec{v}, \vec{w} \in T_p(M)$

$$S_p(\vec{v}) \cdot \vec{w} = S_p(\vec{w}) \cdot \vec{v},$$

Proof. At first, we shall prove that for a curve $\alpha : I \longrightarrow M$ and the unit normal U of $M, S(\alpha') = -U'$, where a' is the derivative of α and U' is the derivative of U.

Let $\alpha(t) = (\alpha_1(t),\ \alpha_2(t),\ \alpha_3(t))$ and $U = (u_1, u_2, u_3)$, where u_i is a function x_1, x_2 and x_3 for $i = 1, 2, 3$. In this case

$$\begin{aligned} S(\alpha') &= -\triangle_{\alpha'}U \\ &= -(\alpha'(t)[u_1]U_1 + \alpha'(t)[u_2]U_2 + \alpha'(t)[u_3]U_3) \\ &= -\left(\sum_{i=1}^{3} \frac{\partial u_1}{\partial x_i} \cdot \frac{dx_i}{dt} U_1 + \sum_{i=1}^{3} \frac{\partial u_2}{\partial x_i} \frac{dx_i}{dt} U_2 + \sum_{i=1}^{3} \frac{\partial u_3}{\partial x_i} \frac{dx_i}{dt} U_3\right) \\ &= -(u_1'U_1 + u_2'U_2 + u_3'U_3) \\ &= -(U') \end{aligned}$$

Suppose $\boldsymbol{X}: D \longrightarrow M$. Let $\boldsymbol{X}_u(u_0, v_0)$ and $\boldsymbol{X}_v(u_0, v_0)$ be a base of $T_P(M)$, where $P=\boldsymbol{X}(u_0, v_0)$. We shall prove that $S_P(\boldsymbol{X}_u(u_0, v_0)) \cdot \boldsymbol{X}_v(u_0, v_0) = S_P(\boldsymbol{X}_v(u_0, v_0)) \cdot \boldsymbol{X}_u(u_0, v_0)$. Note that $\boldsymbol{U}\cdot\boldsymbol{X}_u=0=\boldsymbol{U}\cdot\boldsymbol{X}_v$. Hence, we have

$$\frac{\partial}{\partial_v}\boldsymbol{U}\cdot\boldsymbol{X}_u=\boldsymbol{U}_v\cdot\boldsymbol{X}_u+\boldsymbol{U}\cdot\boldsymbol{X}_{uv}=0$$

and

$$\frac{\partial}{\partial u}\boldsymbol{U}\cdot\boldsymbol{X}_v=\boldsymbol{U}_u\cdot\boldsymbol{X}_v+\boldsymbol{U}\cdot\boldsymbol{X}_{vu}=0.$$

Consider each v-parameter curve $u=u_0$ and the covariant derivative of the vector field $v \longmapsto \boldsymbol{U}(u_0, v)$ on $u=u_0$. By the first description in this proof $\boldsymbol{U}_v=-S(\boldsymbol{X}_v)$ and $\boldsymbol{U}_u=-S(\boldsymbol{X}_u)$. Hence we have

$$\boldsymbol{U}\cdot\boldsymbol{X}_{uv}=S(\boldsymbol{X}_v)\cdot\boldsymbol{X}_u$$
$$\boldsymbol{U}\cdot\boldsymbol{X}_{vu}=S(\boldsymbol{X}_u)\cdot\boldsymbol{X}_v.$$

For $\boldsymbol{X}_{uv}=\boldsymbol{X}_{vu}$ as in Theorem 3.3 we shall prove our assertion using $S(\boldsymbol{X}_v)\cdot\boldsymbol{X}_u=S(\boldsymbol{X}_u)\cdot\boldsymbol{X}_v$ as follows. We put

$$\vec{v}=v_1X_u(u_0, v_0)+v_2X_v(u_0, v_0) \text{ and } \vec{w}=w_1X_u(u_0, v_0)+w_2X_v(u_0, v_0).$$

Since

$$\begin{aligned} S_P(\vec{v}) &= -\nabla_{\vec{v}}\boldsymbol{U} = -\nabla_{v_1x_u(u_0,v_0)+v_2x_v(u_0,v_0)}\boldsymbol{U} \\ &= -v_1\nabla_{x_u(u_0,v_0)}\boldsymbol{U} - v_2\nabla_{x_v(u_0,v_0)}\boldsymbol{U} \\ &= v_1S_P(\boldsymbol{X}_u(u_0, v_0)) + v_2S_P(\boldsymbol{X}_v(u_0, v_0)). \end{aligned}$$

we have

$$\begin{aligned} S_P(\vec{v})\cdot\vec{w} &= (v_1S_P(\boldsymbol{X}_u(u_0, v_0)) + v_2S_P(\boldsymbol{X}_v(u_0, v_0))\cdot(w_1\boldsymbol{X}_u(u_0, v_0) \\ &\quad + w_2\boldsymbol{X}_v(u_0, v_0)) \\ &= v_1w_1S_P(\boldsymbol{X}_u(u_0, v_0))\cdot\boldsymbol{X}_u(u_0, v_0) + v_2w_1S_P(\boldsymbol{X}_v(u_0, v_0))\cdot\boldsymbol{X}_u(u_0, v_0) \\ &\quad + v_1w_2S_P(\boldsymbol{X}_u(u_0, v_0))\cdot\boldsymbol{X}_v(u_0, v_0) + v_2w_2S_P(\boldsymbol{X}_v(u_0, v_0))\cdot\boldsymbol{X}_v(u_0, v_0) \\ &= w_1v_1S_P(\boldsymbol{X}_u(u_0, v_0))\cdot\boldsymbol{X}_u(u_0, v_0) + w_2v_1S_P(\boldsymbol{X}_v(u_0, v_0))\cdot\boldsymbol{X}_u(u_0, v_0) \\ &\quad + w_1v_2S_P(\boldsymbol{X}_u(u_0, v_0))\cdot\boldsymbol{X}_v(u_0, v_0) + w_2v_2S_P(\boldsymbol{X}_v(u_0, v_0))\cdot\boldsymbol{X}_v(u_0, v_0) \\ &= (w_1S_P(\boldsymbol{X}_u(u_0, v_0)) + w_2S_P(\boldsymbol{X}_v(u_0, v_0)))\ (v_1\boldsymbol{X}_u(u_0, v_0) + v_2\boldsymbol{X}_v(u_0, v_0)) \\ &= S_P(\vec{w})\cdot\vec{v}. \end{aligned}$$

3. Some Properties of Covariant Derivatives

The covariant derivative ∇ of a surface M(cf. Definition 2.2) may be modified so as to be applied for a vector field Y on a curve α in M. Note that for each $t, Y(t)$ is a tangent vector of M at $\alpha(t)$. Let E_1, E_2 be a frame field on a region of M containing α. Then we can put such that

$$Y(t)=y_1(t)E_1(\alpha(t))+y_2(t)E_2(\alpha(t)).$$

or briefly,

$$Y=y_1E_1+y_2E_2.$$

Definition 3.1. In the above situation we define the covariant derivative Y' of Y by $\triangle\alpha'Y$; *i.e.*, by

$$Y'=\{y_1'+y_2w_{21}(\alpha')\}\ E_1+\{y_2'+y_1w_{12}(\alpha')\}\ E_2.$$

Moreover, if $Y'=0$, then a vector field Y on a curve α in M is said to be parallel (see Proposition 2.3 and note that $\alpha'(t)[y_1]=y_1'$) ([10],[12],[13]).

It is a routine matter to check that this notion of covariant derivative is independent of the choice of frame field and has the same linear and Leibnizian properties as in Definition 2.2.

Let W be a vector field on $\boldsymbol{E}^3$ and let $\vec{v}$ be a tangent vector to $\boldsymbol{E}^3$ at the point $P\in\boldsymbol{E}^3$. The Euclidean covariant derivative of W with respect to $\vec{v}$ is the tangent vector

$$\tilde{\nabla}_{\vec{v}}W=\frac{d}{dt}W(p+t\vec{v})\,|_t=0$$

at th e point p. If $W=\sum_{i=1}^{3} w_iU_i$ where U_1, U_2 and U_3 are the natural frame field of $\boldsymbol{E}^3$, then we can prove that

$$\tilde{\nabla}_{\vec{v}}W=\sum_{i=1}^{3}\vec{v}[w_i]\,U_i(P)$$

([4],[6],[7]). Moreover the Euclidean covariant derivative $\tilde{\nabla}$ satisfies the properties (i)~(v) in Definition 2.2

Teorem 3.2. Let V and W be vector fields on a surface M in $\boldsymbol{E}^3$. If ∇ is the covariant derivative of M as a surface and $\tilde{\nabla}$ is the Euclidean covariant derivative, then

$$\tilde{\nabla}_V W = \nabla_V W + (S(V) \cdot W) U$$

where S is the shape operator derived from a normal unit field U of $T_p(M)$. If α is a curve in M, then

$$\tilde{\alpha}'' = \alpha'' + (S(\alpha') \cdot \alpha') U,$$

where $\tilde{\alpha}'' = \tilde{\nabla}_{\alpha'} \alpha'$ and $\alpha'' = \nabla_{\alpha'} \alpha'$

Proof. Let E_1, E_2, E_3 be an adapted frame field and let w_{ij} $(1 \leq i, j \leq 3)$ be the connection forms of E_1, E_2, E_3. Then we first prove that

$$\tilde{\nabla}_V E_i = \sum_{j=1}^{3} w_{ij}(V)\ F_j (1 \leq i \leq 3).$$

which are called the connection equations of E_1, E_2 and E_3. Note that each w_{ij} is an 1-form $w_{ij} = -w_{ji}$ and at p

$$w_{ij}(V(P)) = \tilde{\nabla}_{V(P)} E_i(p) \cdot E_j(p) \cdots\cdots\cdots\cdots (3\text{—}1)$$

Then it is clear that

$$\tilde{\nabla}_V E_i = \sum_{j=1}^{3} w_{ij}(V) E_j$$

is a solution of (3—1). By the uniqueness of solution (3—1) holds.

We assume that W is one of vector fields E_1, E_2, E_3. Then by the above description for $E_1 = W$

$$\begin{aligned} \tilde{\nabla}_V E_1 &= w_{12}(V) E_2 + w_{13}(V) E_3 \\ &= \nabla_V E_1 + w_{13}(V) E_3 \end{aligned}$$

because $\nabla_V E_1 = w_{12}(V) E_2$ by Proposition 2.3. That is, $\tilde{\nabla}_V E_1$ is $\nabla_V E_1$ plus a vector field normal to M. Here, we note that $E_1(p)$ and $E_2(p)$ are vectors in $T_p(M)$. The same result holds for $E_2 = W$.

In the general case, we can put

$$W = f_1 E_1 + f_2 E_2$$

where $f_1, f_2 : M \longrightarrow \boldsymbol{R}$ are differentiable functions. Then

$$\begin{aligned}\tilde{\nabla}_V W &= \tilde{\nabla}_V(f_1E_1+f_2E_2)=\tilde{\nabla}_V(f_1E_1)+\tilde{\nabla}_V(f_2E_2)\\ &=V[f_1]\cdot E_1+f_1\tilde{\nabla}_V E_1+V[f_2]\cdot E_2+f_2\tilde{\nabla}_V E_2\\ &=(V[f_1]+f_2w_{21}(V))E_1+(V[f_2]+f_1w_{12}(V))E_2\\ &\quad+(f_1w_{13}(V)+f_2w_{23}(V))E_3\\ &=\nabla_V W+(f_1w_{13}(V)+f_2w_{23}(V))E_3\end{aligned}$$

(see Proposition 2.3). Hence

$$\begin{aligned}\tilde{\nabla}_V W &= \nabla_V W-(f_1w_{31}(V)+f_2w_{32}(V))E_3\\ &=\nabla_V W-(f_1\tilde{\nabla}_V E_3\cdot E_1+f_2\tilde{\nabla}_V E_3\cdot E_2)E_3\\ &=\nabla_V W+(S(V)\cdot(f_1E_1+f_2E_2))\boldsymbol{U}\\ &=\nabla_V W+(S(V)\cdot W)\boldsymbol{U}\end{aligned}$$

where $E_3 \perp T_p(M)$ and thus $\pm E_3=\boldsymbol{U}$. Therefore we have for each curve $\alpha : I \longrightarrow M$

$$\tilde{\nabla}_{\alpha'}\alpha'=\nabla_{\alpha'}\alpha'+(S(\alpha')\cdot\alpha')\,\boldsymbol{U}.$$

That is, we have

$$\tilde{\alpha}''=\alpha''+(S(\alpha')\cdot\alpha')\,\boldsymbol{U}.$$

For a surface M an orthogonal coordinate patch

$$\boldsymbol{X} : D \longrightarrow M$$

means that $\boldsymbol{X}_u \perp \boldsymbol{X}_v$ where D is an open subset of $\boldsymbol{E}^2$ with variables u and v.

Thoerem 3.3. For a surface M in $\boldsymbol{E}^3$ let $\boldsymbol{X} : D \longrightarrow M$ be an orthogonal coordinate patch. Then $\boldsymbol{X}_{uv}=\boldsymbol{X}_{vu}$.

Proof. In our situation we can put

$$E_1=\boldsymbol{X}_u/\sqrt{E},\quad E_2=\boldsymbol{X}_v/\sqrt{G}$$

as in Definition 2.5. We shall prove that

$$\nabla_{\boldsymbol{X}_u}\boldsymbol{X}_v=(\boldsymbol{X}_v)_u,\quad \nabla_{\boldsymbol{X}_v}\boldsymbol{X}_u=(\boldsymbol{X}_u)_v.$$

Since for a differentiable function $f : M \longrightarrow \boldsymbol{R}$

$$X_u[f]=\frac{\partial f}{\partial u} \text{ and } X_v[f]=\frac{\partial f}{\partial v}$$

([13],[16]), we shall put

$$X_v=f_1U_1+f_2U_2+f_3U_3$$

where $\{U_1, U_2, U_3\}$ is the natural frame field. Then

$$\begin{aligned}\nabla_{X_u}X_v&=\sum X_u[f_i]U_i\\&=\sum\frac{\partial f_i}{\partial u}U_i\\&=(X_v)_u=X_{vu}.\end{aligned}$$

Similarly, we can prove that $\nabla_{X_v}X_u=(X_u)_v=X_{uv}$.

By our definition in Section 2,

$$\begin{aligned}w_{21}(X_u)&=\nabla_{X_u}E_2\cdot E_1=\nabla X_u\left(\frac{X_v}{\sqrt{G}}\right)\cdot\left(\frac{X_u}{\sqrt{E}}\right)\\&=\left(\frac{X_v}{\sqrt{G}}\right)_u\cdot\frac{X_u}{\sqrt{E}}=X_{vu}\cdot\frac{X_u}{\sqrt{EG}}.\end{aligned}$$

Note that

$$\begin{aligned}\left(\frac{X_v}{\sqrt{G}}\right)_u\cdot\frac{X_u}{\sqrt{E}}&=\frac{X_{vu}\cdot X_u}{\sqrt{EG}}+\left(\frac{1}{\sqrt{G}}\right)_u\frac{1}{\sqrt{E}}X_v\cdot X_u\\&=\frac{1}{\sqrt{EG}}X_{vu}\cdot X_u\end{aligned}$$

since $X_u\perp X_v$. On the other hand, by Proposition 2.6,

$$w_{12}=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

Since du and dv are 1-forms satisfying

$$du(X_u)=1=dv(X_v),\quad du(X_v)=0=dv(X_u)$$

([16]), we have

$$\begin{aligned}w_{21}(X_u)&=(\sqrt{E})_v/\sqrt{G}\\&=\frac{\partial}{\partial v}(X_u\cdot X_u)^{\frac{1}{2}}/\sqrt{G}\end{aligned}$$

$$=\frac{X_{uv}\cdot X_u}{\sqrt{EG}}$$

Thus, we have

$$X_{vu}\cdot\frac{X_u}{\sqrt{EG}}=X_{uv}\cdot\frac{X_u}{\sqrt{EG}}$$

and thus $X_{uv}=X_{vu}$ holds.

4. Geodesics in Surfaces

Throughout this section, we use M as a surface without any statements.

Suppose that $\alpha : I \longrightarrow M$ is a curve in M and E_1, E_2 is a frame field of M, where I is an open interval in E^1.

We put

$$\frac{d\alpha}{dt}=\alpha'=v_1E_1+v_2E_2$$

where $v_1, v_2 : M \longrightarrow E^1$ are diferentiable. For the covariant derivative ∇ of E_1 and E_2 we also put

$$\nabla_{\alpha'}\alpha'=\alpha''=A_1E_1+A_2E_2.$$

Then from

$$\begin{aligned}\nabla_{\alpha'}\alpha' &= \alpha'[v_1]E_1+v_1\nabla_{\alpha'}E_1+\alpha'[v_2]E_2+v_2\nabla_{\alpha'}E_2\\ &=(v_1'+v_2w_{21}(\alpha'))E_1+(v_2'+v_1w_{12}(\alpha'))E_2\end{aligned}$$

we have

$$A_1=v_1'+v_2w_{21}(\alpha'),\quad A_2=v_2'+v_1w_{12}(\alpha'),$$

which are real-valued functions defined on I.

Definiton 4.1. In the above situation α is a geodesic of M if and only if $A_1=0=A_2$, *i.e.*, $\alpha''=0$.

Lemma 4.2. Let $X : D \longrightarrow M$ be an orthogonal coordinate patch (cf. §2). A curve $\alpha(t)=X(a_1(t), a_2(t))$ is a gecodesic of M if and only if

$$a'' + \frac{1}{2E}\{E_u a_1'^2 + 2E_v a_1' a_2' - G_u a_2'^2\} = 0$$

and

$$a_2'' + \frac{1}{2G}\{-E_v a_1'^2 + 2G_u a_1' a_2' + G_v a_2'^2\} = 0.$$

where

$$E_u = \frac{\partial}{\partial u} X_u \cdot X_u, \quad G_u = \frac{\partial}{\partial_u} X_v \cdot X_v$$

etc. ([1],[2],[16]).

Proof.
$$\alpha' = \frac{d\alpha}{dt} = X_u \frac{da_1}{dt} + X_v \frac{da_2}{dt}$$
$$= (a_1'\sqrt{E})E_1 + (a_2'\sqrt{G})E_2$$

by Definition 2.5, where E_1, E_2 is an associated frame field Here we note that $X_u \perp X_v$, $E_1 = X_u/\sqrt{E}$ and $E_2 = X_v/\sqrt{G}$.

Thus we have

$$A_1 = (a_1'\sqrt{E})' + (a_2'\sqrt{G})\, w_{21}(\alpha')$$
$$A_2 = (a_2'\sqrt{G})' + (a_1'\sqrt{E})\, w_{12}(\alpha')$$

by the above description. By Proposition 2.6

$$w_{12}(\alpha') = \frac{-(\sqrt{E})_v}{\sqrt{G}} du(\alpha') + \frac{(\sqrt{G})_u}{\sqrt{E}} dv(\alpha')$$
$$= -\frac{(\sqrt{E})_v}{\sqrt{G}} du\ (a_1'X_u + a_2'X_v) + \frac{(\sqrt{G})_u}{\sqrt{E}} dv\ (a_1'X_u + a_2'X_v)$$
$$= -\frac{(\sqrt{E})_v}{\sqrt{G}} a_1' + \frac{(\sqrt{G})_u}{\sqrt{E}} a_2'.$$

Hence we have

$$A_1 = (a_1'\sqrt{E})' + (\sqrt{E})_v a_1' a_2' - \frac{\sqrt{G}(\sqrt{G})_u}{\sqrt{E}} a_2'^2$$
$$= a_1''\sqrt{E} + a_1' \frac{1}{2\sqrt{E}}(E_u a_1' + E_v a_2') + (\sqrt{E})_v a_1' a_2'$$
$$- \frac{\sqrt{G}(\sqrt{G})_u}{\sqrt{E}} a_2'^2$$
$$= \sqrt{E}\left(a_1'' + \frac{1}{2E}(E_u a_1'^2 + 2E_v a_1' a_2' - G_u a_2'^2)\right).$$

Similarly we have

$$A_2=\sqrt{G}\,(a_2''+\frac{1}{2G}(-E_v a_1'^2+2G_u a_1'a_2'+G_v a_2'^2)).$$

By our definition α is a geodesic of M if and only if $A_1=0=A_2$.

Therefore α is a geodesic of M if and only if

$$\begin{cases} a_1''+\frac{1}{2E}(E_u a_1'^2+2E_v a_1'a_2'-G_u a_2'^2)=0 \\ a_2''+\frac{1}{2G}(-E_v a_1'^2+2G_u a_1'a_2'+G_v a_2'^2)=0 \end{cases}$$

and this completes our proof.

Definition 4.3. A Clairaut parametrization $X: D\longrightarrow M$ is an orthogonal parametrization for which $E_v=G_v=0$ and $F=0$, where $E=X_u\cdot X_u$. $F=X_u\cdot X_v$ and $G=X_v\cdot X_v$.

If $X: D\longrightarrow M$ is a Clairaut parametrization, then we can prove the following ([15] [16]);

Property 4.4. (i) Every u-parameter curve of X has curvature zero. (We shall say that every u-parameter curve of X is pregeodesic in case of (i).)

(ii) A v-parameter curve $u=u_0$, is a geodesic if and only if $G_u(u_0)=0$.

Proposition 4.5. Let $X: D\longrightarrow M$ be a Clairaut parametrization and let $\alpha(t)=X(a_1(t), a_2(t))$ be a curve in M. Then $\alpha(t)$ is a unit speed geodesic if and only if

(i) $C=G(a_1)a_2'=\sqrt{G(a_1)}\sin\varphi$ is a constant,

(ii) $a_1=\pm\sqrt{G-C^2}/\sqrt{EG}$ and $Ga_2'=C$

where φ is the angle from X_u to α'.

Proof($\Longrightarrow$) : Since $X: D\longrightarrow M$ is a Clairaut parametrization by Definition 4.3, $E_v=G_v=0$. Since α is a geodesic of M by Lemma 4.2, we have the following

$$a_2''+\frac{G_u}{G}a_1'a_2'=0.$$

But, since

$$\begin{aligned}(Ga_2')'=G'a_2'+Ga_2''&=G_u a_1'a_2'+G_v a_2'^2+Ga_2''\\ &=\frac{1}{G}(a_2''+\frac{G_u}{G}a_1'a_2')\\ &=0,\end{aligned}$$

$C=Ga_2'$ is a constant.

Next, we want to prove that $C=\sqrt{G}\sin\varphi$. Since

$$\alpha'\cdot X_v=(a_1'X_u+a_2'X_v)\cdot X_v=a_2'X_v\cdot X_v=Ga_2'$$

and

$$\alpha'\cdot X_v=\|\alpha'\|\|X_v\|\cos\left(\frac{\pi}{2}-\varphi\right)=\|X_v\|\sin\varphi=\sqrt{G}\sin\varphi,$$

we get $Ga_2'=\sqrt{G}\sin\varphi=C$.

Since $\alpha'(t)=X_ua_1'+X_va_2'$,

$$\begin{aligned}1=\alpha'(t)\cdot\alpha'(t)&=(X_ua_1'+X_va_2')\cdot(X_ua_1'+X_va_2')\\&=Ea_1'^2+Ga_2'^2=Ea_1'^2+Ca_2'.\end{aligned}$$

Hence $a_1'^2=\dfrac{1-Ca_2'}{E}=\dfrac{G-C^2}{EG}$.

($\Longleftarrow$) : From $C=Ga_2'$ *i.e.*, $a_2'=C/G$ we have

$$a_2''=-\frac{G'C}{G^2}=-\frac{C}{G^2}(G_ua_1'+G_va_2')=-\frac{Ga_2'}{G^2}G_ua_1'=-\frac{G_u}{G}a_1'a_2'.$$

That is, we get

$$a_2''+\frac{G_u}{G}a_1'a_2'=0. \quad\cdots\cdots\cdots(4-1)$$

Next, from

$$\pm a_1'=\sqrt{G-C^2}/\sqrt{EG}$$

we have the following :

$$\begin{aligned}\pm a_1''&=\frac{\dfrac{G'}{2\sqrt{G-C^2}}\sqrt{EG}-\sqrt{G-C^2}\cdot\dfrac{E'G+EG'}{2\sqrt{EG}}}{EG}\\&=\pm\frac{G_u}{2EG}\mp\frac{G-C^2}{2(EG)^2}(E_uG+EG_u)\\&=\mp\frac{1}{2E}\frac{G-C^2}{EG}E_u\pm\frac{1}{2E}\frac{C^2}{G^2}G_u.\end{aligned}$$

Therefore we have

$$a_1''+\frac{1}{2E}(E_ua_1'^2-G_ua_2'^2)=0\quad\cdots\cdots\cdots(4-2)$$

Since $E_v=0=G_v$, by Lemma 4.2, (4—1) and (4—2) imply that $\alpha(t)$ is a geodesic of M.

That α is a unit speed curve is proved as follows. Since $\alpha'(t)=a_1'X_u+a_2'X_v$ and $X_u \perp X_v$, we we have the following :

$$\begin{aligned}\alpha'(t)\cdot\alpha'(t) &= X_u\cdot X_u a_1'^2+X_v\cdot X_v a_2'^2\\ &= Ea_1'^2+Ga_2'^2\\ &= E\left(\frac{\sqrt{G-C^2}}{\sqrt{EG}}\right)^2+G\left(\frac{C}{G}\right)^2\\ &= \frac{G-C^2}{G}+\frac{C^2}{G}=1.\end{aligned}$$

Theorem 4.6. Let α be a unit speed curve such that α' is not collinear with E_1 and let E_1, E_2 be a frame field. In $\alpha''=A_1E_1+A_2E_2$, if $A_1=0$, then α is a geodesic of M.

Proof. As before, we can put such that

$$\alpha'=v_1E_1+v_2E_2.$$

Therefore

$$\begin{aligned}\alpha'' &= \nabla_{\alpha'}\alpha'=\alpha'[v_1]E_1+v_1\nabla_{\alpha'}E_1+\alpha'[v_2]E_2+v_2\nabla_{\alpha'}E_2\\ &= (v_1'+v_2w_{21}(\alpha'))\ E_1+(v_2'+v_1w_{12}(\alpha'))E_2\end{aligned}$$

and thus

$$A_1=v_1'+v_2w_{21}(\alpha').\quad A_2=v_2'+v_1w_{12}(\alpha').$$

By our assumption $A_1=0$, *i.e.*,

$$v_1'+v_2w_{21}(\alpha')=0 \cdots\cdots\cdots (4\text{—}3)$$

On the other hand, from

$$\alpha'\cdot\alpha'=v_1^2+v_2^2=1$$

we get

$$v_1v_1'+v_2v_2'=0,\ \ i.e.,\ \ v_1v_1'=-v_2v_2'.$$

By (4—3)

$$v_1v_1'+v_1v_2w_{21}(\alpha')=0$$

implies

$$-v_2v_2' - v_1v_2w_{12}(\alpha') = 0$$

which in turn implies

$$v_2' + v_1w_{12}(\alpha') = 0.$$

Therefore we get $\alpha'' = 0$. By Definition 4.1 α is a geodesic of M.

Let $X : D \longrightarrow M$ be a Clairaut parametrization and let $\alpha : I \longrightarrow M$ be a unit speed geodesic of M with slant $C = Ga_2'$ where $\alpha(t) = X(a_1(t), a_2(t))$. Suppose

$$\alpha(0) = X(a_1(0), a_2(0)) = X(u_0, v_0)$$

with $a_1'(0) > 0$. Assume that there exists $u > u_0$ such that $G(u) = C^2$. Let u_1 be the smallest such numbers $u > u_0$ with $G(u) = C^2$. Then

$$\beta(v) = X(u_1, v)$$

is called a barrier curve for α.

Theorem 4.7. Let $\beta(v) = X(u_1, v)$ be a barrier curve for a unit speed geodesic $\alpha(t)$ of M as in the above description.

If β is a geodesic of M, then α and β are not intersected.

Proof. At first we note that

$$\alpha' = a_1'X_u + a_2'X_v, \quad \beta' = a_2'X_v.$$

By Proposition 4.5

$$a_1'(t^*) = \frac{\sqrt{G(u_1) - C^2}}{\sqrt{EG}} = 0$$

and thus

$$\alpha' |t^* = \beta' |u_1 = a_2'(t^*)X_v(u_1)$$

where $a_1(t^*) = u_1$. If $\alpha \cap \beta$ is not empty, then there exists v_1 such that $X(u_1, v_1) \in \alpha \cap \beta$. Note that α and β are geodesics of M by our hypothesis. Moreover, at the point $X(u_1, v_1) = \alpha(t_1) = \beta(v_1)$

$$\alpha'(t_1) = \beta'(v_1).$$

By the uniqueness of geodesic ([3],[15]) we have $\alpha=\beta$. This is imposible, sine $\alpha(t)$ passes through the point $X(u_0, v_0)\in M$, but not β.

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