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On the Cohomology of Affine Schemes

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I. Introduction

Since the 1960's algebraic geometry has come to be a recognized branch of mathematics and has become especially advanced in its investigations of the properties of algebraic manifolds and algebraic curves ([7], [12], [16]). Much progress has also been made in relational studies between the k-theory (of the early 1960's) and algebraic geometry([10]).

As is generally known, algebraic geometry has been developed from the sheaf theory ([14], [15]). Moreover, the cohomology theory of sheaves is an indispensable device of algebraic geometry ([5], [6], [9]).

This paper is based upon notes of the Seminar on Algebraic Geometry held at the Hanyang University Graduate School in 1988, where [9] is used as a textbook.

The contents of this paper are as follows:

In Section II we discuss the general necessary terminology used in this paper and proves basic properties which are not sufficiently proved in Propositions 2.2 and 2.4.

In Section III we summarize sheaf cohomology and prove the following:

Theorem 3.6 Let $(X,\mathcal{O}_X)$ be a ringed space and let

$$0\longrightarrow\mathcal{F}\xrightarrow{\varepsilon}\mathcal{G}^0\xrightarrow{\sigma^0}\mathcal{G}^1\xrightarrow{\sigma^1}\cdots$$

be an exact sequence of $\mathcal{O}_X$-modules. If $H^q(X,\mathcal{G}^i)=0$ whenever $q>0$ and $i\geq 0$, then each $H^q(X,\mathcal{F})$ $(q\geq 0)$ is isomorphic to the q^{th} cohomology group of the complex

$$0\longrightarrow\Gamma(X,\mathcal{G}^0)\xrightarrow{\sigma^0(X)}\Gamma(X,\mathcal{G}^1)\xrightarrow{\sigma^1(X)}\cdots.$$

In Section IV we discuss the cohomology of affine schemes and prove the following:

Theorem 4.10 Let X and Y be noetherian separated schemes, and let $f: X\longrightarrow Y$ be an affine morphism. For any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms

$$H^i(X,\mathcal{F})\cong H^i(Y,f_*\mathcal{F})$$

for all $i\geq 0$.

II. Preliminaries

Let A be a commutative ring with 1. The *spectrum* of A will be denoted by $X=\mathrm{Spec}(A)$, which consists of all prime ideals of A. That is, each element $x\varepsilon X$ is a prime ideal $\mathscr{P}(x)$ of A. X is a topological space with *Zariski topology* (or *spectral topology*). For each subset E of A,

$$V(E)=\{x\varepsilon X \mid E\subset\mathscr{P}(x)\}$$

is a closed subset of X, and for each $f\varepsilon A$

$$D(f)=X-V(f)$$

is an open subset of X, where $V(f)=V(\{f\})=V((f))$.

For an ideal $\mathfrak{a}$ of A the *radical* $r(\mathfrak{a})$ of $\mathfrak{a}$ is the set of all $f\varepsilon A$ such that some power of f lies in $\mathfrak{a}$; it is also the intersection of all prime ideals of A containing $\mathfrak{a}$. In particular, the radical $r(0)$ of the zero ideal is the set of all nilpotent elements of A, which is called the *nilradical* of A. We shall use the following notations:

Let $x\varepsilon X=\mathrm{Spec}(A)$ and $f\varepsilon A$

$A_x=A_{\mathscr{P}(x)}$=the local ring of A with respect to the prime ideal $\mathscr{P}(x)$,
$\mathfrak{M}(x)=\mathscr{P}(x)A_x$=the maximal ideal of A_x,
$k(x)=A_x/\mathfrak{M}(x)$=the residue field of $A_x\cong$the field of fractions of $A/\mathscr{P}(x)$,
$f(x)$=the class of f mod $\mathscr{P}(x)$ in $A/\mathscr{P}(x)\subset k(x)$.
Thus $f(x)=0$ if and only if $f\in\mathscr{P}(x)$.

Throughout this paper we mean by a ring a commutative ring with 1.

Let A and B be two rings, and let $\varphi: A\longrightarrow B$ be a ring homomrphism with $\varphi(1)=1$. Then we have a continuous map

$$^a\varphi : \mathrm{Spec}(B)=Y\longrightarrow \mathrm{Spec}(A)=X,$$

which is defined by $^a\varphi(y)=\varphi^{-1}(\mathscr{Q}(y))$, where $y\varepsilon Y$ and $\mathscr{Q}(y)$ is the prime ideal of B represented by y. This continuous map $^a\varphi$ is said to be *associated* with φ. For each

$y \in Y$, let φ^y denote the embedding of $A/\varphi^{-1}(\mathscr{Q}(y))$ in $B/\mathscr{Q}(y)$ induced by φ; then φ^y is extended to a field monomorphism

$$\varphi^y : k({}^a\varphi(y)) \longrightarrow k(y).$$

In the above situation we have the following property ([5], [6], [9]):

Property 2.1.

(i) For any subset E of A, $({}^a\varphi)^{-1}(V(E)) = V(\varphi(E))$.

(ii) For every ideal $\mathscr{B}$ of B,

$${}^a\varphi(V(\mathscr{B})) = V(\varphi^{-1}(\mathscr{B})).$$

(iii) If φ is injective, then ${}^a\varphi$ is dominant (i.e. ${}^a\varphi(Y)$ is dense in X).

Proposition 2.2. For a ring A and an element $f \in A$, f is nilpotent if and only if $D(f)$ is empty.

Proof. Let $\mathfrak{N}$ be the nilradical of A. Since $\mathfrak{N}$ is the intersection of all prime ideals of A, $f \in \mathfrak{N}$ implies that $f \in \mathscr{P}$ for every prime ideal of A. Therefore $D(f) = \phi$.

Conversely, $D(f) = \phi$ implies that f is contained in every prime ideal of A. That is, $f \in \mathfrak{N}$ and hence f is a nilpotent element. ///

A non-empty topological space X is said to be *irreducible* if every pair of non-empty open subsets in X intersects.

We have the following equivalent conditions ([3], [9]):

X is irreducible

$\Longleftrightarrow X$ is not the union of two proper closed subsets

$\Longleftrightarrow$ Every non-empty set is dense in X

$\Longleftrightarrow$ Every open set in X is connected.

For a subset Y of X, Y is *irreducible* if it is irreducible in the induced topology. In particular, every topological space has maximal irreducible subsets such that they are all closed and cover X ([1]).

A topological space X is *Noetherian* if the closed subsets of X satisfy the descending chain condition. There are following equivalent conditions([1], [3], [9]):

A topological space X is noetherian

$\Longleftrightarrow$ The open subsets in X satisfy the ascending chain conditions

$\Longleftrightarrow$Every open subset of X is *quasi-compact*

$\Longleftrightarrow$Every subset of X is quasi-compact.

Let X be a topological space. Let $\Omega(X)$ be the category which consists of all open subsets of X and all inclusion mappings between open subsets in X. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of rings on X, which is called the *structure sheaf* of the ringed space. For each $U \varepsilon \Omega(X)$, let $\Gamma(U, \mathcal{O}_X)$ be the set of all sections $U \longrightarrow \mathcal{O}_X | U$ which is isomorphic to $\mathcal{O}_X(U)$.

A morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair (ψ, ψ^*), where $\psi : X \longrightarrow Y$ is a continuous map and $\psi^* : \mathcal{O}_Y \longrightarrow \psi_* \mathcal{O}_X$ is a sheaf homomorphism compatible with the restriction homomorphism; that is to say, whenever $V \supseteq U$ in $\Omega(Y)$, the following diagram is commutative.

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_Y) & \xrightarrow{\psi^*(V)} & \Gamma(\psi^{-1}(V), \mathcal{O}_X) \\ \text{restriction} \downarrow & & \downarrow \text{restriction} \\ \Gamma(U, \mathcal{O}_Y) & \xrightarrow{\psi^*(U)} & \Gamma(\psi^{-1}(U), \mathcal{O}_X) \end{array}$$

For each $x \varepsilon X$, ψ^* then induces a homomorphism of the stalks

$$\psi^* : \mathcal{O}_{Y, \psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

by taking direct limits ([8], [9]).

Let $\varphi : A \longrightarrow B$ be a ring homomorphism. Recall the continuous map ${}^a\varphi : \mathrm{Spec}(B) = Y \longrightarrow \mathrm{Spec}(A) = X$. Suppose $f, g \varepsilon A$ are such that $D(f) \supseteq D(g)$. Then $r((f)) = r(f) \supseteq r(g)$, and so there exists an element $s \varepsilon A$ and a positive integer n such that $g^n = sf$. We define

$$\rho_{g,f} : A_f \longrightarrow A_g$$

by $$\rho_{g,f}(a/f^m) = as^m/g^{mn}$$

It is clear that $\rho_{g,f}$ is a well-defined ring homomorphism depending only on f and g, and that for $D(f) \supseteq D(g) \supseteq D(h)$, $\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}$. Then the assignment

$$D(f) \longrightarrow A_f$$

and the homomorphisms $\rho_{g,f}$ form a presheaf on the basis $\mathfrak{B} = \{D(f) | f \varepsilon A\}$. This presheaf on $\mathfrak{B}$ determines the sheaf on X, denoted by $\mathcal{O}_X$ or $\tilde{A}$. We can prove that

$(D(f), \mathcal{O}_X) \cong A_f$ and $\Gamma(X, \mathcal{O}_X) \cong A$ for all $f \varepsilon A$ ([5], [6], [13]). Moreover for all $x \varepsilon X$, We have $\mathcal{O}_{X,x} \cong A_x = A_{\mathscr{P}(x)}$.

Recall a ring homomorphism $\varphi : A \longrightarrow B$ and the continuous function ${}^a\varphi$: Spec(B) $= Y \longrightarrow$ Spec$(A) = X$. The morphism $({}^a\varphi, {}^a\varphi^\sharp) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ is defined as follows:

For each $g \varepsilon A$, φ induces a ring homomorphism

$$A_g \longrightarrow B_{\varphi(g)},\ a/g^n \longmapsto \varphi(a)/\varphi(g)^n$$

and thus we have a homomorphism

$$\begin{array}{ccc} {}^a\varphi^\sharp_{D(g)} : \Gamma(D(g), \mathcal{O}_X) & \longrightarrow & \Gamma({}^a\varphi^{-1}(D(g), \mathcal{O}_Y) = \Gamma(D(\varphi(g)), \mathcal{O}_Y). \\ \wr\Vert & & \wr\Vert \\ A_g & & B_{\varphi(g)} \end{array}$$

Thus we have the sheaf homomorphism

$${}^a\varphi^\sharp :\ \mathcal{O}_X \longrightarrow \mathcal{O}_Y.$$

Moreover, since

$$\Gamma({}^a\varphi^{-1}(D(g), \mathcal{O}_Y) = \mathcal{O}_Y(f^{-1}(D(g))) = {}^a\varphi_* \mathcal{O}_Y(D(g))$$

(direct image sheaf with respect to ${}^a\varphi$), we have the sheaf homomorphism

$${}^a\varphi^\sharp :\ \mathcal{O}_X \longrightarrow {}^a\varphi_* \mathcal{O}_Y$$

([5], [6], [9]). In particular, for $y \varepsilon Y$ and ${}^a\varphi(y) = x \varepsilon X$ $({}^a\varphi(y) = \varphi^{-1}(\mathscr{Q}(y)) = \mathscr{P}(x) = x)$, we have the following ring homomorphism:

$$\begin{array}{ccc} {}^a\varphi^\sharp{}_Y :\ \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{Y,y} \\ \wr\Vert & & \wr\Vert \\ A_x & & B_y \\ \omega & & \omega \\ a/s & \longmapsto & \varphi(a)/\varphi(s) \end{array}$$

where $s \varepsilon A - \mathscr{P}(x)$

Property 2.3. Let X be a topological space, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves defined on X. For a sheaf homomorphism φ: $\mathscr{F} \longrightarrow \mathscr{G}$, the following conditions hold ([6], [8]):

(i) φ is injective if for each $U \varepsilon \Omega(X)$,

$$\varphi(U) :\ \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$$

is injective.

(ii) φ is surjective if for each $x\varepsilon X$,

$$\varphi_x: \mathscr{F}_x \longrightarrow \mathscr{G}_x$$

is surjective, where $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x.

Proposition 2.4. For a ring homomorphism φ: $A \longrightarrow B$ and the induced morphism ${}^a\varphi$: Spec$(B)=Y \longrightarrow$ Spec$(A)=X$, the following conditions hold:

(i) φ is injective$\Longleftrightarrow {}^a\varphi^\sharp$: $\mathcal{O}_X \longrightarrow {}^a\varphi_*\mathcal{O}_Y$ is injective,

(ii) If φ is surjective, then ${}^a\varphi$ is a homeomorphism of Y onto a closed subset of X, and ${}^a\varphi^\sharp$: $\mathcal{O}_X \longrightarrow {}^a\varphi_*\mathcal{O}_Y$ is surjective.

(iii) If ${}^a\varphi$: $Y \longrightarrow X$ is a homeomorphism onto a closed subset of X and ${}^a\varphi^\sharp$: $\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is surjective, then φ is surjective.

Proof. (i) For each $D(g)\varepsilon\Omega(X)$ $(g\varepsilon A)$, we shall prove that

$$\varphi: A \longrightarrow B \text{ is injective if and only if } {}^a\varphi^\sharp_{D(g)}: \mathcal{O}_X(D(g)) \longrightarrow \mathcal{O}_Y({}^a\varphi^{-1}(D(g))) \text{ is injective;}$$

that is,

$$\varphi: A \longrightarrow B \text{ is injective if and only if } {}^a\varphi^\sharp_{D(g)}: A_g \longrightarrow B_{\varphi(g)}$$

is injective, where ${}^a\varphi^\sharp_{D(g)}(a/g^n)=\varphi(a)/\varphi(g)^n$.

($\Longrightarrow$): Suppose $\varphi(a)/\varphi(g)^m=\varphi(a')/\varphi(g)^n$ in $B_{\varphi(g)}$, where a, a' and g are in A. Put $n=\max\{m,n\}$; then

$$\varphi(g)^{n-m}\varphi(a)-\varphi(a')=\varphi(g^{n-m}a-a')=0.$$

Since φ is injective, we have $g^{n-m}a=a'$, which means that

$$a/g^m=a'/g^n \text{ in } A_g.$$

Therefore ${}^a\varphi^\sharp|D(g)$ is injective. Since $\{D(g)|g\varepsilon A\}$ is an open basis of X, ${}^a\varphi^\sharp$ is injective.

($\Longleftarrow$): Since ${}^a\varphi^\sharp$ is injective for every open set $U\varepsilon\Omega(X)$, the continuous map

$$^a\varphi^\sharp(U):\ \mathcal{O}_X(U)\longrightarrow\mathcal{O}_Y(\ ^a\varphi^{-1}(U))$$

is injective. But if $X=U$, then $\mathcal{O}_X(X)\cong\Gamma(\mathrm{Spec}(A),\mathcal{O}_X)\cong A$ ([5]).
Therefore, φ: A$\longrightarrow B$ is injective since $^a\varphi^\sharp$ is induced by a homomorphism.

(ii) Suppose φ: $A\longrightarrow B$ is a surjective ring homomorphism and put Ker $\varphi=\mathfrak{a}$, which is an ideal of A. Then $A/\mathfrak{a}\cong B$ as rings. Hence Spec (B)$\approx$Spec $(A/\mathfrak{a})=V(\mathfrak{a})$, which is a closed subset of X. Hence $^a\varphi$: $Y\longrightarrow X$ is an homeomorphism onto a closed subset $V(\mathfrak{a})$ of X.

Next we shall prove that $^a\varphi^\sharp$: $\mathcal{O}_X\longrightarrow{}^a\varphi_*\mathcal{O}_Y$ is surjective; i.e., by property 2.3, it suffices to prove that for each $x\varepsilon X$ with $^a\varphi(y)=x$ $(y\,\varepsilon\, Y)$,

$$^a\varphi^\sharp_y:\ \mathcal{O}_{X,x}\ \longrightarrow\ (^a\varphi_*\mathcal{O}_Y)_x=\mathcal{O}_{Y,y}$$

is surjective.

Recall that for a prime ideal $y=\mathscr{Q}(y)$ of B,

$$^a\varphi(y)=\varphi^{-1}(\mathscr{Q}(y))=\mathscr{P}(x)$$

is a prime ideal of A,
and

$$^a\varphi^\sharp_y:\ A_{\mathscr{P}(x)}\longrightarrow B_{\mathscr{Q}(y)}$$

is defined by

$$^a\varphi^\sharp_y\,(a/s)=\varphi(a)/\varphi(s),$$

where $a\varepsilon A$ and $s\varepsilon A-\mathscr{P}(x)$. Since φ is surjective it is clear that $^a\varphi^\sharp_y$ is surjective.

(iii) Note that every closed subset of X is of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Thus we may assume that

$$^a\varphi:\ Y\approx V(\mathfrak{a})\subset X$$

This means that for each prime ideal $\mathscr{Q}$ of B, the prime ideal $\varphi^{-1}(\mathscr{Q})=\mathscr{P}$ contains the ideal $\mathfrak{a}$. Thus φ: $\mathscr{P} \longrightarrow \mathscr{Q}$ is surjective.

Next, by our hypothesis,

$$ {}^a\varphi_y^\sharp : A_{\mathscr{P}} \longrightarrow B_{\mathscr{Q}} $$

is surjective where $\mathscr{Q}=\mathscr{Q}(y)$ and ${}^a\varphi(y)=\varphi^{-1}(\mathscr{Q})=\mathscr{P}(x)=\mathscr{P}$.
This implies that ${}^a\varphi_y^\sharp=\varphi|\mathrm{A}-\mathscr{P}$: $A-\mathscr{P}\longrightarrow B-\mathscr{Q}$ is surjective.
Thus φ: $A\longrightarrow B$ is surjective. ///
By Proposition 2.4 and (iii) of Property 2.1 we have the following:

Corollary 2.5. For a ring homomorphism φ: $A\longrightarrow B$, let ${}^a\varphi$: $\mathrm{Spec}(B)=Y\longrightarrow \mathrm{Spec}(A)=X$ be the induced map. Then

φ is injective if and only if ${}^a\varphi^\sharp$: $\mathscr{O}_X \longrightarrow {}^a\varphi_*\mathscr{O}_Y$ is injective, and if φ is injective, then ${}^a\varphi(Y)$ is dense in X.

Ⅲ. Cohomology of Sheaves

Let X be a topological space, and let $\mathscr{O}=\{\mathscr{O}(U),\ \rho_{V,U}\}$, where $V\subseteq U$ in $\Omega(X)$, bea sheaf of rings. For a sheaf $\mathscr{F}=\{\mathscr{F}(U),\ \rho'_{V,U}\}$ of abelian groups on X, if for each $U\varepsilon\Omega(X)$, $\mathscr{F}(U)$ is a $\mathscr{O}(U)$-module and $\rho'_{V,U}(af)=\rho_{V,U}(a)\cdot\rho'_{V,U}(f)$ for $a\varepsilon\mathscr{O}(U)$, $f\varepsilon\mathscr{F}(U)$ and $V\subseteq U$ in $\Omega(X)$, then $\mathscr{F}$ is said to be a *sheaf of* $\mathscr{O}$*-modules* or simply an $\mathscr{O}$*-module.*

For a ringed space $(X,\mathscr{O}_X)$, if all stalks $\mathscr{O}_{X,x}(x\varepsilon X)$ of $(X,\mathscr{O}_X)$ are local rings then $(X,\mathscr{O}_X)$ is called a *local ringed space.* Let $\mathscr{F},\mathscr{G}$ and $\mathscr{H}$ denote $\mathscr{O}_X$-modules. An $\mathscr{O}_X$-module $\mathscr{I}$ is said to be *injective* if whenever any sequence of $\mathscr{O}_X$-modules of the form

$$ 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} $$

is exact, then

$$ \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{I})\longrightarrow \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})\longrightarrow 0 $$

is also exact, where for each $U\varepsilon\Omega(X)$,

$$ \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{I})(U)=\mathrm{Hom}_{\mathscr{O}_X|U}(\mathscr{G}|U,\mathscr{I}|U); $$

Thus $\mathscr{O}_U\varepsilon\mathrm{Hom}_{\mathscr{O}_X|U}(\mathscr{G}|U,\mathscr{I}|U)$ if and only if for every $V\subseteq U$ in $\Omega(X)$, the diagram

$$\begin{array}{ccc} \mathscr{G}(U) & \xrightarrow{\theta_U} & \mathscr{I}(U) \\ \rho_{V,U} \downarrow & & \downarrow \rho'_{V,U} \\ \mathscr{G}(V) & \xrightarrow{\theta_V} & \mathscr{I}(V) \end{array}$$

is commutative, where $\rho_{V,U}$ and $\rho'_{V,U}$ are restrictions of $\mathscr{G}$ and $\mathscr{I}$, respectively.

An injective $\mathscr{O}_X$-module is referred as an injective sheaf. It is well known that every $\mathscr{O}_X$-module is extended to some injective sheaf ([11], [13]), and that any $\mathscr{O}_X$-module $\mathscr{F}$ has an injective resolution, which is an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{I}^0 \xrightarrow{\delta^0} \mathscr{I}^1 \xrightarrow{\delta^1} \cdots,$$

where $\mathscr{I}^i (i=0,1,2,\cdots)$ is an injective sheaf. Note that these sheaves are defined on X. Since the function Γ is left exact ([6], [8], [9]), we have the complex

$$0 \longrightarrow \Gamma(X, \mathscr{I}^0) \xrightarrow{\delta^0(X)} \Gamma(X, \mathscr{I}^1) \xrightarrow{\delta^1(X)} \cdots.$$

We define

$$H^p(X, \mathscr{F}) = \mathrm{Ker}\ \delta^p(X)/\mathrm{Im}\ \delta^{p-1}(X)$$

for $p=0,1,2,\ldots$, which is called the p^{th} *cohomology group* of $\mathscr{F}$, where $\delta^{-1}(X)=0$.

Definition 3.1. A sheaf $\mathscr{F}$ of abelian groups on a topological space X is said to be *flasque* if for every inclusion $V \subseteq U$ of open subsets of X the restriction map $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ is surjective.

Example 3.2. A constant sheaf of abelian groups on an irreducible topological space is flasque.

Proof. Let U and V be open subsets such that $V \subseteq U$, and let $\mathscr{A}$ be a constant sheaf defined on X. Since X is irreducible, U and V are connected (see II), and thus we have

$$\mathscr{A}(U) = A = \mathscr{A}(V),$$

where A is a fixed abelian group. Therefore, the inclusion

$$\mathscr{A}(U) \longrightarrow \mathscr{A}(V) \quad (a \longmapsto a)$$

is an isomorphism. ///

In the remaining part of this section we mean a sheaf of abelian groups by a sheaf. We have the following properties with respect to flasque sheaves ([5],[7]):

Property 3.3. (i) Let $0\longrightarrow\mathscr{F}'\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}''\longrightarrow 0$ be an exact sequence of sheaves on a topological space X. If $\mathscr{F}'$ is flasque, then for each $U\varepsilon\Omega(X)$ the sequence

$$0\longrightarrow\mathscr{F}'(U)\longrightarrow\mathscr{F}(U)\longrightarrow\mathscr{F}''(U)\longrightarrow 0$$

of abelian groups is exact.

(ii) In an exact sequence

$$0\longrightarrow\mathscr{F}'\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}''\longrightarrow 0$$

of sheaves on a topological space X, if $\mathscr{F}'$ and $\mathscr{F}$ are flasque then $\mathscr{F}''$ is also flasque.

(iii) Let $f: X\longrightarrow Y$ be a continuous map, and let $\mathscr{F}$ be a flasque sheaf on X. Then $f_*\mathscr{F}$ is also a flasque sheaf on Y.

(iv) If $(X,\mathscr{O}_X)$ is a ringed space, then any injective $\mathscr{O}_X$-module is flasque.

Proposition 3.4. For a sheaf $\mathscr{F}$ on a topological space X, let $\mathscr{G}$ be the sheaf of discontinuous sections of $\mathscr{F}$; i.e. for each $U\varepsilon\Omega(X)$,

$$\mathscr{G}(U)=\prod_{p\in U}\mathscr{F}_p \text{ (direct product).}$$

Then $\mathscr{G}$ is a flasque sheaf and there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Proof. It is clear that $\mathscr{G}$ is a sheaf on X ([8]). For each open set $U\subseteq X$, we have $\mathscr{G}(U)\cong\{s: U\longrightarrow\prod_{p\in U}\mathscr{F}_p \mid s(p)\varepsilon\mathscr{F}_p,\ p\varepsilon U\}$,
since for every inclusion $V\subseteq U$ of open subsets of X, $\mathscr{G}(U)=\prod_{p\in U}\mathscr{F}_p$ and $\mathscr{G}(V)=\prod_{p\in V}\mathscr{F}_p$. If for $x\varepsilon U-V$ we put $s(x)=0\,\varepsilon\mathscr{F}_x$, then the restriction map $\mathscr{G}(U)\longrightarrow\mathscr{G}(V)$ is surjective. Hence $\mathscr{G}$ is a flasque sheaf.

For each $U\varepsilon\Omega(X)$ we define

$$\mu_U:\ \mathscr{F}(U)\longrightarrow\mathscr{G}(U)$$

is defined by

$$\mu_U(s)=(s_x\mid x\varepsilon U),$$

where s_x is the germ of s at x. Since for $s, t \in \mathscr{F}(U)$, $s+t \in \mathscr{F}(U)$ and $(s+t)_x = s_x + t_x \in \mathscr{F}_x$ $(x \in U)$, the mapping μ_U is an injective group homomorphism. For $V \subseteq U$ in $\Omega(X)$ the restriction map

$$\rho'_{V,U}: \mathscr{G}(U) \longrightarrow \mathscr{G}(V)$$

is defined by $(s_x | s \in \mathscr{F}(U), x \in U) \longmapsto ((s|V)_x | x \in V)$. Hence we have the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\mu_U} & \mathscr{G}(U) \\ \rho_{V,U} \downarrow & & \downarrow \rho'_{V,U} \\ \mathscr{F}(V) & \xrightarrow{\mu_V} & \mathscr{G}(V). \end{array} \quad ///$$

Let $(X, \mathscr{O}_X)$ be a ringed space. For an exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow H^0(X, \mathscr{F}) \cong \Gamma(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{G}) \cong \Gamma(X, \mathscr{G}) \longrightarrow$$
$$H^0(X, \mathscr{H}) \cong \Gamma(X, \mathscr{H}) \xrightarrow{\delta^0} H^1(X, \mathscr{F}) \longrightarrow H^1(X, \mathscr{G}) \longrightarrow \cdots,$$

where δ^q: $H^q(X, \mathscr{H}) \longrightarrow H^{q+1}(X, \mathscr{F})$ is called the q^{th} *connecting homomorphism*. The connecting homomorphism is natural, i.e., for a commutative diagram of exact sequence of $\mathscr{O}_X$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{G}' & \longrightarrow & \mathscr{H}' & \longrightarrow & 0, \end{array}$$

the diagram

$$\begin{array}{ccc} H^q(X, \mathscr{H}) & \xrightarrow{\delta^q} & H^{q+1}(X, \mathscr{F}) \\ \downarrow & & \downarrow \\ H^q(X, \mathscr{H}') & \xrightarrow{\delta^q} & H^{q+1}(X, \mathscr{F}') \end{array}$$

is commutative for all $q \geq 0$ ([2], [4], [6]).

Lemma 3.5. Let $(X, \mathscr{O}_X)$ be a ringed space, if $\mathscr{F}$ is a flasque $\mathscr{O}_X$-module, then $H^i(X, \mathscr{F}) = 0$ for all $i \geq 1$.

Proof. We have an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{H} \longrightarrow 0,$$

where $\mathscr{I}$ is an injective sheaf and $\mathscr{H}=\mathscr{I}/\mathscr{F}$ is the quotient sheaf. By the above description we have the long exact sequence

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{I}) \longrightarrow \Gamma(X,\mathscr{H}) \longrightarrow H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{I}) \longrightarrow \cdots.$$

Since $\mathscr{I}$ is an injective sheaf, we have

$$H^i(X,\mathscr{I})=0 \quad \text{for all } i \geq 1.$$

Noting $\Gamma(X,\mathscr{F})=\mathscr{F}(X)$ for each sheaf on X by (i) of Property 3.3, we have the exact sequence

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow (X,\mathscr{I}) \longrightarrow (X,\mathscr{H}) \longrightarrow 0,$$

since $\mathscr{F}$ is flasque. From the exact sequences

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{I}) \longrightarrow \Gamma(X,\mathscr{H}) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{I}) \longrightarrow \Gamma(X,\mathscr{H}) \longrightarrow H^1(X,\mathscr{F}) \longrightarrow 0,$$

we have $H^1(X,\mathscr{F})=0$. That is, for every flasque $\mathcal{O}_X$-module $\mathscr{F}$ we have $H^1(X,\mathscr{F})=0$. By (iv) of Property 3.3, $\mathscr{I}$ is flasque. Since $\mathscr{F}$ and $\mathscr{I}$ are flasque by (ii) of Property 3.3 $\mathscr{H}$ is also a flasque $\mathcal{O}_X$-module. Thus $H^1(X,\mathscr{H})=0$.

In the above long cohomology exact sequence, we have

$$H^i(X,\mathscr{H}) \cong H^{i+1}(X,\mathscr{F})$$

since $H^i(X,\mathscr{I})=0$ for $i>0$, Hence

$$H^1(X,\mathscr{H})=0=H^2(X,\mathscr{F}), \text{ i.e. for every flasque } \mathcal{O}_X\text{-module } \mathscr{F},$$
$$H^2(X,\mathscr{F})=0$$
$$H^2(X,\mathscr{H})=0=H^3(X,\mathscr{F}), \cdots$$

Hence we have $H^i(X,\mathscr{F})=0$ for all $i \geq 1$. ///

Let $(X,\mathcal{O}_X)$ be a ringed space, and let φ: $\mathscr{F} \longrightarrow \mathscr{G}$ be an $\mathcal{O}_X$-module homomorphism. For each $U \in \Omega(X)$, let $\mathscr{F}'(U)$ be the kernel of $\varphi(U)$: $\mathscr{F}(U) \longrightarrow \mathscr{G}(U)$.

Then $U \longrightarrow \mathscr{F}'(U)$ is a $\mathcal{O}_X$-module $\mathscr{F}'$, called the *kernel* of φ and is denoted by $\mathscr{F}' =$ Ker φ. Let

$$\text{Im } \varphi(U) = \varphi(U)(\mathscr{F}(U)),$$

which is a submodule of $\mathscr{G}(U)$. Then $U \longmapsto \text{Im } \varphi(U)$ is a presheaf. We shall denote the sheafification of this presheaf by Im φ, called the *image* of φ. It is clear that Im φ is a subsheaf of $\mathscr{G}$. Moreover, for each $U \varepsilon \Omega(X)$,

$$U \longrightarrow \mathscr{G}(U)/\text{Im } \varphi(U)$$

defines a presheaf, called the *quotient presheaf* of φ. The sheafification of this presheaf is called the *cokernel* of φ, written Coker φ. We must note that for each $x \varepsilon Y$,

$$(\text{Ker } \varphi)_x \cong \text{Ker } \varphi_x, \ (\text{Im } \varphi)_x \cong \text{Im } \varphi_x, \text{ and}$$
$$(\text{Coker } \varphi)_x \cong \text{Coker } \varphi_x,$$

where φ_x is defined as in the commutative diagram:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \searrow & & \searrow \\ \varinjlim_{x \varepsilon U} \mathscr{F}(U) = \mathscr{F}_x & \xrightarrow{\varphi_x} & \varinjlim_{x \varepsilon U} \mathscr{G}(U) = \mathscr{G}_x. \end{array}$$

Theorem 3.6. Let $(X, \mathcal{O}_X)$ be a ringed space and let

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{G}^0 \xrightarrow{\sigma^0} \mathscr{G}^1 \xrightarrow{\sigma^1} \cdots$$

be an exact sequence of $\mathcal{O}_X$-modules. If $H^q(X, \mathscr{G}^i) = 0$ whenever $q > 0$ and $i \geq 0$, then each $H^q(X, \mathscr{F})$ $(q \geq 0)$ is isomorphic to the q^{th} cohomology group of the complex

$$0 \longrightarrow \Gamma(X, \mathscr{G}^0) \xrightarrow{\sigma^0(X)} \Gamma(X, \mathscr{G}^1) \xrightarrow{\sigma^1(X)} \cdots.$$

Proof. From the given exact sequence of $\mathcal{O}_X$-modules we have short exact sequences as follows:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{G}^0 \longrightarrow \text{Im } \sigma^0 \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im}\ \sigma^0 \longrightarrow \mathscr{G}^1 \longrightarrow \mathrm{Im}\ \sigma^1 \longrightarrow 0$$

....

$$0 \longrightarrow \mathrm{Im}\ \sigma^{n-1} \longrightarrow \mathscr{G}^n \longrightarrow \mathrm{Im}\ \sigma^n \longrightarrow 0$$

....

Since Γ is a left exact functor, the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \xrightarrow{\varepsilon(X)} \Gamma(X, \mathscr{G}^0) \xrightarrow{\sigma^0(X)} \Gamma(X, \mathscr{G}^1)$$

is exact by our assumption. Hence

$$\mathrm{Ker}\ \sigma^0(X) \cong \Gamma(X, \mathscr{F}) \cong H^0(X, \mathscr{F}).$$

By our hypothesis $H^1(X, \mathscr{G}^0)=0$ in the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}^0 \longrightarrow \mathrm{Im}\ \sigma^0 \longrightarrow 0,$$

and we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}^0) \xrightarrow{\sigma^0(X)} \Gamma(X, \mathrm{Im}\ \sigma^0) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow 0,$$

Thus

$$H^1(X, \mathscr{F}) \cong \Gamma(X,\ \mathrm{Im}\ \sigma^0)/\mathrm{Im}\ \sigma^0(X).$$

On the other hand, from the exact sequence

$$0 \longrightarrow \mathrm{Im}\ \sigma^0 \longrightarrow \mathscr{G}^1 \xrightarrow{\sigma^1} \mathscr{G}^2 \longrightarrow \cdots,$$

we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathrm{Im}\ \sigma^0) \longrightarrow \Gamma(X, \mathscr{G}^1) \xrightarrow{\sigma^1(X)} \Gamma(X, \mathscr{G}^2).$$

Thus $\mathrm{Ker}\ \sigma^1(X) \cong \Gamma(X, \mathrm{Im}\ \sigma^0)$. That is, we have

$$H^1(X, \mathscr{F}) \cong \mathrm{Ker}\ \sigma^1(X)/\mathrm{Im}\ \sigma^0(X).$$

Since $H^i(X, \mathscr{G}^0)=0$ for $i \geq 1$, we have

$$H^q(X,\ \mathrm{Im}\ \sigma^0) \cong H^{q+1}(X, \mathscr{F})$$

for $q \geq 1$. By the above exact sequences we have the following isomorphisms:

$$\begin{aligned} H^{q+1}(X, \mathscr{F}) \cong H^q(X, \operatorname{Im} \sigma^0) \cong H^{q-1}(X, \operatorname{Im} \sigma^1) \\ \cong H^{q-2}(X, \operatorname{Im} \sigma^2) \\ \cong \cdots \\ \cong H^1(X, \operatorname{Im} \sigma^{q-1}). \end{aligned}$$

Since

$$0 \longrightarrow \operatorname{Im} \sigma^{q-1} \longrightarrow \mathscr{G}^q \longrightarrow \operatorname{Im} \sigma^q \longrightarrow 0$$

is exact and $H^1(X, \mathscr{G}^q) = 0$, the sequence

$$0 \longrightarrow \Gamma(X, \operatorname{Im} \sigma^{q-1}) \longrightarrow \Gamma(X, \mathscr{G}^q) \xrightarrow{\sigma^q(X)} \Gamma(X, \operatorname{Im} \sigma^q) \longrightarrow H^1(X, \operatorname{Im} \sigma^{q-1}) \longrightarrow 0$$

is exact. Also, since

$$0 \longrightarrow \operatorname{Im} \sigma^q \longrightarrow \mathscr{G}^{q+1} \xrightarrow{\sigma^{q+1}} \mathscr{G}^{q+2} \longrightarrow \cdots$$

is exact, the sequence

$$0 \longrightarrow \Gamma(X, \operatorname{Im} \sigma^q) \longrightarrow \Gamma(X, \mathscr{G}^{q+1}) \xrightarrow{\sigma^{q+1}(X)} \Gamma(X, \mathscr{G}^{q+2})$$

is exact.

Therefore

$$\begin{aligned} H^{q+1}(X, \mathscr{F}) \cong H^1(X, \operatorname{Im} \sigma^{q-1}) \cong \Gamma(X, \operatorname{Im} \sigma^q)/\operatorname{Im} \sigma^q(X) \\ \cong \operatorname{Ker} \sigma^{q+1}(X)/\operatorname{Im} \sigma^q(X). \quad /// \end{aligned}$$

IV. Cohomology of Affine Schemes

Let A be a commutative ring with 1. We put $X = \operatorname{Spec}(A)$. Let $\mathcal{O}_X = \tilde{A}$ be the sheaf on X such that for all $f \in A$ and $x \in X$,

$$\mathcal{O}_X(D(f)) = A_f, \quad \mathcal{O}_{X,x} = A_x = A_{\mathscr{P}(x)},$$

where $\mathscr{P}(x)$ is the prime ideal of A represented by x. Then the locally ringed space $(X, \mathcal{O}_X)$ ($\forall x \in X$, $\mathcal{O}_{X,x}$ is a local ring) is called an *affine scheme* associated with the ring A. The sheaf $\tilde{A} = \mathcal{O}_X$ is said to be the *structure sheaf* of the affine scheme

$(X, \mathcal{O}_X)$.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point of X has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|U$ is an affine scheme.

We should note that a morphism between locally ringed spaces is a local homomorphism. That is, let $(\varphi, \varphi^{\#})$: $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism between locally ringed spaces. Then φ: $X \longrightarrow Y$ is a continuous function, and $\varphi^{\#}$: $\mathcal{O}_Y \longrightarrow \varphi_*\mathcal{O}_X$ is a sheaf homomorphism such that for $\varphi(x)=y$, $\varphi_x^{\#}$: $\mathcal{O}_{Y,\varphi(x)} \longrightarrow \mathcal{O}_{X,x}$ is a local homomorphism; i.e. for the maximal ideal $\mathfrak{M}_{X,x}$ of $\mathcal{O}_{X,x}$, $\varphi_x^{\# -1}(\mathfrak{M}_{X,x})$ is the maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$.

Let $(X, \mathcal{O}_X)$ be a scheme. Then there exists an open covering $X=\bigcup U$ such that for each open subset U, $(U, \mathcal{O}_X|U)$ is an affine scheme. In this case $\{U\}$ is called an *open affine cover* of X and each U is called an *open affine subset*.

Let f: $X \longrightarrow Y$ be a morphism of schemes. If there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is an affine for each i, then f is said to be an *affine morphism*. Moreover, f is said to be *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Thus we can prove that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact ([5]).

Definition 4.1. A scheme X is *locally noetherian* if it can be covered by open affine subsets $\mathrm{Spec}(A_i)$, where each A_i is a noetherian ring. X is *noetherian* if it is locally noetherian and quasi-compact.

It is easily proved that

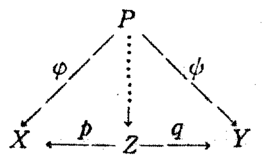
> X is noetherian if and only if X can be covered by a finite number of open affine subsets $\mathrm{Spec}(A_i)$, where each A_i is a noetherian ring.

Let $(X, \mathcal{O}_X)$ be a scheme, and let $\mathscr{F}$ be a $\mathcal{O}_X$-module. $\mathscr{F}$ is said to be *quasi-coherent* if $X=\bigcup\{U_i \mid U_i=\mathrm{Spec}(A_i)$ and A_i is a ring$\}$ such that $\mathscr{F}|U_i=\tilde{M}_i$, M_i is an A_i-module.

Definition 4.2. Let S be a scheme. An S-scheme (or a scheme over S) is a pair (X, f), where X is a scheme and f: $X \longrightarrow S$ is a scheme morphism. If (Y, g) is another S-scheme, a morphism φ: $(X, f) \longrightarrow (Y, g)$ of the S-scheme is a scheme morphism such that $f = g \circ \varphi$.

This is often called an S-*morphism*.

Definition 4.3. Let S be a scheme, and let X, Y be S-schemes. The S-product of X and Y is a triple $(Z : p, q)$, where Z is an S-scheme and $p: Z \longrightarrow X$, $q: Z \longrightarrow Y$ are S-morphisms, such that given any S-scheme P and any S-morphism $\varphi: P \longrightarrow X$, $\psi: P \longrightarrow Y$, there exists a unique S-morphism $f: P \longrightarrow Z$ which commutes the following diagram:



We shall put $Z = X \times_S Y$.

It has been proved that with the above notations,

(i) the S-product $X \times_S Y$ of X and Y is unique up to the S-isomorphism, and

(ii) $X \times_S Y$ always exists ([6]).

Example 4.4. Let R, A and B be rings and let $S = \text{Spec}(R)$. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ be S-Schemes. Then

$$(X \times_S Y,\ p, q) = (\text{Spec}\ (A \otimes_R B),\ {}^a\Pi_1,\ {}^a\Pi_2),$$

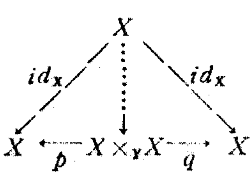
where

$$\Pi_1: A \longrightarrow A \otimes_R B \text{ and } \Pi_2: B \longrightarrow A \otimes_R B$$

are defined by

$$\Pi_1(a) = a \otimes 1 \quad \text{and} \quad \Pi_2(b) = 1 \otimes b.$$

Definition 4.5. A morphism $f: X \longrightarrow Y$ of schemes is said to be *separated* if $\Delta_{X/Y}(X)$ is a closed subscheme of the Y-product $(X \times_Y X,\ p, q)$, where the Y-morphism $\Delta_{X/Y}: X \longrightarrow X \times_Y X$ is defined as in the commutative diagram



In this case, it is also said that the scheme X is *separated over* Y, or is *Y-separated*. If X is separated over Spec($\boldsymbol{Z}$), where $\boldsymbol{Z}$ is the ring of integers, then we say that X is *separated*.

Proposition 4.6. Let X be a separated scheme over an affine scheme S. If U and V are open affine subsets of X, then $U\cap V$ is also open affine.

Proof. Since X is a separated scheme over an affine scheme S, we have

$$U\cap V\cong \Delta_{X/S}(X)\cap(U\times_S V)$$

for affine open subsets U and V of X.

Therefore $U\cap V$ is a closed subset of $U\times_S V$, since for ring A and B, Spec(A)$=U$ and Spec(B)$=V$. If for a ring R, Spec(R)$=S$, then by Example 4.5,

$$U\times_S V=\mathrm{Spec}(A\otimes_R B).$$

Therefore there exists an ideal $\mathfrak{a}$ of the ring $A\otimes_R B$ such that $U\cap V=\mathrm{Spec}(A\otimes_R B/\mathfrak{a})$. Thus $U\cap V$ is an open affine subset of X. ///

Lemma 4.7. Let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be schemes. For a morphism $f: X\longrightarrow Y$ of schemes, assume that either X is noetherian or f is quasi-compact and separated over Y. Then if $\mathscr{F}$ is quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathscr{F}$ is also a quasi-coherent sheaf $\mathcal{O}_Y$-modules.

Proof. Since our question is local on Y only, we may assume that $Y$is an affine scheme. Since X is quasi-compact (since if X is noetherian then X is quasi-compact by Definition 4.1), we can cover X with a finite number of open affine subsets U_i. By our hypothesis, X is separated over Y, so that $U_i\cap U_j$ is also an open affine subset of X by Proposition 4.6. We put $U_i\cap U_j=U_{ij}$. For any open subset V of Y, giving a section s of $\mathscr{F}$ over $f^{-1}(V)$ is the same thing as giving a collection of sections s_i of $\mathscr{F}$ over $f^{-1}(V)\cap U_i$, whose restrictions to the open sets $f^{-1}(V)\cap U_{ij}$ are equal. Hence we have an exact sequence of sheaves on Y:

$$0\longrightarrow f_*\mathscr{F}\longrightarrow\bigoplus_i f_*(\mathscr{F}|U_i)\longrightarrow\bigoplus_{i,j} f_*(\mathscr{F}|U_{ij}),$$

where by abuse of notation we denote by f the induced morphisms $U_i\longrightarrow Y$ and $U_{ij}\longrightarrow Y$. Since $\mathscr{F}|U_i$ and $\mathscr{F}|U_{ij}$ are $\mathcal{O}_X|U_i$-module and $\mathcal{O}_X|U_{ij}$-module respectively, it is clear that $f_*(\mathscr{F}|U_i)$ and $f_*(\mathscr{F}|U_{ij})$ are quasi-coherent ([5]). Since the kernel of morphism of quasi-coherent sheaves is also quasi-coherent ([5]), in the above exact sequence $f_*\mathscr{F}$ is quasi-coherent. ///

Let X be a topological space, and let $\mathscr{U}=\{U_i\,|\,i\in I\}$ be an open covering of X, where I is an index set. We define a well-ordering of the index set I. For a sheaf $\mathscr{F}$ of abelian groups on X we can define two cohomologies:

(i) $H^p(X,\mathscr{F})=p^{th}$ cohomology group of $\mathscr{F}$ $(p\geq 0)$ (see III)

(ii) $\check{H}^p(\mathscr{U},\mathscr{F})=p^{th}$ Čech cohomology group of $\mathscr{F}$ with respect to the covering $\mathscr{U}$ $(p\geq 0)$ ([5]).

If $(X,\mathcal{O}_X)$ is an affine scheme, $H^p(X,\mathscr{F})$ is called the *p^{th} cohomology group of an affine scheme X with coefficients in the $\mathcal{O}_X$-module $\mathscr{F}$.* Thus we have the following ([5]):

Property 4.8. With the above notations

(i) $\check{H}^0(\mathscr{U},\mathscr{F})\cong\Gamma(X,\mathscr{F})$, and

(ii) there is a natural map $\check{H}^p(\mathscr{U},\mathscr{F})\longrightarrow H^p(X,\mathscr{F})$ for $p\geq 0$ which is functorial in $\mathscr{F}$.

Lemma 4.9. Let X be a noetherian separated scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for an open affine covering $\mathscr{U}$ of X and for $p\geq 0$,

$$\check{H}^p(\mathscr{U},\mathscr{F})\cong H^p(X,\mathscr{F}).$$

Proof. When $p=0$ we have

$$H^0(X,\mathscr{F})\cong\Gamma(X,\mathscr{F})\cong H^0(X,\mathscr{F}) \text{ (by (i) of Property 4.8).}$$

Now assume that $p>0$, by our assumption X is a noetherian scheme and $\mathscr{F}$ is a quasi-coherent sheaf on X. Thus $\mathscr{F}$ can be embedded in a flasque and quasi-coherent sheaf $\mathscr{G}$ ([5]). If we put $\mathscr{R}=\mathscr{G}/\mathscr{F}$, then we have the following exact sequence of sheaves:

$$0\longrightarrow\mathscr{F}\longrightarrow\mathscr{G}\longrightarrow\mathscr{R}\longrightarrow 0 \qquad (*)$$

For each $i_0<i_1<\cdots<i_p$, we put

$$U_{i_0i_1\cdots i_p}=U_{i_0}\cap\cdots\cap U_{i_p},$$

where $U_{i_0},\ldots,U_{i_p}\in\mathscr{U}$. Then, by Proposition 4.6, $U_{i_0\cdots i_p}$ is also affine. Since $\mathscr{F}$ is quasi-coherent, so is $\mathscr{F}|U_{i_0\cdots i_p}$. Hence

$$H^i(U_{i_0\cdots i_p}, \mathscr{F}|U_{i_0\cdots i_p})=0,$$

for all $i>0$ ([5]). This implies that

$$0\longrightarrow\mathscr{F}(U_{i_0\cdots i_p})\longrightarrow\mathscr{G}(U_{i_0\cdots i_p})\longrightarrow\mathscr{R}(U_{i_0\cdots i_p})\longrightarrow 0$$

is exact. Thus, putting

$$C^p(\mathscr{U},\mathscr{F})=\prod_{i_0<\cdots<i_p}\mathscr{F}(U_{i_0\cdots i_p}),$$

we have the exact sequences

$$0\longrightarrow C^\cdot(\mathscr{U},\mathscr{F})\longrightarrow C^\cdot(\mathscr{U},\mathscr{G})\longrightarrow C^\cdot(\mathscr{U},\mathscr{R})\longrightarrow 0$$

of Čech complexes. Since $\mathscr{G}$ is flasque by Lemma 3.5, we have $H^p(X,\mathscr{G})=0$ for all $p>0$. Therefore $\check{H}^p(\mathscr{U},\mathscr{G})=0$ for all $p>0$ ([5]). Therefore we have the exact sequence

$$0\longrightarrow\check{H}^0(\mathscr{U},\mathscr{F})\longrightarrow\check{H}^0(\mathscr{U},\mathscr{G})\longrightarrow\check{H}^0(\mathscr{U},\mathscr{R})\longrightarrow\check{H}^1(\mathscr{U},\mathscr{F})\longrightarrow 0$$

and isomorphisms

$$\check{H}^p(\mathscr{U},\mathscr{R})\cong\check{H}^{p+1}(\mathscr{U},\mathscr{F})$$

for all $p\geq 1$.

From (*) above we have the exact sequence

$$0\longrightarrow H^0(X,\mathscr{F})\longrightarrow H^0(X,\mathscr{G})\longrightarrow H^0(X,\mathscr{R})\longrightarrow H^1(X,\mathscr{F})\longrightarrow 0$$

and isomorphisms $H^p(X,\mathscr{R})\cong H^{p+1}(X,\mathscr{F})$ for all $p\geq 1$.

By (ii) of Property 4.8, the diagram

$$\begin{array}{ccccccccc} 0\longrightarrow & \check{H}^0(\mathscr{U},\mathscr{F}) & \longrightarrow & \check{H}^0(\mathscr{U},\mathscr{G}) & \longrightarrow & \check{H}^0(\mathscr{U},\mathscr{R}) & \longrightarrow & \check{H}^1(\mathscr{U},\mathscr{F}) & \longrightarrow 0 \ \text{(exact)} \\ & \cong\downarrow & © & \cong\downarrow & © & \cong\downarrow & © & \downarrow & \\ 0\longrightarrow & H^0(X,\mathscr{F}) & \longrightarrow & H^0(X,\mathscr{G}) & \longrightarrow & H^0(X,\mathscr{R}) & \longrightarrow & H^1(X,\mathscr{F}) & \longrightarrow 0 \ \text{(exact)} \end{array}$$

is commutative and we get the natural isomorphism

$$\check{H}^1(\mathscr{U},\mathscr{F})\cong H^1(X,\mathscr{F})$$

for all quasi-coherent sheaves $\mathscr{F}$. Since the cokernel of quasi-coherent sheaves is also quasi-coherent ([5], [6], [9]), $\mathscr{R}$ is a quasi-coherent sheaf. Thus

$$\check{H}^1(\mathscr{U},\mathscr{R})\cong H^1(X,\mathscr{R}).$$

Therefore, if

$$\check{H}^p(\mathscr{U},\mathscr{R})\cong H^p(X,\mathscr{R}),$$

then

$$\check{H}^{p+1}(\mathscr{U},\mathscr{F})(\cong\check{H}^p(\mathscr{U},\mathscr{R})\cong H^{p+1}(X,\mathscr{F})\ \ (=H^p(X,\mathscr{R})).$$

Since $\check{H}^1(\mathscr{U},\mathscr{F})\cong H^1(X,\mathscr{F})$, we have natural isomorphisms

$$\check{H}^p(\mathscr{U},\mathscr{F})\cong H^p(X,\mathscr{F})$$

for all $p\geq 2$. ///

Theorem 4.10 Let X and Y be noetherian separated schemes, and let $f: X\longrightarrow Y$ be an affine morphism. For any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms

$$H^i(X,\mathscr{F})\cong H^i(Y,f_*\mathscr{F})$$

for all $i\geq 0$.

Proof. Let $\mathscr{U}=\{U_i\,|\,i\in I\}$ be an open affine covering of Y. By Lemma 4.7, $f_*\mathscr{F}$ is a quasi-coherent sheaf on Y. Since Y is a noetherian separated scheme, we have

$$\check{H}^i(\mathscr{U},f_*\mathscr{F})\cong H^i(Y,f_*\mathscr{F})$$

for all $i\geq 0$ by Lemma 4.9. Put

$$f^{-1}(\mathscr{U})=\{f^{-1}(U)\,|\,U\in\mathscr{U}\}.$$

Then $f^{-1}(\mathscr{U})$ is an open affine covering of X because f is an affine morphism. Since

$$\check{H}^i(\mathscr{U}, f_*\mathscr{F}) \cong \check{H}^i(f^{-1}(\mathscr{U}), \mathscr{F}),$$

and

$$\check{H}^i(f^{-1}(\mathscr{U}), \mathscr{F}) \cong H^i(X, \mathscr{F})$$

for all $i \geq 0$, by Lemma 4.9, we have

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

for all $i \geq 0$. ///

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