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On Extensions of Banach Algebras

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1. Introduction

It is well-known that a study of Banach algebra has been a central topic in the classical analysis of the twentieth century. Since 1960, some scholars have studied the extension algebra in the area of Banach algebra ([1], [6], [7], [9], [10]).

Following this trend, graduate students who want to major in such a field, have debated textbooks [2] and [12], under the supervision of Professor Jeon Won-Kee. They have also held a seminar on articles [6] and [14].

This dissertation examines some properties of those discussions. The more concrete contents of this study are as follows.

In §2, separable algebras over commutative ring are studied with particular reference to their properties and terminology and some terminology and properties about Banach algebra are also explained, which are needed in §3 and §4. §3 proves that, for commutative Banach algebra A with identity, we can define a norm (lemma 3.1) so that ∂B may Banach algebra when $B=A[x]/(\alpha(x))$ ($\alpha(x)$ is a monic polynomial in $A[x]$).

When we also assume that ∂B is the Shilov boundary of B, if A is either regular or self-adjoint, then $\partial B=\Phi_B$(Φ_B means the Carrier space of B) as proven in Theorem 3.7.

§4 aims to determine the properties of B, under the assumption that A has no idempotent but 0 and 1, and B is a separable projective extension over A.

In Theorm 4.4., the following contents are proven;

i) $\Pi : \Phi_B \longrightarrow \Phi_A$ is open and contineuous

ii) For each $\psi_1 \in \Phi_B$, there is neighborhood U of $\Pi(\psi_1)$ such that

a) $\Pi^{-1}(\Pi(\psi_1))=\{\psi_1, \cdots, \psi_m\}$

b) there is disjoint neighborhoods V_i of ψ_i $(i=1, \cdots, m)$ with $\Pi_i(V_i)=U$ and $\Pi^{-1}(U)=\bigcup_{i=1}^{n} V_i$.

2. Preliminaries

Let A be a commutative ring with 1 in this paper. For any A-algebra B, $B\otimes_A B^\circ$ is called the *enveloping algebra* of B, where B has identity and B° denotes the A-alebra oppsite to B. We shall put $B^e=B\otimes_A B^\circ$. We have the left B^e-module homomorphism

$$\mu : B^e=B\otimes_A B^\circ \longrightarrow B$$

defined by $\mu(\sum_i b_i\otimes b_i')=\sum_i b_i b_i'$. Setting $J=\text{Ker } \mu$, we have the exact sequence of left B^emodules :

$$0\longrightarrow J \longrightarrow B^e \xrightarrow{\mu} B \longrightarrow 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Proposition 2. 1. In the above situation J is an ideal generated by $\{b\otimes 1-1\otimes b \mid b\in B\}$.

Proof. Since $\mu(b\otimes 1-1\otimes b)=0$, we assume that $\mu(\sum_i b_i\otimes b_i')=0$. Since $\sum_i b_i b_i'=0$ and

$$\sum_i b_i\otimes b_i'=\sum_i (b_i\otimes 1)\ (1\otimes b_i'-b_i'\otimes 1),$$

the proposition is proved. ■

We have the following equivalence relations ([4]) :

(i) B is a left projective B^e-module under the μ-structure.

(ii) The sequence (1) of left B^e-modules splits.

(iii) There exists an element $e\in B^e$ such that $\mu(e)=1$ and $Je=0$.

Definition 2. 2. An A-algebra B is said to be *separable* if it statisfies the equivalent conditions (i)~(iii) as above.

An element $e\in B^e$ as in (iii) above is called a *separability idempotent* for B.

Proposition 2. 3. Every separability idempotent is an idempotent. If B is commutative then there is a unique separability idempotent for B.

Proof. Let $e\in B^e$ be a separability idempotent. $\mu(e-1\otimes 1)=0$ implies that e-$1\otimes 1\in J$. Since $Je=0$

$$(e-1\otimes 1)e=e^2-e=0,$$

and thus e is an idempotent of B^e.

Next, we assume that B is commutative, and e_1, e_2 are separability idempotents.

Since $e_1-e_2\in J$

$$(e_1-e_2)e_1=0=(e_1-e_2)^r(-e_2)$$

i. e.,

$$e_1-e_2e_1=-e_1e_2+e_2,$$

and thus $e_1=e_2$.

Lemma 2.4. Let B be a separable A-algebra which is projective as an A-module. Then B is a finitely generated A-module ([4], [15], [16]).

Proof. Since B is A-projective B° is also A-projective. Hence we have a dual basis $\{(f_i, b_i)\in \mathrm{Hom}_A(B^\circ, A)\times B^\circ\}$ such that for all $b\in B^\circ$

$$b=\sum_i f_i(b)b_i$$

where $f_i(b)=0$ for all but a finite number of subscription i ([4]).

Since $1_B\otimes f_i\in \mathrm{Hom}_{B^e}(B^e, B)$, where $B=B\otimes_A A$, $\{1_B\otimes f_i,\ 1\otimes b_i\}$ is a dual basis of B^e as a left B-module. Hence B^e is a left projective B-module. That is, for each $u\in B^e$

$$u=\sum_i (1_B\otimes f_i)(u)(1\otimes b_i).$$

For a separability idempotent e for B let $u=(1\otimes b)e$. Then

$$b=\mu(u)=\mu((1\otimes b)e)=\sum_i[(1_B\otimes f_i)\ ((1\otimes b)e)]b_i\cdots\cdots\cdots\cdots(2)$$

Note that $(b'\otimes b'')b=b'bb''$. Since $(1_B\otimes f_i)\ ((1\otimes b)e)=(1\otimes b)(1_B\otimes f_i)e$, the set of subscripts i for which $(1_B\otimes f_i)\ ((1\otimes b)e)\neq 0$ is contained in the finite set of subscripts for which $(1_B\otimes f_i)(e)\neq 0$ and this latter set is independent of b. Hence the summation (2) may be taken over a fixed finite set. Let us put

$$e=\sum_j x_j\otimes y_j\in B^e.$$

From (2) we have

$$b=\sum_{i,j} x_j f_i(y_j b)b_i=\sum_{i,j} f_i(y_j b)x_j b_i,$$

and thus the finite set $\{x_j b_i\}$ generates B° over A. ∎

For each prime ideal $\mathscr{P}$ of A and a finitely generated, projective A-module M,

$$M_{\mathscr{P}}\cong M\otimes_A A_{\mathscr{P}}$$

is a finitely generated, free $A_{\mathscr{P}}$-module ([4]). Hence there is a non-negative integer $n_{\mathscr{P}}$ such that

$$M_{\mathscr{P}}=(A_{\mathscr{P}})^{n_{\mathscr{P}}}$$

If for every prime ideal $\mathscr{P}$ of A, $n=n_{\mathscr{P}}$ then n *is* called the *rank* of M and we write

$$n=\operatorname{rank}_A(M).$$

The collection of all prime ideals of A is called the *spectrum* of A, written Spec(A). By the *Zariski topology* Spec(A) becomes a topological space. This topological space will be denoted by the same notation Spec(A) as above. If A has the only idempotents 0 and 1, then Spec(A) is connected, and the converse is true ([4]). Moreover, for a finitely generated, projective A-module M and a prime ideal $\mathscr{P}$ of A there exists an element $a\in A-\mathscr{P}$ such that $M_{(a)}$ is free as an $A_{(a)}$-module; where$(a)=(a,a^2,\cdots)\subset A$ ([4]).

Proposition 2.5. If A has the only two idempotents 0 and 1 then for any finitely generated, pojective A-module M $\operatorname{rank}_A(M)$ is well defined.

Proof. By the above description there is $a\in A$ such that $M_{(a)}$ is a free $A_{(a)}$-module of rank m. Then, for any prime ideal $\mathscr{P}$ of A not containing a it is clear that $A_{\mathscr{P}}$ is an $A_{(a)}$-algebra and

$$\begin{aligned}M_{\mathscr{P}}&\cong A_{\mathscr{P}}\otimes_A M\cong A_{\mathscr{P}}\otimes_{A(a)}(A_{(a)}\otimes_A M)\\&\cong A_{\mathscr{P}}\otimes_{A(a)}M_{(a)}\\&\cong A_{\mathscr{P}}\otimes_{A(a)}(A_{(a)})^m\\&\cong(A_{\mathscr{P}})^m\end{aligned}$$

and thus $\operatorname{rank}_q(M)=m$.

Let q be a prime ideal of A such that rank $q(M)=n$. Then, by the preceding paragraph, we have an element $a\in A-q$ such that $\operatorname{rank}_{A(a)}(M_{(a)})=n$. We put $V(a)=\{\mathscr{P}\in\operatorname{Spec}(A)\,|\,a\in\mathscr{P}\}$, which is a closed subset of Spec(A). Therefore $\operatorname{Spec}(A)-V(a)$ is open and $q\in\operatorname{Spec}(A)-V(a)$. Again by the above paragraph, if $q\in\operatorname{Spec}(A)-V(a)$ then $\operatorname{rank}_q(M)=n=\operatorname{rank}_{\mathscr{P}}(M)$. We regard $\{0,1,2,\cdots\}$ as a topological space with the discrete topology. Then

$$\operatorname{Spec}(A)\longrightarrow\{0,1,2,\cdots\}\quad(\mathscr{P}\longrightarrow\operatorname{rank}_{\mathscr{P}}(M))$$

is contineuous map.

On the other hand, since there are only two idempotents 0 and 1 in A, Spec(A) is connected. Hence the map

$$\text{Spec}(A) \longrightarrow \{0, 1, 2, \cdots\}$$

is a constant map. ■

In the sequel, we assume that A is a commutative Banach algebra with identity 1 and its norm $\|\cdot\|$. Let Φ_A be the *Carrier space* of A with the *Gelfand topology.* Then $\Phi_A(\neq\phi)$ is a compact Hausdorff space ([2], [8]).

For each $x \in A$ let $\hat{x}$ be the Gelfand transform of x, and $\hat{A}$ is the *Gelfand representation* of A. We have the following ([12], [13]) :

Property 2.6. (i) For each $x \in A$ $\hat{x} : \Phi_A \longrightarrow C$ (Complexes) is a contineous map, *i. e.*, $\hat{x} \in C(\Phi_A) = \{f : \Phi_A \longrightarrow C \mid f \text{ is contineous}\}$.

(ii) $\hat{A}$ separates the points of Φ_A and it contains the constant functions.

(iii) $\Phi_A \approx \Phi_{C(\Phi_A)}$ (homeomorphic) ($C(\Phi_A)$ is a Banach algebra with sup norm $\|\cdot\|_\infty$)

(iv) If A is semi-simple, then $A \longrightarrow \hat{A}(x \longrightarrow \hat{x})$ is an algebra isomorphism.

Definition 2.7. (i) A is said to be *self-adjoint* if whenever $\hat{x} \in \hat{A}$ then $\bar{\hat{x}} \in \hat{A}$, where the bar denotes complex conjugation.

(ii) A is said to be *regular* for each closed subset $E \subset \Phi_A$ and $\tau \notin E (\tau \in \Phi_A)$ there exists an element $x \in A$ such that

$$\hat{x}(\tau) = 1, \ {}^{\forall}\omega \in {}'E \quad \hat{x}(\omega) = 0.$$

Let F be a subset of $C(\Phi_A)$. *A maximizing set* for F is a closed subset E of Φ_A such that

$$|f|_E = |f|_{\Phi_A} ({}^{\forall} f \in F)$$

where $|f|_E = \sup_{x \in E} |f(x)|$.

Definition 2.8. For a commutative Banach algebra A with identity the *Shilov boundary* ∂A of A is the intersection of all maximizing set of $\hat{A} \subset C(\Phi_A)$.

Proposition 2.9. (i) ∂A is a maximizing set for $\hat{A}$.

(ii) $\tau \in \partial A$ if and only if there exists an open neighborhood U of τ and there exists $\hat{a} \in \hat{A}$ such that

$$|\hat{a}(\tau)| = \|\hat{a}\|_\infty = 1 \text{ and } |\hat{a}(\eta)| < 1 \text{ for } \eta \in \Phi_A - U.$$

Proof. (i) At first, we have to note that for each $x_0 \in \Phi_A - \partial A$ there exists an open neighborhood U of x_0 such that for each maximizing set E for $\hat{A}$ $E-U$ is also a maximizing set for $\hat{A}$ ([2]).

For $\hat{x} \in \hat{A}$ we put $K = \{\tau \in \Phi_A \mid |\hat{x}(\tau)| = |\tau(x)| = |\hat{x}|_{\Phi_A}\}$. To prove that ∂A is a maximizing set for $\hat{A}$ it is enough to show that $\partial A \cap K = \phi$. Since Φ_A is a compact and Hasudorff space with the Gelfand topology, K is compact and closed.

We assume $K \cap \partial A \neq \phi$. Then for each $\tau_0 \in K \subset \Phi_A - \partial A$ there exists an open neighborhood U_0 to τ_0 such that $\Phi_A - U_0$ is a maximizing set for $\hat{A}$ by the above description because of that Φ_A is itself a maximizing set for $\hat{A}$. That is, for each $\tau_\lambda \in K$ there exists an open neighborhood U_λ of τ_λ such that $\Phi_A - U_\lambda$ is a maximizing set for $\hat{A}$. Since K is compact there exists a positive integer n such that $K = \bigcup_{i=1}^{n} U_i$, where $\{U_1, \cdots, U_n\} \subset \{U_\lambda \mid \tau_\lambda \in K\}$. Thus $\Phi_A - U_1, \cdots, \Phi_A - \bigcup_{i=1}^{n} U_i = E$ are maximizing sets for $\hat{A}$ (Take $\tau_2 \in \Phi_A - U_i$ then $(\Phi_A - U_1) - U_2$ is a maximizing set for $\hat{A}$). Since $K \cap E = \phi$ $|\hat{x}|_E < |\hat{x}|_{\Phi_A}$, which is a contradiction.

ii) For $\tau \in \partial A$, suppose $\tau \neq \eta$ in Φ_A. Since $\hat{A}$ separates the points of Φ_A, $\exists x \in A$ such that $\hat{x}(\tau) = 1$ and $\hat{x}(\eta) = \eta(x) = 0$ (*i.e.* $x \in \ker \eta$).

Put $C = \{\tau \in \Phi_A \mid |\hat{x}(\tau)| \leq \frac{1}{2}\}$. Then $U = \Phi_A - C$ is an open neighborhood of τ.

Conversely, if there exist an open neighborhood U of τ and an element $\hat{a} \in \hat{A}$ such that $|\hat{a}(\tau)| = \|\hat{a}\|_\infty = 1$ and $|\hat{a}(\eta)| < 1$ for $\eta \in \Phi_A - U$, then we can easily see $\tau \in \partial A$.

3. The Banach Algebras $A[x]/(\alpha(x))$

Let A be a commutative Banach algebra with identity.

For a monic polynomial $\alpha(x) \in A[x]$ with degree n, we have the following.

Lemma 3.1. We put $B = A[x]/(\alpha(x))$. Then the following hold :

(i) B is a finitely generated and projective A-module,

(ii) There is a norm in B such that

(a) B is a commutative Banach algebra with identity,

(b) the inclusion $A \hookrightarrow B$ is an isometry and an into isomorphism.

Proof. (i) We have

$$B \cong A \oplus Ax \oplus \cdots \oplus Ax^{n-1} \quad \text{as } A\text{-modules.}$$

Hence B is a free A-module with a basis $\{1, x, \cdots, x^{n-1}\}$.

(ii) For each $a(x)+(\alpha(x))\in B$, where $a(x)=\sum_{i=0}^{n-1} a_i x^i \in A[x]$, we define the following:

$$\|a(x)+(\alpha(x))\|=\sum_{i=0}^{n-1}\|a_i\|$$

where $\alpha(x)=\sum_{i=0}^{n}\alpha_i x^i$ and $\sum_{i=1}^{n-1}\|\alpha_i\|\leq 1$. Then this is a norm defined in B (for details see [1] or [9]). Suppose a Cauchy sequence

$$\{a^{(m)}(x)+(\alpha(x))\,|\,a^{(m)}(x)=\sum_{i=0}^{n-1} a_i^{(m)}x^i\in A[x]\} \text{ in } B.$$

For a large number m, a positive number $\varepsilon>0$ and a positive integer l we have

$$\|a^{(m)}(x)-a^{(m+l)}(x)+(\alpha(x))\|=\sum_{i=0}^{n-1}\|a_i^{(m)}-a_i^{(m+l)}\|<\varepsilon.$$

Hence, for each $i=0, 1, \cdots, n-1$

$$\|a_i^{(m)}-a_i^{(m+l)}\|<\varepsilon.$$

Since A is a Banach algebra there exists an element $a_i\in A$ such that $a_i^{(m)}\longrightarrow a_i$. We put $a(x)=\sum_{i=0}^{n-1} a_i x^i$. Then

$$a^{(m)}(x)+(\alpha(x))\longrightarrow a(x)+(\alpha(x))\in B.$$

In consequence, B is a complete normed algebra. It is easy to prove that

$$\begin{aligned}\|(a^{(1)}(x)+(\alpha(x)))\ (a^{(2)}(x)+(\alpha(x)))\|&=\|a^{(1)}(x)a^{(2)}(x)+(\alpha(x))\|\\ &\leq\|a^{(1)}(x)+(\alpha(x))\|\ \|a^{(2)}(x)+(\alpha(x))\|\end{aligned}$$

and that the inclusion $A\hookrightarrow B$ is an isometry and an into isomorphism. ■

Proposition 3.2. (i) $\Phi_B=\{\varphi\times\lambda\in\Phi_A\times \boldsymbol{C}\,|\,\sum_{i=0}^{n}\varphi(\alpha_i)\lambda^i=0\}$

(ii) If $(\varphi, \lambda)\in\Phi_B$ then $|\lambda|\leq 1$.

Proof. (i) Since $\psi\in\Phi_B$ is a complex homomorphism

$$\psi: B=A[x]/(\alpha(x))\longrightarrow \boldsymbol{C}$$

ψ is a complex homomorphism from $A[x]$ to $\boldsymbol{C}$ satisfying $\psi(\alpha(x))=0$. This implies that $\psi|_A: A\longrightarrow \boldsymbol{C}$ is a complex homomorphism, and thus $\psi|_A\in\Phi_A$. Moreover,

$$\psi(\alpha(x)) = \psi(\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} + x^n)$$
$$= \psi(\alpha_0) + \psi(\alpha_1)\,\psi(x) + \cdots + \psi(\alpha_{n-1})\,\psi(x)^{n-1} + \psi(x)^n = 0.$$

Thus

$$-\psi(x) = \sum_{i=0}^{n-1} (\psi|_A)(\alpha_i)\,\psi(x)^i \in C.$$

Thus if we put $\psi|_A = \varphi \in \Phi_A$ and $\psi(x) = \lambda$ then we have $\psi = (\varphi, \lambda)$.
Conversely for $(\varphi, \lambda) \in \Phi_A \times C$ and $\sum_{i=0}^{n-1} a_i x^i + (\alpha(x)) \in B$ we define

$$(\varphi, \lambda)\left(\sum_{i=0}^{n-1} a_i x^i + (\alpha(x))\right) = \sum_{i=0}^{n-1} \varphi(a_i)\lambda^i$$

then $(\varphi, \lambda) \in \Phi_B$. Moreover, for $\psi = (\varphi, \lambda)$

$$\|\psi\|_\infty = \|(\varphi, \lambda)\|_\infty$$
$$= \operatorname{Sup}_{\|a(x)+(\alpha(x))\|=1} \left|\sum_{i=0}^{n-1} \varphi(a_i)\lambda^i\right|$$
$$= \operatorname{Sup}_{\|a(x)+(\alpha(x))\|=1} \left|\sum_{i=0}^{n-1} \hat{a}_i(\varphi)\lambda^i\right|$$
$$\leq \sum_{i=0}^{n-1} \|a_i\| |\lambda^i| \leq 1$$

because that $\|a(x) + (\alpha(x))\| = 1$, $\|a_i\| \geq \|\hat{a}_i\|_\infty$ and $|\alpha| \leq 1$ (see (ii) below).

(ii) We first prove that $\|\varphi\| \leq 1$ for $\varphi \in \Phi_A$. For $a \in A$ $(a \neq 0)$ we assume that $|\varphi(a)| > \|a\|$, *i.e.*, $\varphi(a) = \zeta \in C \Longrightarrow \|a/\zeta\| < 1$. Thus a/ζ is quasi-regular ([8]), and thus there exists $b \in A$ such that

$$a/\zeta + b - ab/\zeta = 0.$$

Hence $\varphi(a)/\zeta + \varphi(b) - \varphi(a)\varphi(b)/\zeta = 0$. Since $\varphi(a) = \zeta$ we have $1 + \varphi(b) - \varphi(b) = 0$, which is an absurdity. Thus, for all $a \in A$ $|\varphi(a)| \leq \|a\|$ and $\|\varphi\| \leq 1$.

Since λ is a root of $\sum_{i=0}^{n} \varphi(\alpha_i)\lambda^i = 0$ we have

$$\left|\sum_{i=0}^{n-1} \varphi(\alpha_i)\lambda^i\right| = |\lambda^n|$$

We assume that $|\lambda| > 1$. Since $\sum_{i=0}^{n-1} \|\alpha\| \leq 1$ and $\|\varphi\| \leq 1$

$$|\sum_{i=0}^{n-1}\varphi(\alpha)\lambda^i| = |\sum_{i=0}^{n-1}\hat{a}_i(\varphi)\lambda^i| \leq \sum_{i=0}^{n-1}\|\hat{a}_i\|_\infty|\lambda|^i$$

$$\leq \sum_{i=0}^{n-1}\|\alpha_i\||\lambda|^i$$

$$< \sum_{i=0}^{n-1}\|\alpha_i\||\lambda|^n < |\lambda|^n$$

which is a contradiction. ■

The *multiplicity function* M of $\alpha(x)$ is defined as follows. For each $(\varphi,\lambda)\in\Phi_B$ $M(\varphi,\lambda)$ is equal to the multiplicity of λ as a root of

$$\alpha_\varphi(x)=\varphi(\alpha_o)+\varphi(\alpha_1)x+\cdots+\varphi(\alpha_{n-1})x^{n-1}+x^n=0.$$

Definition 3.3. *A* neighborhood W in Φ_B of $(\varphi_o,\lambda_o)\in\Phi_B$ is called a *M-neighborhood* of (φ_o,λ_o) if for each $\varphi\in\Pi(W)$ ($\Pi:\Phi_B\longrightarrow\Phi_A$ is defined by $\Pi(\varphi,\lambda)=\varphi$)

$$M(\varphi_o,\lambda_o)=\sum_{(\varphi,\lambda)\in\pi^{-1}(\varphi)\cap W}M(\varphi,\lambda)$$

Note that the Gelfand topology of Φ_B is the product topology of the Carrier space Φ_A of A and C with the usual topology.

Lemma 3.4. If W is a neighborhood of $(\varphi_o,\lambda_o)\in\Phi_B$, then there eixsts a M-neighborhood $W_o\subset W$ of (φ_o,λ_o) ([10]).

Proof. We may put

$$W=V\times\{\lambda\in C\mid|\lambda-\lambda_o|<\varepsilon\},$$

where V is an open neighborhood φ_o in Φ_A and $\varepsilon>0$. Suppose that λ_o is a root of

$$\varphi_o(\alpha_o)+\varphi_o(\alpha_1)x+\cdots+\varphi_o(\alpha_{n-1})x^{n-1}+x^n=0$$

with its multiplicity m. By the Weierstrass preparation theorem there exists $t>0$ and $s>0$ such that an equation

$$z_o+z_1x+\cdots+z_{n-1}x^{n-1}+x^n=0\quad(z_i\in C)$$

has just m roots (miltiplicities counted) in $\{z\in C\mid |z-\lambda_o|<s\}$ if $|z_i-\varphi_o(\alpha_i)|<t$ for $i=0,1,\cdots,n-1$.

We take an open neighborhood V_o of φ_o such that

$$V_o=\{\varphi\in\Phi_A\mid |\varphi(\alpha_i)-\varphi_o(\alpha_i)|<t \text{ for } i=0,\cdots,n-1\},$$

then $V \cap V_o$ is also an open neighborhood of φ_o. Note that for each $\varphi \in V \cap V_o$ the number of roots (multiplicities counted) of the equation

$$\varphi(\alpha_o)+\varphi(\alpha_1)x+\cdots+\varphi(\alpha_{n-1})x^{n-1}+x^n=0$$

such that $|\text{each root}-\lambda_o|<\varepsilon=s$ is just m. We put

$$W_o=V \cap V_o \times \{\lambda \in C \mid |\lambda-\lambda_o|<\varepsilon\},$$

then W_o is a required M-neighborhood of (φ_o, λ_o). ■

Proposition 3.5. $\Pi : \Phi_B \longrightarrow \Phi_A((\varphi,\lambda) \longmapsto \varphi)$ is a contineous and open map. Furthermore $\Pi(\Phi_B)=\Phi_A$.

Proof. For each $\varphi \in \Phi_A$ we always have a root λ in C of

$$\varphi(\alpha_o)+\varphi(\alpha_1)x+\cdots+\varphi(\alpha_{n-1})x^{n-1}+x^n=0.$$

Then $(\varphi,\lambda) \in \Phi_B$, and thus $\Pi : \Phi_B \longrightarrow \Phi_A$ is surjective.

For each $\varphi \in \Phi_A$ take an open neighborhood

$$\begin{aligned} &U(\varphi : \varepsilon : x_1,\cdots,x_n) \quad (\varepsilon>0, x_1,\cdots,x_n \in A) \\ &=\{\eta \in \Phi_A \mid |\varphi(x_i)-\eta(x_i)|<\varepsilon \text{ for } i=1,\cdots,n\} \text{ of } \varphi. \end{aligned}$$

Then

$$\begin{aligned} &\Pi^{-1}(U(\varphi : \varepsilon : x_1,\cdots,x_n)) \\ &=\{(\omega,\lambda) \in \Phi_B \mid |\omega(x_i)-\varphi(x_i)|<\varepsilon \ {}^{\forall} i=1,\cdots,n \\ &\quad \text{and } \alpha_\omega(\lambda)=\sum_{i=0}^{n}\omega(\alpha_i)\lambda^i=0\} \end{aligned}$$

which is an open neighborhood of $(\varphi,\lambda_o) \in \Pi^{-1}(\varphi) \subset \Phi_B$, where $\alpha_\varphi(\lambda_o)=0$. This implies that Π is contineous.

Let W be an open neighborhood of $(\varphi,\lambda) \in \Phi_B$. Then, by Lemma 3.4, there exists a M-neighborhood $W_o(\subset W)$ of (φ,λ) which was constructed as in the proof of Lemma 3.4. Then $\Pi(W_o)=V_o$ which is an open neighborhood of $\Pi(\varphi,\lambda)=\varphi$. ■

Proposition 3.6. (i) For each $\varphi \in \Phi_A(\alpha_\varphi(x) \not\equiv 0)$ there are disjoint M-neighborhoods $V_1,\cdots,V_m (m \leq n)$ in Φ_B of the points $\Pi^{-1}(\varphi)=\{(\varphi,\lambda_1),\cdots,(\varphi,\lambda_m)\}$ such that $\Pi(V_1)=\Pi(V_i)$, $i=2,\cdots,m$ and $\Pi^{-1}(\Pi(V_1))=\bigcup_{i=1}^{m} V_i\}$. (ii) $\partial B=\Pi^{-1}(\partial A)$.

Proof. (i) Since

$$\alpha_\varphi(x)=\varphi(\alpha_0)+\varphi(\alpha_1)x+\cdots+\varphi(\alpha_{n-1})x^{n-1}+x^n \not\equiv 0.$$

We have different roots $\lambda_1, \cdots, \lambda_m (m \leq n)$ of the equation $a_\varphi(x) = 0$. Then $\Pi^{-1}(\varphi) = \{(\varphi, \lambda_1), \cdots, (\varphi, \lambda_m)\} \subset \Phi_B$.

Since Φ_A and Φ_B are compact and Hasudorff spaces ([2],[8],[12]) there exist open M-neighborhoods W_i of (φ, λ_i) $(i=1,2,\cdots,m)$ such that

$$W_i \cap W_j = \begin{cases} \phi \text{ if } i \neq j \\ W_i \text{ if } i = j. \end{cases}$$

Since Π is an open mapping

$$\bigcap_{i=1}^{m} \Pi(W_i) = \Omega \subset \Phi_A$$

is an open neighborhood of φ. If we put

$$\Pi^{-1}(\Omega) \cap W_i = V_i$$

then $V_1, \cdots$ and V_m are required M-neighborhoods.

(ii) At first, we shall prove that $\Pi^{-1}(\partial A) \subset \partial B$. For $\varphi_o \in \partial A$ we take an open neighborhood W_o of $(\varphi_o, \lambda_o^{(1)}) \in \Pi^{-1}(\varphi_o)$ and $\hat{g} \in \hat{B}$ such that

$$\hat{g}(\varphi_o, \lambda_o^{(1)}) = 1, \ \hat{g}(\varphi_o, \lambda_o^{(i)}) = 0 \ (i=2,3,\cdots,m)$$

where $\lambda_o^{(1)}, \cdots,$ and $\lambda_o^{(m)}$ are different roots of

$$\varphi_o(\alpha_o) + \varphi_o(\alpha_1)x + \cdots + \varphi_o(\alpha_{n-1})x^{n-1} + x^n = 0.$$

Then, we have an open neighborhood $W_1 (\subset W_o)$ of $(\varphi_o, \lambda_o^{(1)})$ such that for all $(\varphi, \lambda) \in W_1$ $|g(\varphi, \lambda)| > \frac{1}{2}$. Let W_i $(i=2,\cdots,m)$ be an open neighborhood of $(\varphi_o, \lambda_o^{(i)})$ such that for all $(\varphi, \lambda) \in W_i$ $|\hat{g}\varphi, \lambda| < \frac{1}{2}$. Put

$$V_o = \bigcap_{i=1}^{m} \Pi(W_i) \text{ and } V_i = W_i \cap \Pi^{-1}(V_o) \text{ for } i=1,2,\cdots,m.$$

Then V_i is an open neighborhood of $(\varphi_o, \lambda_o^{(i)})$ $(i=1,2,\cdots,m)$. By (ii) of Proposition 2.9, we have a function $f \in \hat{A}$ such that $\|f\|_\infty = 1$, $|f(\varphi)| = 1$ for $\varphi \in V_o$ and for $\varphi \in \Phi_A - V_o$ $|f(\varphi)| < 1$. Since $\Phi_A - V_o$ is compact and thus there exists a positive integer N such that

$$|f(\varphi)|^N \leq \frac{1}{2\|g\|_\infty} \text{ for all } \varphi \in \Phi_A - V_o.$$

We can define f, g as follows :

$$\text{for any } (\varphi,\lambda)\in\Phi_B \ (fg)(\varphi,\lambda)=f(\varphi)g(\varphi,\lambda)$$

Thus, for $\varphi\notin V_o$ and $(\varphi,\lambda)\in\Phi_B$ we have the following :

$$\begin{aligned}|(f^N g)(\varphi,\lambda)| &= |f(\varphi)|^N \ |g(\varphi,\lambda)| \\ &\le \frac{1}{2\|g\|_\infty} \|g\|_\infty = \frac{1}{2}.\end{aligned}$$

For $\varphi\in V_o$ and $(\varphi,\lambda)\notin V_1$

$$|(f^N g)(\varphi,\lambda)| = |f^N(\varphi)| \, |g(\varphi,\lambda)| \le \frac{1}{2}.$$

But, for $(\varphi_1,\lambda_1)\in V_1$

$$|(f^N g)(\varphi_1,\lambda_1)| = |f(\varphi_1)|^N \ |g(\varphi_1,\lambda_1)| > \frac{1}{2}.$$

This implies that the maximal value of $|f^N g|$ is on V_1, and hence on W_1. On the other hand, the value of $|f^N g|$ outside W_1 is less than its maximal value. By (ii) of Proposition 2. 9, $(\varphi_o,\lambda_o^{(1)})\in\partial B$ and thus $\Pi^{-1}(\partial A)\subset\partial B$.

Conversely, we take $(\varphi_o,\lambda_o)\in\partial B$ and an open neighborhood V_o of $\varphi\in\Phi_A$. Let W be an open neighborhood of (φ_o,λ_o) such that $\Pi(W)\subset V_o$ and $(\varphi_o,\lambda_o')\notin W$ if $\lambda_o\neq\lambda_o'$. Then by (ii) of Proposition 2. 9, there exists a function $g\in\hat{B}$ such that

$$\|g\|_\infty = |g(\varphi,\lambda)| \ \text{ for } (\varphi,\lambda)\in W \text{ and}$$
$$|g| < \|g\|_\infty \text{ outside } W.$$

We may assume that

$$\forall(\varphi,\lambda)\in\Phi_B - W \quad |g(\varphi,\lambda)| < \frac{1}{2n}.$$

Let f be a function defined by

$$f(\varphi)=\sum_{i=1}^{n} g(\varphi,\lambda_i(\varphi)),$$

where $\varphi\in\Phi_A$ and the $\lambda_i(\varphi)$ $(i=1,\cdots,n)$ denote all roots of

$$\varphi(\alpha_0)+\varphi(\alpha_1)x+\cdots+\varphi(\alpha_{n-1})x^{n-1}+x^n=0.$$

Then there exists an element $a\in A$ such that $\hat{a}=f$ ([9]). It is clear that for $\varphi\notin\Pi(W)$

$$|f(\varphi)|=|\sum_{i=1}^{n} g(\varphi,\lambda_i(\varphi))|\leq\sum_{i=1}^{n} g(\varphi,\lambda_i(\varphi))|<\frac{1}{2}.$$

We assume that $\|g\|_\infty=|g(\varphi_1,\lambda_1(\varphi_1))|\quad((\varphi_1,\lambda_1(\varphi_1))\in W)$.

Then we have the following

$$\begin{aligned}|f(\varphi_1)|&=|g(\varphi_1,\lambda_1(\varphi_1))+\sum_{i=2}^{n} g(\varphi_i,\lambda_i(\varphi_i))|\\&>|g(\varphi_1,\lambda_1(\varphi_1))|-\sum_{i=2}^{n} g(\varphi_1,\lambda_i(\varphi_1))\\&>1-\frac{n-1}{2n}>\frac{1}{2}.\end{aligned}$$

Thus $\varphi_0\in\partial A$ and $\Pi(\partial B)\subset\partial A$. ■

Theorem 3.7. If A is regular or self-adjoint then $\partial B=\Phi_B$.

Proof. It is sufficient to prove $\partial A=\Phi_A$ because of that by proposition 3.6, $\Pi^{-1}(\partial A)=\Pi^{-1}(\Phi_A)=\Phi_B=\partial B$.

Let A be regular. Then $\hat{A}$ is dense in $C(\Phi_A)$ (see property 2.6) ([2],[12]). We have to note that $C(\Phi_A)$ is a regular commutative Banach algebra with sup norm ([8]).

For an element $\varphi\in\Phi_A$ and an open neighborhood U of φ we have $g\in C(\Phi_A)$ such that $g(\varphi)=1$ and for $\eta\in\Phi_A-U\quad g(\eta)=0$ by (ii) of Definition 2.7, because that $C(\Phi_A)$ is regular and $\Phi_A=\Phi_{C(\Phi_A)}$. Since $\hat{A}$ is dense in $C(\Phi_A)$ we have $f\in\hat{A}$ such that $\|f-g\|_\infty<\frac{1}{2}$. Hence, for $\eta\in\Phi_A-U\quad|f(\eta)|<\frac{1}{2}$. By (ii) of Proposition 2.9, $\varphi\in\partial A$. That is, $\partial A=\Phi_A$.

Let A be self-adjoint. We have to note that if A is closed under complex conjugation then $\partial A=\Phi_A$ ([8]). By Definition 2.7, since A is closed under complex conjugation we have $\partial A=\Phi_A$. ■

4. Separable Extensions over A

Let B be a commutative separable A-algebra such that it is projective as an A-module in this section. By Lemma 2.4, B is a finitely generated, projective A-module. Throughout this section we assume that A has only two idempotents 0 and 1 and A is a commutative Banach algebra.

Proposition 4.1. There are at most $\text{rank}_A(B)$ A-algebra homomorphisms from B to A.

Proof. By Proposition 2.5, $\mathrm{rank}_A(B)$ is well defined. At first, for each A-algebra homomorphism $f_i : B \longrightarrow A (i=1,2,\cdots)$ we shall prove that there exists an idempotent $e_i (i=1,2,\cdots)$ such that (a) $^{\forall}b \in B$ $be_i = f_i(b)e_i$ and $f_i(e_i)=1$

(b) $e_i e_j = \delta_{ij} e_j$ and $f_i(e_j) = \delta_{ij}$.

Let $e = \sum_j x_j \otimes y_j$ be the separability idempotent for B(see Proposition 2.3) and let $\mu : B \otimes_A B \longrightarrow B$ defined by $\mu(x \otimes y) = xy$ for all $x \otimes y \in B \otimes_A B$. For an A-algebra homomorphism $f_i : B \longrightarrow A$ we put

$$e_i = \sum_j f_i(x_j) y.$$

Suppose an A-algebra homomorphism $\beta : B \otimes_A B \longrightarrow B$ defined by $\beta(b \otimes b') = f(b)b'$ for each $b \otimes b' \in B \otimes_A B$. Since e is an idempotent of $B \otimes_A B$ by Proposition 2.3 form

$$\beta(e^2) = \beta(e)\,\beta(e) = \beta(e) \text{ and } \beta(e) = e_i$$

e_i is an idempotent of B. Moreover, for any $b \in B$

$$\begin{aligned} be_i &= 1 \otimes b\mu(\textstyle\sum_j f_i(x_j) \otimes y_j) \quad (B \cong 1 \otimes_A B) \\ &= \mu(f_i \otimes 1)\,(\textstyle\sum_j x_j \otimes by_j) \\ &= \mu(f_i \otimes 1)\,(\textstyle\sum_j bx_j y_j) \quad ((1 \otimes b)e = (b \otimes 1)e) \\ &= \textstyle\sum_j f_i(x_j b) y_j = f_i(b) \sum_j f_i(x_j) y_j = f_i(b) e_i. \end{aligned}$$

Next,

$$\begin{aligned} f_i(e_i) &= f_i(\textstyle\sum_j f_i(x_j) y_j) = \sum_j f_i(x_j)\, f_i(y_j) \\ &= f_i(\textstyle\sum_j x_j y_i) = f_i(1) = 1 \quad \text{(Definition 2.2)}. \end{aligned}$$

Let f_i $(i=1,2,\cdots)$ be A-algebra homomorphisms from B to A and let $e_i (i=1,2,\cdots)$ be the corresponding idempotents. Then $f_i(e_j)$ is an idemptent of A and thus $f_i(e_j) = 1$ or 0. If $f_i(e_j) = 1$ $(i \neq j)$, for each $b \in B$

$$\begin{aligned} f_i(b) &= f_i(b \cdot 1) = f_i(b)\, f_i(1) = f_i(b)\, f_i(e_j) \\ &= f_i(be_j) = f_i(f_j(b)e_j) \quad \text{(by (a) above)} \\ &= f_j(b)\, f_i(e_j) = f_j(b). \end{aligned}$$

Hence we have $f_i = f_j$. Thus $f_i(e_j) = \delta_{ij}$. By (a) above

$$e_i e_j = f_j(e_i) e_j = \delta_{ij} e_j.$$

Let $f_1, \cdots$, and f_m be distinct algebra homomorphisms from B to A. Then, by the above description we have idempotents $e_1, \cdots, e_m$ such that

$$f_i(e_j)=\delta_{ij},\ e_ie_j=\delta_{ij}e_j,\ be_i=f_i(b)e_i\quad (1\leq i,\ j\leq m \text{ and } b\in B).$$

Since $\quad {}^{\forall}a\in A\quad ae_i=0 \Longleftrightarrow af_i(e_i)=a=0\quad (f_i(e_i)=1\in A)$

it is clear that

$$Be_i\cong Ae_i\cong A.$$

Thus, we have the following decomposition of B :

$$\begin{aligned}B&=Be_1\oplus\cdots\oplus Be_m\oplus B(1-e_1,\cdots,e_m)\\ &\cong \underbrace{A\oplus\cdots\oplus A}_{m\text{-times}}\oplus B(1-e_1,\cdots,e_m)\end{aligned}$$

Hence, for any prime ideal $\mathscr{P}(\mathscr{P}\neq 0,\ \mathscr{P}\neq A)$ of A

$$B_{\mathscr{P}}=A_{\mathscr{P}}\otimes_A B\cong \underbrace{A_{\mathscr{P}}\oplus\cdots\oplus A_{\mathscr{P}}}_{m\text{-times}}\oplus A_{\mathscr{P}}\otimes_A B(1-e_1-\cdots-e_m)$$

That is, $\mathrm{rank}_A(B)\geq m$. In consequence the number of A-algebra homomorphisms from B to $A\leq \mathrm{rank}_A(B)$. ∎

In general, we can write

$$B=Be_1\oplus\cdots\oplus Be_m \quad\cdots\cdots\cdots\cdots(3)$$

where $e_1, \cdots$, and e_m are idempotents of B such that $e_1+\cdots+e_m=1$ and $m\leq \mathrm{rank}_A(B)$. In this case Be_i $(i=1,\cdots,m)$ is a projective separable extension over Ae_i with no idempotents other than 0 and $e_i (i=1,\cdots,m)$.

Theorem 4.2. There is a norm in B such that B is a Banach algebra.

Proof. Since B is a finitely generated A-module (as before) each element of B is integral over A([11]). Hence, there exist elements $b_1,\cdots,b_m$ in B that generate B as an A-algebra and there are monic polynomials $f_1(x),\cdots,f_m(x)$ in $A[x]$ such that $f_i(b_i)=0$ $(i=1,\cdots,m)$. We put

the degree of $f_i(x)=d_i$ $(i=1,\cdots,m)$

and

$$B_0=A,\ B_1=B_0[x]/(f_1(x)),\cdots,\ B_i=B_{i-1}[x]/(f_i(x)) \text{ for } i=1,2,\cdots,m.$$

Then, by (ii) of Lemma 3.1 we have a norm in B_1 such that B_1 is a Banach algebra. Repeatting of (ii) of Lemma 3.1 $B_0, B_1, \cdots$, and B_m are Banach algebras.

If we put $A^{(n)}=A\oplus\cdots\oplus A$ (n-times) and define

$$\|(a_1,\cdots,a_n)\|=\|a_1\|+\cdots+\|a_n\| \quad ((a_1,\cdots,a_n)\in A^{(n)})$$

then it is obvious that $A^{(n)}$ is a Banach algebra. By Lemma 3.1 it is clear that B_1 is isometric to $A^{(d_1)}$. Similarly, B is isometric to $A^{(d_1\cdot d_2\cdots d_i)}(i=0,1,\cdots,m)$. That is, $B_m\cong A^{(d_1\cdots d_m)}$. Since

$$A^{(d_1\cdots d_m)}\cong A[x_1,\cdots,x_m]/(f_1(x_1),\cdots,f_m(x_m))$$

we have an A-algebra homomorphism

$$f: B_m\cong A[x_1,\cdots,x_m]/(f_1(x_1),\cdots,f_m(x_m))\longrightarrow B$$

defined by $x_i\longmapsto b_i$ $(i=1,\cdots,m)$. Then f is an epimorphism. Since B is a projective A-module the sequence of A-modules :

$$0\longrightarrow \text{Ker } f\longrightarrow B_m\xrightarrow{\ f\ }B\longrightarrow 0 \quad \text{is split.}$$

Since Ker f is a closed ideal of B_m

$$B\cong B_m/\text{Ker } f$$

is a Banach algebra with its norm $\|\cdot\|$ such that

$$^{\forall}b\in B \quad b=b_m+\text{Ker } f\ (b_m\in B_m)\Longrightarrow \|b\|=\inf_{a\in \text{Ker } f}\|b_m+a\| \quad ([8]). \quad \blacksquare$$

Recall that Φ_B is the Carrier space of B. Then the projection

$$\Pi : \Phi_B \longrightarrow \Phi_A \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

is contineous (a proof is the same as the proof of Lemma 3.5). For each element $\psi\in\Phi_B$ we put $\varphi=\Pi(\psi) : A\longrightarrow \boldsymbol{C}$, put Ker $\varphi=m_\varphi$. By (3) above

$$B/m_\varphi B=e_1(B/m_\varphi B)\oplus\cdots\oplus e_m(B/m_\varphi B)$$

where e_i is an idempotent such that $e_1+\cdots+e_m=1$. In particular $e_i(B/m_\varphi B)$ is a finite dimensional $e_i(A/m_\varphi)$ $(\cong C)$-algebra. Since $\Pi^{-1}(\varphi)=\{\psi\in\Phi_B \mid m_\varphi B\subset \text{Ker}\,\psi\}$ we have the Carrier space of $e_i(B/m_\varphi B)=e_i(\Pi^{-1}(\varphi))\approx\{e_i\psi \mid \psi\in\Pi^{-1}(\varphi)\}\approx\Pi^{-1}(\varphi)$. For $\psi\in\Phi_B$ we assume that $e_i\notin$ Ker ψ and $\{e_1,\cdots,e_{i-1}, e_{i+1},\cdots,e_m\}\subset$Ker ψ, then the support of $\hat{e}_i$ is just $\{\psi\}$. In this case we put $m(\psi)=$the complex dimension of $e_i(B/m_\varphi B)$.

For a monic polynomial $\alpha(x)=\alpha_0+\alpha_1 x+\cdots+\alpha_{n-1}x^{n-1}+x^n\in A[x]$ we put the Banach algebra $A[x]/(\alpha(x))=A_\alpha$(see §3). Moreover we put the following :

(i) $\Pi_\alpha : \Phi_{A_\alpha}\longrightarrow\Phi_A$ is open and contineous (Proposition 3.5)

(ii) ${}^{\forall}\varphi\in\Phi_A$ $Z(\alpha_\varphi)=\{\lambda\in C \mid \alpha_\varphi(\lambda)=\varphi(\alpha_0)+\cdots+\varphi(\alpha_{n-1})\lambda^{n-1}+\lambda^n=0\}$

(iii) $M(\lambda,\alpha_\varphi)=$the multiplicity of λ in $\alpha_\varphi(\lambda)=0$.

Lemma 4.3. For each $b\in B$ there exists a monic polynomial $\alpha(x)\in A[x]$ with degree$=\text{rank}_A(B)=n$, and a contineous and open map $f^* : \Phi_B\longrightarrow\Phi_A$ such that

(a) f^* is onto and $\Pi_\alpha\circ f^*=\Pi$

(b) $M(\psi(b), \alpha_{\pi(\psi)})=\Sigma(\theta)$, where $\psi\in\Phi_B$, the sum being taken over those $\theta\in\Phi_B$ satisfying $f^*(\theta)=f^*(\psi)$. In partcular, Π is open and contineous.

Proof. Let us define an A-endomorphism $u_b : B\longrightarrow B$ by $u_b(c)=cb$ for all $c\in B$, since B is a finitely generated projective A-module with rank n and u_b is an A-module homomorphism, we have the characteristic polynomial of u $\alpha(x)=\det(xI-\imath_b)$ such that $\alpha(b)=0$ and the degree of $\alpha(x)=n$ ([7]).

Define an A-algebra homomorphism

$$f : A_\alpha\longrightarrow B \text{ by } f(x+(\alpha(x))=b.$$

Since

$$(\varphi,\lambda)\in\Phi_{A_\alpha} \text{ and } a(x)+(\alpha(x))\in A_\alpha,\ a(x)=\sum_{i=0}^{n-1}a_i x^i\in A[x],$$

$$(\varphi,\lambda)\ (a(x)+(\alpha(x)))=\sum_{i=0}^{n-1}\varphi(a_i)\lambda^i,$$

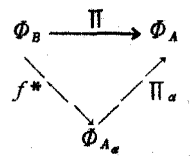
we know that for each $\varphi\in\Phi_B$

$$f^*(\psi)\ (a(x)+(\alpha(x)))=\psi\Big(\sum_{i=0}^{n-1}a_i b^i\Big)$$

$$=\sum_{i=0}^{n-1}(\Pi(\psi)a_i)\ (\psi(b))^i$$

$$=(\Pi(\psi),\ \psi(b))\ (a(x)+(\alpha(x))).$$

Therefore $f^*(\psi)=(\Pi(\psi),\ \psi(b))\in\Phi_{A_\alpha}$ and we have the following commutative diagram



Put $\Pi(\psi)=\varphi\in\Phi_A$. Then, as before,

$$B/m_\varphi B=e_1(B/m_\varphi B)+\cdots+e_m(B/m_\varphi B)$$

(e_i : idempotent, $e_1+\cdots+e_m=1$, $i\neq j\Longrightarrow e_ie_j=0$). Then we have get *a* C-endomorphism

$$u_b : B/m_\varphi B\longrightarrow B/m_\varphi B \text{ induced from the}$$

A-endomorphism u_b such that $u_b(e_i(B/m_\varphi B))\subset e_i(B/m_\varphi B)$.
We assume that $\psi_i\in\Pi^{-1}(\varphi)$ is the support of $\hat{e}_i$.

Moreover, we can regard such that

$$Ce_i\subset e_i(B/m_\varphi B) \text{ and } u_b(e_i)=be_i=\psi_i(b)e_i=\lambda_ie_i,$$

and thus there is a unique eigenvalue of $\lambda_i=\psi_i(b)$ of $u_b|e_i(B/m_\varphi B)$. Thus

$$a_\varphi(x)=\prod_{i=1}^{m}(x-\psi_i(b))^{n(\psi_i)} \quad\cdots\cdots(5)$$

([6],[9],[14]). Since f^* is surjective if and only if for each element $(\varphi,\lambda)\in\Phi_{A_a}$ $\lambda=\psi_i(b)$ for some $\psi_i\in\Pi^{-1}(\varphi)$

$$(\varphi,\lambda)\in\Phi_{A_a}\Longleftrightarrow a_\varphi(\lambda)=0\Longleftrightarrow\exists i(1\leq i\leq m)\ \lambda=\psi_i(b)\ (\psi_i\in\Pi^{-1}(\varphi)).$$

Thus (5) proves (a). Since for ψ and $\theta\in\Phi_B$ $f^*(\psi)=f^*(\theta)$ is equivalent to $\psi(b)=\theta(b)$ and $\Pi(\psi)=\Pi(\theta)$, it is clear that (b) is true.

Since Π is contineous and Π_a is open (Proposition 3.5 and (4) above) f^* is contineuous. Moreover, since

$$f^*(U(\psi:\varepsilon:b_1,\cdots,b_l))=U\ ((\Pi(\psi),\psi(b)):\varepsilon:a^{(1)}(x)$$
$$+(\alpha(x)),\cdots,a^{(l)}(x)+(\alpha(x))),$$

where $f(a^{(i)}(x)+(\alpha(x)))=b_i$, f^* is an open map. ∎

Theorem 4.4. For each $\psi_1\in\Phi_B$, $\Pi^{-1}(\Pi(\psi_i))=\{\psi_1,\cdots,\psi_m\}$ for some positive integer m. In this case, given any neighborhood W of ψ_1 there exist disjoint neighborhoods V_i of $\psi_i(i=1,\cdots,m)$ and a neighborhood U of $\Pi(\psi_i)$ such that

(i) $\Pi(V_i)=U$ for all $i=1,\cdots,m$, $\Pi^{-1}(U)=\bigcup_{i=1}^{m} V_i$ and $V_1\subset W$,

(ii) $m(\psi_i)=\sum_{\theta\in\pi^{-1}(\varphi)\cap V_i} m(\theta)$, $\varphi\in U$, $i=1,\cdots,m$.

Moreover $\Pi:\Phi_B\longrightarrow\Phi_A$ is open and contineuous.

Proof. Since B is separable and projective extension of A, by (3) above there are mutually orthogonal idempotents $e_1,\cdots,e_p$ of B such that $e_1+\cdots+e_p=1$. In this case e_iB is a finitely generated projective extension of e_iA with a well defined rank over e_iA $(i=1,2,\cdots,p)$ (Proposition 2.5. Note that the idempotents of e_iA are only two elements 0 and e_i). We may assume that $\mathrm{rank}_A(B)=n$.

Put $B_0=B\otimes_A C(\Phi_A)$. Then $\Phi_{B_0}\approx\Phi_B\otimes_{\Phi_A}\Phi_{C(\Phi_A)}$ ([3], [11]). By (iii) of Property 2.6, since $\Phi_{C(\Phi)}\approx\Phi_A$ we have $\Phi_{B_0}=\Phi_B$. Moreover, the algebra homomorphism

$$g:B\longrightarrow B_0=B\otimes_A C(\Phi_A)\quad(b\longmapsto b\otimes 1)$$

preserves the multiplicity and B_0 is a finitely generated projective extension over $C(\Phi_A)$. Thus we may assume that $A=C(\Phi_A)$ ([14]).

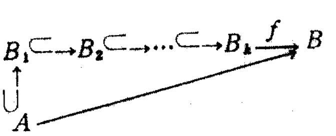
As in the proof of Theorem 4.2, let $\{b_1,\cdots,b_k\}$ be a set of generators of B over A, and let $\alpha_1(x),\cdots,\alpha_k(x)$ be monic polynomials in $A[x]$ such that $\alpha_i(b_i)=0$ $(i=1,\cdots,k)$ We put as follows :

$$B_0=A,\ B_1=B_0[x]/(\alpha_1(x)),\cdots,\ B_i=B_{i-1}[x]/(\alpha_1(x))$$

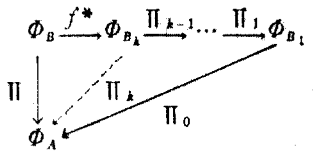
for $i=1,\cdots,k$. Then we have a contineous homomorphism $B_k\longrightarrow B$. Since $\hat{B}_1$ is uniformly dense in $C(\Phi_{B_1})$, we see that $\hat{B}_k$ is uniformly dense in $C(\Phi_{B_k})$ using k-times the above fact. From this it follows that $\hat{B}$ is uniformly dence in $C(\Phi_B)$ ([14]). Hence for an element $b\in B$ and $\varepsilon>0$ we may consider that $\hat{b}$ separates the points of $\Pi^{-1}(\Pi(\psi_1))$. Assume that

$$U(\psi:\varepsilon:b)=\{\theta\in\Phi_B\mid\ |\psi_1(b)-\theta(b)|<\varepsilon\}\subset W.$$

From the commutative diagram



we have the commutative diagram



Thus $\Pi^{-1}(\Pi(\psi_1))=f^{*^{-1}}\cdots\Pi_1^{-1}\ \Pi_0^{-1}(\Pi(\psi))=\{\psi_1,\cdots,\psi_m\}$. By Lemma 4.3, for each $b\in B$ there exists a contineous and open map $f^*:\Phi_B\longrightarrow\Phi_{B_k}$ such that

$$U(\psi_1:\varepsilon,b)=f^{*^{-1}}(U(f^*(\psi_1):\varepsilon:[x]=x+(\alpha_m(x)).$$

We put $\Pi_\alpha=\Pi_0\Pi_1\cdots\Pi_{k-1}$. Then By (i) of Proposition 3.6 there are mutually disjoint M-neighborhoods $V_1^*,\cdots,V_m^*$ of the points $f^*(\psi_1),\cdots,f^*(\psi_m)$, respectively, such that

$$\begin{cases}\Pi_\alpha(V_i^*)=\Pi_\alpha(V_1^*)\ \text{for}\ i=2,3,\cdots,m.\\ \Pi^{-1}(\Pi_\alpha(V_1^*))=V_1^*\cup\cdots\cup V_m^*\\ V_1^*=U(f^*(\psi_1);\ \varepsilon;\ [x])\end{cases}$$

If we put such $U=\Pi_\alpha(V_1^*)$, $V_i=f^{*^{-1}}(V_i^*)$ $(i=1,\cdots,m)$ then (i) is proved. (Note that Π is contineuous and open by Proposition 3.5, and thus U is open in Φ_A). By (ii) of Property 2.6 $\hat{b}(b\in B)$ separates the points of $\Pi^{-1}(\Pi(\psi_1))$, for each element $\varphi\in U$

$$m(\psi_i)=M(\psi_i(b),\ \alpha_{\pi(\psi_i)})=\sum_{\substack{\lambda=\theta(b)\\ \theta\in\pi^{-1}(\varphi)\cap V_i}} M(\lambda,\alpha_\varphi)$$

by Definition 3.4 of M-neighborhood.

Moreover, by (ii) of Lemma 4.3

$$M(\lambda,\alpha_\varphi)=\sum_{\eta\in\{\theta\in\pi^{-1}(h)\mid\theta(b)=\lambda\}} m(\eta)=\sum_{\theta\in\pi^{-1}(\varphi)\cap V_i} m(\theta)$$

and which completes the proof of (ii).

By Lemma 4.3

$$\Pi=\Pi_\alpha\circ f^*,$$

and since Π_α and f^* are open and contineous Π is also a contineous and open map. ■

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