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On The Number of Replications in Simulatioin Study

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ABSTRACT

A method which determines the number of replications in the simulation is proposed, particularly for small-sample comparison of estimators. This method takes the smallest number of replications that makes the difference of mean square errors be statistically significant and provides an efficient algorithm for calculating the standard error of the mean square error. Two examples are illustrated, the first one is on comparison of mean and median ; the second, the Kaplan-Meier type and Buckley-James type estimators of a quantile function with censored data.

1. Introduction

Let θ be a parameter of interest, and let $\hat{\theta} = \hat{\theta}(X_1, \cdots, X_n)$ be an estimator of θ based on a random sample of size n, $X_1, X_2, \cdots, X_n$. When the small-sample properties of $\hat{\theta}$ are compared with those of other estimators, the Monte Carlo simulation is widely used. In this case the following question is naturally arisen : How many replications are performed in simulation ?

The purpose of this note is to answer partially the above question through comparing the mean square error(MSE) of $\hat{\theta}$ with those of others. In section 2 a method which determines the number of replications is proposed, two examples are given for illustrating the proposed method in section 3.

2. A Proposed Method

Let $\hat{\theta} = \hat{\theta}(X_1, \cdots, X_n)$ be an estimator of the parameter of interest θ based on a random sample $X_1, X_2, \cdots, X_n$. If we make R replications of

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$\hat{\theta},\ \hat{\theta}(1),\ \hat{\theta}(2),\ \cdots,\hat{\theta}(R)$, by simulation, the estimator of MSE($\hat{\theta}$) is given by

$$\widehat{MSE}(\hat{\theta}) = \hat{E}(\hat{\theta}-\theta)^2 = \frac{1}{R}\sum_{r=1}^{R}[\hat{\theta}(r)-\theta]^2 . \quad (1)$$

As $R \to \infty$, $\widehat{MSE}(\hat{\theta})$ converges to MSE($\hat{\theta}$) and is asymptotically normally distributed with the asymptotic variance $\frac{Var(\hat{\theta}-\theta)^2}{R}$ provided that $Var(\hat{\theta}-\theta)^2$ is finite.

Consider an estimator of $Var(\hat{\theta}-\theta)^2$ defined by

$$\widehat{Var}(\hat{\theta}-\theta)^2 = \frac{1}{R-1}\sum_{r=1}^{R}[(\hat{\theta}(r)-\theta)^2 - \widehat{MSE}(\hat{\theta})]^2 . \quad (2)$$

Then we have the following results.
Fact : If the 8-th moment of $\hat{\theta}$ is finite, then $\sqrt{R}[\widehat{Var}(\hat{\theta}-\theta)^2 - Var(\hat{\theta}-\theta)^2]$ weakly converges to $N(0, E(\hat{\theta}-\theta)^8 - (Var(\hat{\theta}-\theta)^2)^2)$ as R goes to infinity.

Suppose we have two alternative estimators, say $\hat{\theta}_1$ and $\hat{\theta}_2$. Then an estimator $\hat{\theta}_1$ is said to be " better " one if $MSE(\hat{\theta}_1) < MSE(\hat{\theta}_2)$. Here if we substitute $\widehat{MSE}(\hat{\theta}_i)$ for $MSE(\hat{\theta}_i)$, $i = 1,2$, it should be checked that $MSE(\hat{\theta}_1)$ is really smaller than $MSE(\hat{\theta}_2)$.

Let $\hat{\theta}_1(1),\ \hat{\theta}_1(2),\ \cdots,\hat{\theta}_1(R)$ and $\hat{\theta}_2(1),\ \hat{\theta}_2(2),\ \cdots,\hat{\theta}_2(R)$ be the independent replications of $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively. From equations (1) and (2), we consider the test statistic T defined by

$$T = \frac{\sqrt{R}|\widehat{MSE}(\hat{\theta}_1) - \widehat{MSE}(\hat{\theta}_2)|}{\sqrt{\widehat{Var}(\hat{\theta}_1-\theta)^2 + \widehat{Var}(\hat{\theta}_2-\theta)^2}}. \quad (3)$$

Then T is asymptotically t-distributed with degrees of freedom 2R - 2. Therefore for given $\alpha > 0$, we reject the hypothesis that $MSE(\hat{\theta}_1)$ is equal to $MSE(\hat{\theta}_2)$ if $T > t_{(2R-2,\frac{\alpha}{2})}$ at the level of significance α, where $t_{(\nu,\alpha)}$ is the upper α-th percentile point of the t-distribution with degrees of freedom ν.

Now we propose a method that determines the number of replications R in the following way : Take R as the smallest number that makes the difference of

$MSE(\hat{\theta}_1)$ and $MSE(\hat{\theta}_2)$ be statistically significant. Here is an algorithm for determining R :

Step 1. Generate $X_1, X_2, \cdots, X_n$ from $f(x;\theta)$.

Step 2. Calculate $\hat{\theta}_1$ and $\hat{\theta}_2$.

Step 3. Independently repeat step 1 and 2 an initial number of replications R_0 times, obtaining independent copies $\hat{\theta}_1(1), \cdots, \hat{\theta}_1(R_0)$ and $\hat{\theta}_2(1), \cdots, \hat{\theta}_2(R_0)$.

Step 4. Calculate T defined in (3) and perform the T − test for the hypothesis that two MSE's are equal. If do not rejected, then go to step 5, otherwise R is the sum of all replications $R_0, R_1, \cdots,$ and stop.

Step 5. Go to step 3 with an additional number of replications R_k obtaining independent copies $\hat{\theta}_1(\sum_{i=0}^{k-1} R_i + 1), \cdots, \hat{\theta}_1(\sum_{i=0}^{k} R_i)$ and $\hat{\theta}_2(\sum_{i=0}^{k-1} R_i + 1), \cdots, \hat{\theta}_2(\sum_{i=0}^{k} R_i)$.

Step 6. Go to step 4.

3. Two Examples

(1) Estimation of mean

Let θ be the population mean, and let $\hat{\theta}_1$ and $\hat{\theta}_2$ be the sample mean and median based on a random sample $X_1, \cdots, X_n$ from a d.f. $F(x;\theta)$. Then the asymptotic relative efficiencies of $\hat{\theta}_2$ to $\hat{\theta}_1$ were calculated for various F with $F(0) = \frac{1}{2}$ and $f(0) > 0$ (see Lehmann (1983)). For finite sample size n=20(20)300, the MSE's of $\hat{\theta}_1$ and $\hat{\theta}_2$ are compared and R($\leq$1000) is determined by the procedure described in section 2 with R_0=20 and $R_1 = \cdots = R_{98} = 10$. Table 1 summarizes the results of simulations. From Table 1, we see that

(i) the sample median $\hat{\theta}_2$ performs better than $\hat{\theta}_1$ for the cases of (4),(5),(6) and (7), i.e., for heavy-tailed distributions or t-distributions with small degrees of freedom.

(ii) in the case that two estimators highly significantly differ, R tends to be small, and when R becomes too small (< 20), we use the smaller α in order that R is not less than 20.

(iii) the ratio of variances gets close to the asymptotic relative efficiency as n increases.

(2) Estimation of quantiles with censored data

Let $T_1, T_2, \cdots, T_n$ be iid random variables with d.f. F, and let $C_1, C_2, \cdots, C_n$ be iid random variables with d.f. G. Suppose that two sequences $\{T_i\}_{i=1}^n$ and $\{C_i\}_{i=1}^n$ are independent. By random censoring model the true survival times T_i's are censored on the right by the censoring times C_i's so that we only observe the pairs $\{(X_i, \delta_i)\}_{i=1}^n$, where

$$X_i = min(T_i, C_i) \ \ and \ \ \delta_i = I(T_i \leq C_i), \quad i = 1, 2, \cdots, n \ .$$

Let Q(y) be the quantile function of the underlying d.f. F(t) defined by

$$Q(y) = inf\{t : F(t) \geq y\}$$

In this example, two estimators are compared : the first one is the Kaplan-Meier (1958) type estimator $\hat{Q}_1(y)$ defined by

$$\hat{Q}_1(y) = inf\{t : \hat{F}_n(t) \geq y\} \ ,$$

where $\hat{F}_n(t)$ is the product-limit estimator of F(t), and the second, the Buckley-James (1979) type estimator $\hat{Q}_2(y)$ given by

$$\hat{Q}_2(y) = inf\{t : \hat{F_n}^*(t) \geq y\}$$

where

$$\hat{F_n}^*(t) = \frac{1}{n}\sum_{i=1}^{n} I(\hat{X_i}^* \leq t) \quad and$$

$$\hat{X_i}^* = X_i\delta_i \ + \ \hat{E}(T_i|T_i > C_i)(1 - \delta_i),$$

where $\hat{E}(T_i|T_i > C_i) = C_i + \frac{\int_{c_i}^{\infty}(1-\hat{F}_n(x))\,dx}{1-\hat{F}_n(x))}$. The simulations are performed for the following three distributions (1) F : Exp(1) and G : Exp(0.111) (censoring rate = 10 %), (2) F : Weib(1,0.5) and G : Exp(0.374) (censoring rate = 30 %), (3) F : Weib(2,4) and G : Exp(0.936) (censoring rate = 50 %). Here Exp(λ) stands for the exponential distribution with mean $\frac{1}{\lambda}$ and Weib(λ, α), the Weibull distribution whose d.f. are defined by

$$F(t;\lambda,\alpha) = 1 - e^{-\lambda t^{\alpha}}, \quad t > 0,\ \alpha,\ \lambda > 0 .$$

For n=20, 40, 60, 100, $\hat{Q}_1(y)$ is compared with $\hat{Q}_2(y)$ for y=0.1(0.05)0.95 and R($\leq$1000) is determined with R_0=30 and $R_1 = \cdots = R_{97}$=10. (See Table 2)

From Table2, we see that

(i) $\hat{Q}_2(y)$ is better than $\hat{Q}_1(y)$ for large y in cases of (1) and (2), and for middly y in (3).

(ii) R tends to decrease as n increases.

4. Discussions

We, in this note, proposed the method to determine the number of replications R in simulation, and the method was applied to two examples. We see that the method may not be used only for determining R since R depends on the underlying distribution, the sample size, the level of significance and an estimator of interest, and fluctuates simulation by simulation. However we can remarkably reduce computing time and can compare a relative accuracy of estimators. Futhermore the efficient algorithm for calculating an estimator of the standard error of the mean square error is, by product, produced. The Fortran programs are available to Author.

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Table 1. R for Comparisons of Mean and Median

(1) F = Uniform dist. on (-1,1)
(alpha = 0.001)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
60	20	.015	.025	.01688	.04866	.003345	.008613
60	40	-.008	-.009	.00746	.02032	.001203	.003491
50	60	.015	.031	.00555	.01492	.000923	.002385
40	80	-.004	-.005	.00433	.01353	.000609	.002071
60	100	-.005	-.008	.00386	.01000	.000612	.001688
60	120	.001	-.007	.00274	.00789	.000505	.001247
70	140	.009	.014	.00228	.00716	.000422	.001381
70	160	.002	-.004	.00173	.00516	.000321	.000926
70	180	.001	-.001	.00219	.00642	.000305	.001107
50	200	.007	.006	.00125	.00406	.000227	.000739
60	220	-.003	-.001	.00136	.00410	.000211	.000737
50	240	.005	.011	.00146	.00418	.000222	.000674
30	260	.000	-.008	.00135	.00367	.000232	.000601
30	280	-.002	-.008	.00144	.00488	.000300	.000935
40	300	.001	.007	.00125	.00391	.000214	.000716

(2) F = Standard Normal dist.
(alpha = 0.001)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
460	20	.003	.005	.05171	.07172	.003445	.004870
140	40	.003	.008	.02270	.04422	.002919	.005506
150	60	.004	.006	.01422	.02639	.001708	.003205
210	80	.002	-.004	.01084	.01696	.001011	.001440
270	100	.004	.008	.00929	.01418	.000785	.001245
190	120	.005	.006	.00782	.01292	.000819	.001221
250	140	-.001	-.002	.00767	.01180	.000663	.001016
150	160	.001	-.003	.00511	.00794	.000531	.000811
210	180	.002	-.004	.00563	.00836	.000567	.000803
210	200	.006	.006	.00551	.00900	.000554	.000884
40	220	.015	.014	.00274	.00371	.000616	.000711
90	240	.005	.003	.00338	.00575	.000489	.000823
50	260	.012	.004	.00298	.00517	.000532	.000836
30	280	.001	-.009	.00231	.00300	.000563	.000585
20	300	.000	.015	.00159	.00262	.000522	.000728

(3) F = Standard Logistic dist.
(alpha = 0.05)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
500	20	.011	.016	.15094	.18425	.009633	.013287
820	40	.009	.009	.07361	.08507	.003685	.004378
430	60	.005	-.003	.05421	.06602	.003929	.004404
570	80	.003	.010	.04271	.05099	.002557	.003296
780	100	-.004	-.004	.03182	.03665	.001626	.001823
130	120	-.012	-.014	.02191	.03256	.002686	.004403
100	140	.018	.018	.01988	.03099	.002694	.004020
40	160	.003	-.032	.01154	.02298	.002514	.004024
270	180	-.001	.005	.01744	.02221	.001565	.001797
530	200	.005	.013	.01704	.02040	.001039	.001297
360	220	-.003	-.006	.01504	.01850	.001167	.001289
450	240	-.003	-.004	.01259	.01539	.000873	.001077
430	260	.002	.003	.01251	.01536	.000904	.001115
550	280	-.001	-.002	.01179	.01401	.000707	.000841
80	300	-.001	-.010	.00932	.01369	.001231	.001685

(4) F = Double exponential dist.
(alpha = 0.005)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
380	20	.007	.010	.09343	.06540	.007118	.006639
190	40	-.006	.005	.05056	.03194	.005069	.003669
110	60	.002	-.012	.02890	.01614	.004024	.001999
170	80	.019	.014	.02052	.01233	.002361	.001654
50	100	.001	.007	.01906	.00785	.003320	.001438
60	120	-.014	-.013	.01576	.00688	.002658	.001249
120	140	.016	.008	.01719	.00940	.002286	.001136
20	160	-.026	-.023	.01082	.00369	.002143	.000869
70	180	.001	.007	.01014	.00508	.001466	.000873
90	200	.002	.006	.01337	.00655	.001703	.001189
150	220	-.005	.001	.00981	.00613	.001132	.000632
40	240	-.011	-.013	.00790	.00325	.001439	.000531
140	260	.013	.002	.00729	.00414	.000929	.000474
80	280	-.021	-.012	.00662	.00337	.000946	.000595
200	300	-.001	-.004	.00575	.00358	.000664	.000362

Table 1. R for Comparisons of Mean and Median(Continued)

(5) F : Tukey model T(0.10,4.0)
(alpha : 0.01)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
240	20	-.024	-.030	.12466	.08463	.012708	.008333
210	40	.008	-.001	.05798	.04045	.005227	.003399
60	60	-.038	-.017	.04736	.02307	.008048	.003422
50	80	.041	.020	.04242	.01921	.007916	.003555
240	100	.001	-.004	.02772	.01809	.003278	.001736
110	120	-.015	-.002	.02001	.01171	.002665	.001624
400	140	.002	-.002	.01606	.01231	.001147	.000882
620	160	.004	.002	.01652	.01350	.000890	.000734
390	180	.008	.010	.01344	.01028	.000979	.000692
380	200	.009	.006	.01331	.00999	.001047	.000721
210	220	-.007	-.008	.01321	.00908	.001218	.000969
210	240	.000	.006	.00995	.00674	.000928	.000663
460	260	-.001	-.006	.00968	.00761	.000585	.000519
360	280	-.001	.004	.00985	.00728	.000729	.000613
280	300	.007	.004	.00859	.00617	.000756	.000512

(6) F : Student's T dist. with df : 3
(alpha : 0.05)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
80	20	.084	.064	.14145	.08686	.021981	.012274
40	40	-.089	-.027	.06201	.03416	.010630	.006092
120	60	-.012	.008	.05946	.03459	.011626	.004636
160	80	-.012	.004	.03183	.02181	.003924	.002604
50	100	.005	.023	.02580	.01208	.005432	.002613
170	120	-.006	-.004	.02423	.01721	.002651	.002040
230	140	.000	-.015	.01661	.01251	.001630	.001289
90	160	.006	.018	.02351	.01338	.004138	.001699
260	180	.006	-.002	.01249	.00947	.001194	.000856
70	200	.001	-.010	.01597	.00896	.002460	.001818
30	220	-.039	-.030	.00932	.00419	.002228	.001058
100	240	.010	.014	.01052	.00690	.001536	.000865
90	260	-.008	.000	.01312	.00864	.001768	.001026
160	280	-.009	-.007	.01042	.00728	.001259	.000954
60	300	-.009	.003	.01033	.00552	.001992	.000865

(7) F : Student's T dist. with df : 4
(alpha : 0.05)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
1000	20	.008	.004	.09305	.08333	.004321	.003868
630	40	.011	.004	.05088	.04293	.003051	.002376
640	60	-.001	.001	.03172	.02707	.001807	.001414
1000	80	-.002	-.005	.02514	.02293	.001184	.000992
1000	100	.002	.002	.02045	.01974	.000947	.000879
1000	120	-.004	-.004	.01597	.01423	.000785	.000667
610	140	-.002	-.007	.01529	.01293	.000921	.000704
20	160	.027	.000	.01283	.00474	.003428	.001209
350	180	-.003	.005	.01093	00885	.000811	.000671
1000	200	-.005	-.002	.01022	.00924	.000471	.000422
730	220	-.003	-.002	.00897	.00772	.000461	.000427
1000	240	.001	.002	.00855	.00773	.000380	.000338
1000	260	.003	.003	.00730	.00679	.000328	.000301
140	280	-.008	-.013	.00702	.00481	.000877	.000575
140	300	.000	-.005	.00708	.00479	.000934	.000555

(8) F : Student's T dist. with df : 8
(alpha : 0.10)

R	n	$\bar{X}$	$\tilde{X}$	MSE1	MSE2	SEMSE1	SEMSE2
210	20	.005	.017	.06589	.08270	.006339	.007327
570	40	.003	.007	.03788	.04413	.002257	.002621
410	60	-.005	-.008	.02440	.02870	.001628	.001882
150	80	.024	.030	.01740	.02363	.002225	.002834
90	100	.016	.004	.01019	.01625	.001866	.002942
490	120	-.001	.001	.01058	.01237	.000646	.000769
140	140	.001	.001	.00748	.01013	.000939	.001221
70	160	.009	.003	.00751	.01104	.001057	.001816
230	180	.007	.012	.00829	.01028	.000718	.000955
200	200	.007	.001	.00666	.00876	.000652	.001044
380	220	.004	.005	.00610	.00746	.000494	.000568
340	240	-.001	.001	.00534	.00671	.000480	.000645
140	260	-.005	.000	.00438	.00604	.000457	.000767
60	280	-.004	-.002	.00313	.00572	.000585	.001205
260	300	-.003	-.008	.00392	.00498	.000347	.000523

Table 2. R for Comparisons of $Q_1(y)$ and $Q_2(y)$

(1) F = EXP(1.0) and G = EXP(.111)

(n = 20 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
140	.10	.105	.107	.107	.00493	.00493	.000690	.000690
530	.15	.163	.202	.158	.01233	.00917	.001195	.000795
670	.20	.223	.263	.218	.01551	.01185	.001366	.000964
1000	.25	.288	.284	.284	.01714	.01714	.000987	.000987
1000	.30	.357	.354	.354	.02058	.02062	.001176	.001179
1000	.35	.431	.429	.430	.02889	.02959	.001627	.001768
1000	.40	.511	.550	.518	.03974	.03777	.002701	.002600
70	.45	.598	.639	.715	.03738	.06828	.006943	.011131
1000	.50	.693	.691	.698	.05702	.06017	.002765	.002909
1000	.55	.799	.804	.812	.06688	.07039	.003617	.003886
1000	.60	.916	.923	.934	.08175	.08096	.004433	.004196
780	.65	1.050	1.098	1.191	.10980	.13445	.007216	.008595
1000	.70	1.204	1.239	1.219	.14274	.12368	.008746	.006959
920	.75	1.386	1.380	1.334	.18001	.15022	.011686	.007132
620	.80	1.609	1.610	1.506	.20946	.17201	.014585	.008965
400	.85	1.897	1.918	1.728	.37172	.26580	.043508	.019590
1000	.90	2.303	2.485	2.520	.78339	.77544	.056443	.056960
110	.95	2.996	3.262	2.498	2.30853	.87622	.642355	.080268

(n = 40 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.105	.097	.097	.00229	.00229	.000602	.000602
100	.15	.163	.167	.153	.00471	.00426	.000758	.000561
60	.20	.223	.216	.203	.00456	.00495	.000638	.000690
130	.25	.288	.274	.274	.00687	.00687	.000705	.000705
420	.30	.357	.357	.357	.01052	.01059	.000692	.000702
1000	.35	.431	.427	.428	.01431	.01454	.000722	.000748
1000	.40	.511	.525	.520	.01811	.01925	.000982	.001047
270	.45	.598	.625	.676	.02524	.03675	.003109	.004189
990	.50	.693	.706	.723	.02681	.03181	.001412	.001795
680	.55	.799	.807	.833	.03345	.04009	.001843	.002288
1000	.60	.916	.925	.953	.04343	.04709	.002112	.002383
580	.65	1.050	1.072	1.166	.05419	.07171	.005485	.005815
1000	.70	1.204	1.225	1.231	.06255	.05720	.003477	.003009
1000	.75	1.386	1.394	1.378	.09089	.08180	.004703	.004183
1000	.80	1.609	1.607	1.549	.10904	.09941	.005480	.004388
900	.85	1.897	1.914	1.803	.17071	.14342	.010055	.007400
1000	.90	2.303	2.343	2.365	.30662	.29073	.019404	.019570
160	.95	2.996	3.113	2.684	.91145	.49424	.173922	.059445

(n = 60 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.105	.115	.115	.00252	.00252	.000695	.000695
30	.15	.163	.163	.158	.00205	.00222	.000438	.000449
60	.20	.223	.229	.225	.00364	.00354	.000710	.000692
260	.25	.288	.293	.293	.00728	.00733	.000686	.000687
290	.30	.357	.357	.358	.00739	.00757	.000682	.000705
350	.35	.431	.439	.441	.00877	.00903	.000688	.000715
390	.40	.511	.519	.523	.00921	.00989	.000679	.000729
270	.45	.598	.634	.676	.01818	.02696	.002216	.003238
790	.50	.693	.704	.724	.01822	.02215	.001038	.001429
510	.55	.799	.801	.836	.02245	.02847	.001511	.002129
740	.60	.916	.931	.975	.02648	.03222	.001603	.001981
280	.65	1.050	1.056	1.136	.02812	.03749	.002393	.003352
1000	.70	1.204	1.205	1.226	.03853	.03869	.001895	.001965
1000	.75	1.386	1.394	1.389	.05373	.04813	.002634	.002210
1000	.80	1.609	1.630	1.591	.08062	.07031	.004163	.003370
1000	.85	1.897	1.917	1.836	.10902	.09617	.006606	.004959
1000	.90	2.303	2.329	2.329	.17796	.16290	.009315	.008145
270	.95	2.996	3.061	2.729	.48822	.36106	.050111	.026709

(n = 100 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.105	.097	.097	.00106	.00106	.000233	.000233
50	.15	.163	.179	.177	.00257	.00259	.000652	.000658
50	.20	.223	.219	.218	.00312	.00317	.000590	.000588
30	.25	.288	.299	.299	.00226	.00230	.000559	.000562
150	.30	.357	.374	.376	.00521	.00529	.000660	.000677
280	.35	.431	.442	.447	.00636	.00693	.000659	.000726
170	.40	.511	.505	.513	.00617	.00681	.000639	.000761
350	.45	.598	.604	.640	.00860	.01146	.000652	.000950
300	.50	.693	.703	.732	.00992	.01340	.000850	.001256
140	.55	.799	.813	.852	.01202	.01772	.001361	.002042
280	.60	.916	.906	.957	.01555	.01983	.001148	.001586
200	.65	1.050	1.050	1.128	.01943	.02601	.001597	.002329
1000	.70	1.204	1.207	1.235	.02513	.02577	.001188	.001258
1000	.75	1.386	1.386	1.395	.03319	.03284	.001605	.001608
1000	.80	1.609	1.615	1.590	.04493	.04181	.002123	.001901
1000	.85	1.897	1.900	1.843	.06210	.05885	.003046	.002653
1000	.90	2.303	2.320	2.301	.11061	.10482	.005904	.005713
1000	.95	2.996	3.007	2.770	.26242	.23035	.013415	.008845

Table 2. R for Comparisons of $Q_1(y)$ and $Q_2(y)$(Continued)

(2) F = WEIB(1.0,0.5) and G = EXP(.374)
(n = 20 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.011	.015	.015	.00058	.00058	.000385	.000385
70	.15	.026	.073	.041	.00865	.00172	.003100	.000426
40	.20	.050	.072	.048	.00304	.00167	.000777	.000487
1000	.25	.083	.100	.100	.01133	.01149	.001532	.001599
1000	.30	.127	.149	.150	.01960	.02036	.002308	.002451
1000	.35	.186	.224	.225	.05002	.05110	.012444	.012480
1000	.40	.261	.341	.340	.08043	.22759	.013051	.116969
80	.45	.357	.494	.773	.17846	.59699	.089928	.160298
1000	.50	.480	.607	.666	.44792	.49077	.134267	.086180
130	.55	.638	.790	.942	.28065	.61608	.048470	.142013
1000	.60	.840	1.057	1.205	.90721	1.06152	.117877	.097052
430	.65	1.102	1.384	1.865	1.20223	1.95857	.234697	.245827
1000	.70	1.450	1.811	1.867	2.35868	1.71608	.323789	.214677
130	.75	1.922	2.790	2.394	4.95824	2.13259	1.140055	.480916
170	.80	2.590	2.815	2.455	2.95776	1.72659	.474759	.297814
390	.85	3.599	3.378	2.827	3.00799	2.31049	.282621	.122367
1000	.90	5.302	3.962	3.743	6.00617	5.92323	.290153	.212146
80	.95	8.974	4.490	3.909	22.94690	28.07369	1.513016	1.503680

(n = 40 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.011	.015	.015	.00012	.00012	.000000	.000000
50	.15	.026	.049	.034	.00229	.00103	.000756	.000571
50	.20	.050	.062	.056	.00174	.00160	.000659	.000653
170	.25	.083	.093	.093	.00365	.00365	.000676	.000676
1000	.30	.127	.143	.144	.00818	.00835	.000851	.000872
1000	.35	.186	.201	.204	.01350	.01530	.001039	.001336
1000	.40	.261	.283	.288	.02195	.02821	.001658	.002592
70	.45	.357	.400	.517	.04785	.11944	.010957	.029122
170	.50	.480	.504	.573	.05492	.12715	.009449	.029668
90	.55	.638	.713	.901	.16737	.44736	.036186	.121460
40	.60	.840	.969	1.371	.21599	.81608	.065789	.224074
180	.65	1.102	1.219	1.786	.56599	1.51199	.169237	.354443
110	.70	1.450	1.587	1.972	.55513	.97938	.106590	.158988
280	.75	1.922	2.358	2.427	2.29017	1.37882	.376382	.182072
190	.80	2.590	3.188	2.928	4.26852	1.97114	.887733	.410480
110	.85	3.599	3.973	3.109	5.08462	1.79404	1.440633	.285032
1000	.90	5.302	4.684	4.092	4.84450	4.41829	.271415	.199059
80	.95	8.974	5.202	4.678	17.42162	21.64361	1.266881	1.400165

(n = 60 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.011	.012	.012	.00007	.00007	.000000	.000000
30	.15	.026	.038	.033	.00041	.00036	.000099	.000094
50	.20	.050	.065	.063	.00207	.00206	.000628	.000628
60	.25	.083	.090	.090	.00268	.00268	.000648	.000648
120	.30	.127	.128	.129	.00390	.00394	.000696	.000699
1000	.35	.186	.204	.208	.00949	.01082	.000696	.000793
500	.40	.261	.288	.301	.01471	.02018	.001334	.002124
110	.45	.357	.377	.457	.02225	.04552	.005062	.009187
320	.50	.480	.517	.595	.04933	.10361	.011798	.020918
40	.55	.638	.691	.849	.05715	.18349	.011728	.054060
30	.60	.840	.926	1.281	.10102	.50299	.032232	.162057
30	.65	1.102	1.140	1.738	.20373	.88294	.090754	.274393
30	.70	1.450	1.389	2.151	.13229	1.11710	.023648	.333382
1000	.75	1.922	2.162	2.475	1.47973	1.17645	.217066	.081957
360	.80	2.590	2.931	2.881	3.13486	1.49504	.681937	.261838
150	.85	3.599	4.066	3.372	5.45467	1.93040	1.446105	.552587
150	.90	5.302	5.208	4.270	5.90985	3.96667	.793956	.386175
60	.95	8.974	6.131	4.980	13.72685	19.15548	1.527663	1.466314

(n = 100 and alpha = 0.03)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.011	.011	.011	.00004	.00004	.000000	.000000
30	.15	.026	.029	.028	.00028	.00029	.000104	.000104
30	.20	.050	.054	.054	.00046	.00048	.000124	.000125
30	.25	.083	.091	.091	.00115	.00115	.000187	.000187
90	.30	.127	.139	.141	.00290	.00302	.000630	.000649
50	.35	.186	.190	.193	.00261	.00330	.000510	.000804
350	.40	.261	.274	.288	.00736	.01048	.000687	.001218
100	.45	.357	.365	.436	.01453	.03736	.002834	.009608
50	.50	.480	.492	.571	.01805	.04226	.003427	.009434
40	.55	.638	.696	.887	.03989	.15994	.011046	.047297
60	.60	.840	.810	1.123	.04884	.30980	.008134	.110182
30	.65	1.102	1.204	1.927	.14645	1.30202	.044865	.376158
30	.70	1.450	1.650	2.277	.28177	1.12940	.083582	.246312
40	.75	1.922	2.084	2.616	.39979	1.12745	.129756	.287831
260	.80	2.590	2.958	3.085	2.02633	1.11056	.396512	.140072
70	.85	3.599	4.166	3.580	4.02962	1.01177	1.285096	.226350
30	.90	5.302	5.848	4.623	4.00824	1.73446	.804835	.342679
30	.95	8.974	6.028	4.923	10.47952	18.00593	1.577453	1.900033

Table 2. R for Comparisons of $Q_1(y)$ and $Q_2(y)$(Continued)

(3) F = WEIB(2.0,4.0) and G = EXP(.936)

(n = 20 and alpha = 0.005)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
450	.10	.479	.489	.495	.01012	.01111	.000672	.000738
140	.15	.534	.545	.586	.01029	.01668	.001258	.001844
190	.20	.578	.587	.636	.00767	.01185	.000839	.001177
80	.25	.616	.621	.694	.00746	.01474	.001299	.002121
50	.30	.650	.667	.738	.00581	.01281	.001110	.002109
290	.35	.681	.687	.746	.00789	.00910	.000744	.000663
90	.40	.711	.728	.763	.00620	.00575	.000669	.000696
160	.45	.739	.734	.759	.00800	.00478	.000903	.000589
50	.50	.767	.780	.768	.00449	.00275	.000765	.000532
100	.55	.795	.800	.766	.00562	.00409	.000808	.000474
130	.60	.823	.832	.785	.00629	.00607	.000675	.000669
160	.65	.851	.858	.813	.00678	.00602	.000700	.000677
60	.70	.881	.868	.793	.00632	.01235	.001118	.001558
170	.75	.912	.906	.823	.00951	.01408	.001124	.001163
90	.80	.947	.947	.846	.00999	.01598	.001174	.001641
100	.85	.987	.983	.871	.01175	.01972	.001594	.002222
610	.90	1.036	1.021	.967	.01307	.01459	.000731	.000670
60	.95	1.106	1.059	.978	.01460	.02697	.002587	.003396

(n = 40 and alpha = 0.005)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
110	.10	.479	.493	.533	.00552	.00918	.000658	.001105
60	.15	.534	.540	.599	.00448	.00961	.000820	.001415
30	.20	.578	.607	.675	.00507	.01673	.000924	.003707
30	.25	.616	.635	.718	.00402	.01557	.000964	.002938
30	.30	.650	.652	.746	.00399	.01210	.000781	.001707
70	.35	.681	.677	.751	.00358	.00743	.000544	.001017
50	.40	.711	.716	.775	.00292	.00606	.000551	.000895
30	.45	.739	.736	.761	.00301	.00168	.000768	.000353
30	.50	.767	.764	.761	.00296	.00235	.000731	.000399
40	.55	.795	.794	.769	.00306	.00239	.000696	.000489
30	.60	.823	.825	.782	.00207	.00262	.000417	.000469
50	.65	.851	.834	.792	.00366	.00572	.000602	.000704
70	.70	.881	.882	.808	.00367	.00781	.000727	.001063
30	.75	.912	.893	.818	.00528	.01197	.001050	.001868
40	.80	.947	.945	.860	.00567	.01109	.001056	.001403
30	.85	.987	.992	.892	.00428	.01160	.000869	.001621
290	.90	1.036	1.050	.967	.00801	.00925	.000759	.000614
30	.95	1.106	1.088	1.000	.00400	.01485	.000966	.002735

(n = 60 and alpha = 0.005)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
50	.10	.479	.485	.522	.00260	.00491	.000439	.000890
40	.15	.534	.536	.598	.00321	.00898	.000606	.001804
30	.20	.578	.570	.664	.00235	.01065	.000499	.001912
30	.25	.616	.631	.724	.00233	.01444	.000609	.002100
30	.30	.650	.674	.767	.00271	.01486	.000623	.001531
30	.35	.681	.683	.761	.00163	.00748	.000323	.000842
30	.40	.711	.710	.761	.00237	.00356	.000570	.000707
30	.45	.739	.745	.764	.00256	.00219	.000602	.000529
30	.50	.767	.766	.769	.00255	.00141	.000591	.000269
30	.55	.795	.791	.768	.00272	.00241	.000486	.000552
40	.60	.823	.823	.776	.00209	.00382	.000477	.000690
40	.65	.851	.840	.795	.00256	.00482	.000516	.000835
30	.70	.881	.879	.805	.00306	.00756	.000643	.001170
30	.75	.912	.906	.826	.00267	.00946	.000612	.001376
30	.80	.947	.953	.864	.00202	.00829	.000553	.001106
30	.85	.987	.993	.894	.00551	.01176	.001217	.001740
60	.90	1.036	1.031	.960	.00429	.00873	.000689	.001141
40	.95	1.106	1.091	1.011	.00550	.01161	.001200	.001671

(n = 100 and alpha = 0.005)

R	y	Q(y)	Q1(y)	Q2(y)	MSE1	MSE2	SEMSE1	SEMSE2
30	.10	.479	.475	.525	.00159	.00384	.000528	.000822
30	.15	.534	.540	.608	.00176	.00769	.000248	.001195
30	.20	.578	.578	.655	.00204	.00780	.000384	.001314
30	.25	.616	.625	.721	.00162	.01258	.000392	.001424
30	.30	.650	.660	.762	.00145	.01332	.000306	.001108
30	.35	.681	.681	.765	.00103	.00760	.000215	.000780
30	.40	.711	.714	.757	.00127	.00294	.000433	.000649
30	.45	.739	.727	.765	.00183	.00158	.000484	.000369
30	.50	.767	.770	.765	.00089	.00054	.000191	.000149
30	.55	.795	.806	.775	.00164	.00133	.000337	.000419
30	.60	.823	.816	.776	.00224	.00353	.000576	.000658
30	.65	.851	.855	.803	.00150	.00343	.000277	.000746
30	.70	.881	.874	.817	.00190	.00547	.000402	.000855
30	.75	.912	.901	.824	.00157	.00882	.000365	.001152
30	.80	.947	.947	.863	.00129	.00829	.000539	.001140
30	.85	.987	.986	.904	.00178	.00868	.000402	.001719
30	.90	1.036	1.038	.954	.00278	.00861	.001051	.001372
30	.95	1.106	1.103	1.016	.00276	.01023	.000640	.001637