ON DILATION THEOREMS OF A
CONTRACTION IN THE CLASSES $A_n$

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A dual algebra is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1\mathcal{H}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $T$ and $I_\mathcal{H}$ and is closed in the ultraweak operator topology. Moreover, let $Q_T$ denote the quotient space $\mathcal{C}_1/\mathcal{A}_T$, where $\mathcal{C}_1$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\text{^}_\perp \mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1$. One knows that $\mathcal{A}_T$ is the dual space of $Q_T$ and that the duality is given by

$$\langle A, [L] \rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$ 

Furthermore, the weak* topology that accrues to $\mathcal{A}_T$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}_T$. For vectors $x$ and $y$ in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by

$$ (x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}. $$

The theory of dual algebras is deeply related to the study of the problem of solving systems of simultaneous equations in the predual of a dual algebra (cf. [1], [3], [5], and [7]). That is the main topic of this work. In this paper, we consider the following question:

**Question 1.** Let $A$ be a normal completely nonunitary contraction acting on an $n$-dimensional Hilbert space such that $\|Ax\| < \|x\|$ for every nonzero vector $x$ and let $T \in \mathcal{A}_m(\mathcal{H})$ (will be defined below), where $m = n(n+1)/2$.

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Is it true that there always exist invariant subspaces $M$ and $N$ for $T$ with $M \supset N$ such that the compression $T_{M \oplus N}$ of $T$ to $M \oplus N$ is unitarily equivalent to $A$?

The notation and terminology employed herein agree with those in [5],[6], and [8]. We shall denote by $D$ the open unit disc in the complex plane $\mathbb{C}$, and we write $T$ for the boundary of $D$. For $1 \leq p < \infty$, we denote by $L^p = L^p(T)$ the Banach space of complex valued, Lebesgue measurable functions $f$ on $T$ such that $|f|^p$ is Lebesgue integrable, and by $L^\infty = L^\infty(T)$ the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on $T$. If for $1 \leq p \leq \infty$ we denote by $H^p = H^p(T)$ the subspace of $L^p$ consisting of those functions whose negative Fourier coefficients vanish, then one knows that the preannihilator $\perp(H^\infty)$ of $H^\infty$ in $L^1$ is the subspace $H^1_0$ consisting of those functions $g$ in $H^1$ whose analytic extension $\tilde{g}$ to $D$ satisfies $\tilde{g}(0) = 0$. It is well known that $H^\infty$ is the dual space of $L^1/H^1_0$, where the duality is given by the pairing

$$
\langle f, [g] \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})g(e^{it})dt, \quad f \in H^\infty, \quad [g] \in L^1/H^1_0.
$$

Recall that any contraction $T$ can be written as a direct sum $T = T_1 \oplus T_2$, where $T_1$ is a completely nonunitary contraction and $T_2$ is a unitary operator. If $T_2$ is absolutely continuous or acts on the space $(0)$, $T$ will be called an absolutely continuous contraction. The following Foias–Sz.–Nagy functional calculus [5, Theorem 4.1] provides a good relationship between the function space $H^\infty$ and a dual algebra $A_T$.

**Theorem 2.** Let $T$ be an absolutely continuous contraction in $L(H)$. Then there is an algebra homomorphism $\Phi_T : H^\infty \rightarrow A_T$ defined by $\Phi_T(f) = f(T)$ that has the following properties:

(a) $\Phi_T(1) = 1_H, \Phi_T(\xi) = T$,

(b) $\|\Phi_T(f)\| \leq \|f\|_\infty, f \in H^\infty$,

(c) $\Phi_T$ is continuous if both $H^\infty$ and $A_T$ are given their weak* topologies,

(d) the range of $\Phi_T$ is weak* dense in $A_T$,

(e) there exists a bounded, linear, one-to-one map $\phi_T : Q_T \rightarrow L^1/H^1_0$ such that $\phi_T^* = \Phi_T$, and

(f) if $\Phi_T$ is an isometry, then $\Phi_T$ is a weak* homeomorphism of $H^\infty$ onto $A_T$ and $\phi_T$ is an isometry of $Q_T$ onto $L^1/H^1_0$.

**Definition 3** (cf. [4]). Let $A \subset L(H)$ be a dual algebra and let $n$ be any cardinal number such that $1 \leq n \leq \aleph_0$. Then $A$ will be said to have
property \((A_n)\) provided every \(n \times n\) system of simultaneous equations of the form
\[
[L_{ij}] = [x_i \otimes y_j], \quad 0 \leq i, j < n
\]
(which the \([L_{ij}]\) are arbitrary but fixed elements from \(Q_A\)) has a solution
\(\{x_i\}_{0 \leq i < n}, \{y_j\}_{0 \leq j < n}\) consisting of a pair of sequences of vectors from \(\mathcal{H}\).

**Definition 4** (cf. [4]). The class \(A(\mathcal{H})\) consists of all those absolutely continuous contraction \(T\) in \(\mathcal{L}(\mathcal{H})\) for which the functional calculus \(\Phi_T : H^\infty \to A_T\) is an isometry. Furthermore, if \(n\) is any cardinal number such that \(1 \leq n \leq \aleph_0\), we denote by \(A_n(\mathcal{H})\) the set of all \(T\) in \(A(\mathcal{H})\) such that the algebra \(A_T\) has property \((A_n)\).

We write simply \(A_n\) for \(A_n(\mathcal{H})\) when there is no confusion. If \(T \in \mathcal{L}(\mathcal{H})\) and \(M \subset \mathcal{H}\) is a semi-invariant subspace for \(T\) (i.e., there exist invariant subspaces \(N_1\) and \(N_2\) for \(T\) with \(N_1 \supset N_2 = N_1 \cap N_2^\perp\)), we write \(T_M\) for the compression of \(T\) to \(M\). In other words, \(T_M = P_M T M\), where \(P_M\) is the orthogonal projection whose range is \(M\). Let \(n\) be any cardinal number such that \(1 \leq n \leq \aleph_0\). Throughout this paper, we write \(\mathbb{C}\) for the complex plane and \(\mathbb{N}\) for the set of natural numbers. Now we are ready to show the main theorem of this paper.

**Theorem 5.** Let \(A\) be a completely nonunitary normal contraction acting on an \(n\)-dimensional Hilbert space \(\mathcal{H}_n\), \(2 \leq n \in \mathbb{N}\), whose matrix relative to some orthonormal basis \(\{u_k\}_{k=1}^n\) for \(\mathcal{H}_n\) is the diagonal matrix
\[
\text{Diag}(\{\lambda_k\})_{k=1}^n
\]
and let \(T \in A_m(\mathcal{H})\), where \(m = n(n + 1)/2\). Then there exist invariant subspaces \(M\) and \(N\) for \(T\) with \(M \supset N\) such that the compression \(T_{M \oplus N}\) of \(T\) to \(M \oplus N\) is unitarily equivalent to \(A\).

**Proof.** Let \(\mathcal{H}_m\) be an \(m\)-dimensional Hilbert space. We define a normal operator \(\widetilde{N} \in \mathcal{L}(\mathcal{H}_m)\) whose matrix relative to some orthonormal basis \(\{u^{(i)}_k\}_{1 \leq i \leq k \leq n}\) for \(\mathcal{H}_m\) is a diagonal matrix
\[
\text{Diag}(\lambda^{(1)}_1, \lambda^{(2)}_2, \lambda^{(3)}_3, \ldots, \lambda^{(n)}_n),
\]
(5)
where \(\lambda^{(1)}_k = \lambda^{(2)}_k = \cdots = \lambda^{(k)}_k = \lambda_k\), for \(k = 1, 2, \ldots, n\). Since \(\widetilde{N}\) is a completely nonunitary contraction, we have \(\{\lambda_k\}_{k=1}^n \subset \mathbb{D}\) and it follows from [4, Corollary 3.5] that there exist invariant subspaces \(M\) and \(N\) for \(T\) with \(M \supset N\) such that \(\dim(M \oplus N) = m\) and \(T_{M \oplus N}\) is similar to \(\widetilde{N}\).
Let $X$ be an invertible operator with $T_{\mathcal{M} \oplus \mathcal{N}} X = X \tilde{N}$. Note that

$$\tilde{N} u_k^{(i)} = \lambda_k^{(i)} u_k^{(i)}, \quad 1 \leq i \leq k, \quad 1 \leq k \leq n. \quad (6)$$

For a brief notation, we write $\tilde{T} = T_{\mathcal{M} \oplus \mathcal{N}}$. Since $X$ is one-to-one, it is easy to show that there exists a linearly independent set $\{ w_k^{(i)} \}_{1 \leq i \leq k \leq n}$ in $\mathcal{M} \oplus \mathcal{N}$ such that $\| w_k^{(i)} \| = 1$ and

$$\tilde{T} w_k^{(i)} = \lambda_k^{(i)} w_k^{(i)}, \quad 1 \leq i \leq k, \quad 1 \leq k \leq n. \quad (7)$$

Taking $f_1 = w_1^{(1)}$, we have $\tilde{T} f_1 = \lambda_1 f_1$. Assume that there exist $f_1, \ldots, f_k$ in $\mathcal{M} \oplus \mathcal{N}$ with $k < n$ such that $\tilde{T} f_i = \lambda_i f_i, i = 1, \ldots, k$. Since $\{ w_k^{(1)}, \ldots, w_{k+1}^{(k+1)} \}$ induces an $(k + 1)$-dimensional Hilbert space $\mathcal{R}$, there exists a normal vector $f_{k+1} \in \mathcal{R}$ such that $(f_i, f_{k+1}) = 0, \quad i = 1, 2, \ldots, k$. Say

$$f_{k+1} = \sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}, \quad (8)$$

where $a_i \in \mathbb{C}, i = 1, \ldots, k+1$. Then we have

$$\tilde{T} f_{k+1} = \tilde{T} \left( \sum_{i=1}^{k+1} a_i w_{k+1}^{(i)} \right)$$

$$= \sum_{i=1}^{k+1} a_i \tilde{T} w_{k+1}^{(i)}$$

$$= \sum_{i=1}^{k+1} a_i \lambda_k^{(i)} w_{k+1}^{(i)}$$

$$= \lambda_{k+1} \sum_{i=1}^{k+1} a_i w_{k+1}^{(i)}$$

$$= \lambda_{k+1} f_{k+1}. \quad (9)$$

Hence by the mathematical induction, there exists a set $\{ f_i \}_{i=1}^n \subset \mathcal{M} \oplus \mathcal{N}$ such that $\tilde{T} f_i = \lambda_i f_i$, for $i = 1, 2, \ldots, n$. Let us denote

$$\mathcal{K} = \sqrt[n]{f_k}. \quad (10)$$

If we define a linear map $Y : \mathcal{H}_n \to \mathcal{K}$ with $Y u_k = f_k, \quad k = 1, 2, \ldots, n$, then it is obvious that $Y$ is onto and isometry. Since $\mathcal{K}$ is an invariant
subspace for $\tilde{T}$, $K$ is a semi-invariant subspace for $T$. Furthermore, we have $T_KY = YA$. Hence $A$ is unitarily equivalent to $T_K$ and the proof is complete.

Remark 6. Theorem 5 gives a solution for Question 1.

Remark 7. It follows from [4, Corollary 3.6] that if $\lambda \in D$ and $T \in A_n$, then there exist invariant subspaces $M$ and $N$ for $T$ with $M \supset N$ such that $\dim(M \oplus N) = n$ and $T_{M \oplus N} = \lambda I$. This statement is a special case for the work of this paper.

References


