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ASYMPTOTIC BEHAVIOR OF PERIODIC LIPSCHITZIAN EVOLUTION OPERATORS IN BANACH SPACES

JAE UG JEONG AND JONG YEOUL PARK

1. Introduction

Let X be a real Banach space and let X^* be its dual, that is, the space of all continuous linear functionals on X. The value of $f \in X^*$ at $x \in X$ will be denoted by $< x, f >$. With each $x \in X$, we associate the set

$$J(x) = \{f \in X^* :< x, f >= \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is readily verified that $J(x) \neq \phi$ for any $x \in X$. The multi-valued map $J : X \to X^*$ is called the duality map of X. Let $B = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then a Banach space X is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t} \tag{1}$$

exists for each $x, h \in B$. In this case, the norm of X is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each x in B, limit (1) is attained uniformly for h in B. The space X is said to have a uniformly Gâteaux differentiable norm if for each $h \in B$, limit (1) is attained uniformly for $x \in B$. The norm of X is said to be uniformly Fréchet differentiable (and X is said to be uniformly smooth) if limit (1) is attained uniformly for (x,h) in $B \times B$. It is well known that if X is smooth, then the duality map J is single valued. It is also known that if X has a Fréchet differentiable norm, J is norm to norm continuous.

Let $\{C_t\}_{t \geq 0}$ be a family of nonempty closed convex subsets of a Banach space X, and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a Lipschitzian

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evolution operator constrained in $\{C_t\}$, i.e., U be a family of mappings $U(t,s): C_s \to C_t$ such that

$$U(t,s)U(s,r) = U(t,r), U(r,r) = I,$$
$$\|U(mT+t,s)x - U(mT+t,s)y\| \le k_m\|x-y\|$$

for $0 \le r \le s \le t$ and $x, y \in C_s$, where $m \in N$, T is a fixed positive number, and $k_m > 0$. Such a Lipschitzian evolution operator U is said to be T-periodic($T > 0$) if

$$C_{t+T} = C_t \quad \text{and} \quad U(t+T, s+T) = U(t,s)$$

for $0 \le s \le t$.

A function $u : [0,\infty) \to X$ is said to be an almost semitrajectory of U if for each s in $[0,\infty)$ and $u(s) \in C_s$,

$$\lim_{s\to\infty} \sup_{t\ge s} \|u(t) - U(t,s)u(s)\| = 0.$$

In what follows, let U be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ and set $u_n(t) = u(nT+t)$ for $t \in [0,T]$, $n \in N$.

If $F(U_t) = \{z : U(T+t,t)z = z \quad \text{for} \quad 0 \le t \le T\}$ is nonempty, then we have that $F(U_t)$ is a closed convex subset of C_t, and we see that

$$\lim_{n\to\infty} \|u_n(t) - z\| = \rho(t)$$

exists for every $z \in F(U_t)$.

The objective of this chapter is to study the asymptotic behavior of a T-periodic Lipschitzian operator with $\limsup_n k_n \le 1$ where k_n is the Lipschitzian constant of $U(nT+t,s)$. We prove that if u is an almost semitrajectory of U and $u_n(t) = u(nT+t)$, then the closed convex set

$$\bigcap_k \overline{co}\{u_n(t) : n \le k\} \bigcap F(U_t)$$

consists of at most one point, where $\overline{co}\{u_n(t) : n \ge k\}$ is the closed convex hull of $\{u_n(t) : n \ge k\}$. We also prove that if P is the metric projection of X onto $F(U_t)$, then the strong limit of $Pu_n(t)$ exists.

2. Lemmas

THEOREM 1. *Let* $\{C_t\}_{t\geq 0}$ *be a family of nonempty closed convex subsets of a uniformly convex Banach space* X, *and let* $U = \{U(t,s) : 0 \leq s \leq t\}$ *be a* T*-periodic Lipschitzian operator constrained in* $\{C_t\}$ *with* $\limsup_t k_t \leq 1$. *Then* $F(U_t)$ *is a closed and convex subset of* C_t.

Proof. The closedness of $F(U_t)$ is obvious. To show convexity it is sufficient to show that $z = (u+v)/2 \in F(U_t)$ for all $u, v \in F(U_t)$. If $\lim_{n\to\infty} U(nT+t,t)z = z$, then

$$\begin{aligned} U(T+t,t)z &= \lim_{n\to\infty} U((n+1)T+t, nT+t)U(nT+t,t)z \\ &= \lim_{n\to\infty} U((n+1)T+t,t)z \\ &= z. \end{aligned}$$

i.e., $z \in F(U_t)$. Hence, it suffices to prove that $\lim_{n\to\infty} U(nT+t,t)z = z$. If not, there exists $\varepsilon > 0$ such that for any $n \geq 0$, there is n' with $n' \geq n$ and

$$\begin{aligned} 4\|U(n'T+t,t)z - z\| &= \|2(U(n'T+t,t)z - u) \\ &\qquad - 2(v - U(n'T+t,t)z)\| \\ &\geq \varepsilon. \end{aligned}$$

Choose $d > 0$ so small that

$$(R+d)(1-\delta(\frac{\varepsilon}{R+d})) < R,$$

where $R = \|u-v\| > 0$ and δ is the modulus of convexity of X. Since $\limsup_t k_t \leq 1$, there is n_0 such that

$$k_n\|u-v\| \leq (\|u-v\| + d)$$

for $n \geq n_0$. Put $u' = 2(U(nT+t,t)z-u)$, $v' = 2(v - U(nT+t,t)z)$. Then

$$\|u'-v'\| = 4\|U(nT+t,t)z - z\| \geq \varepsilon.$$

Furthermore, if $n \geq n_0$, then we have

$$\begin{aligned}\|u'\| &= 2\|U(nT+t,t)z - u\| \\ &\leq 2k_n\|z-u\| \\ &\leq R+d,\end{aligned}$$

$$\begin{aligned}\|v'\| &= 2\|v - U(nT+t,t)z\| \\ &\leq 2k_n\|z-v\| \\ &\leq R+d.\end{aligned}$$

So, we have

$$\|\frac{u'+v'}{2}\| \leq (R+d)(1-\delta(\frac{\varepsilon}{R+d}))$$

and hence

$$\begin{aligned}\|u-v\| &= \|\frac{u'+v'}{2}\| \\ &\leq (R+d)(1-\delta(\frac{\varepsilon}{R+d})) \\ &< R = \|u-v\|.\end{aligned}$$

This is a contradiction. Therefore, $\lim_{n\to\infty} U(nT+t,t)z = z$. The proof is completed.

We prove some lemmas which are crucial for our argument.

LEMMA 1. *Let $\{C_t\}_{t\geq 0}$ be a family of nonempty closed convex subsets of a uniformly convex Banach space X and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ with $\limsup_t k_t \leq 1$. If u is an almost semitrajectory of U and $z \in F(U_t)$, then the limit of $\|u_n(t) - z\|$ exists.*

Proof. Put

$$\phi(n) = \sup_m \|u_{m+n}(t) - U((m+n)T+t, nT+t)u_n(t)\|$$

for $n, m \geq 0$. Then, by the definition of semitrajectory, $\lim_{n\to\infty} \phi(n) = 0$. Since, for any $n, m \geq 0$,

$$\begin{aligned}\|u_{m+n}(t) - z\| &\leq \|u_{m+n}(t) - U((m+n)T + t, nT + t)u_n(t)\| \\ &\quad + \|U((m+n)T + t, nT + t)u_n(t) - z\| \\ &\leq \phi(n) + k_m\|u_n(t) - z\|,\end{aligned}$$

we have

$$\begin{aligned}\inf_m \sup_{m\leq r} \|u_r(t) - z\| &\leq \phi(n) + (\inf_m \sup_{m\leq r} k_r)\|u_n(t) - z\| \\ &\leq \phi(n) + \|u_n(t) - z\|,\end{aligned}$$

and then

$$\inf_m \sup_{m\leq r} \|u_r(t) - z\| \leq \sup_m \inf_{m\leq n} \|u_n(t) - z\|$$

Thus, $\lim_{n\to\infty} \|u_n(t) - z\|$ exists.

LEMMA 2. *Let X be a uniformly convex Banach space and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ with $\limsup_t k_t \leq 1$. Let $F(U_t) \neq \phi$, $y \in F(U_t)$, $0 < \alpha \leq \beta < 1$, $r = \lim_{n\to\infty} \|u_n(t) - y\|$, and u is an almost semitrajectory of U. Then, for any $\varepsilon > 0$, there exists $n_0 \geq 0$ such that*

$$\begin{aligned}\|U(mT + t, t)(\lambda u_n(t) + (1-\lambda)y) - (\lambda U(mT + t, t)u_n(t) \\ + (1-\lambda)y)\| < \varepsilon,\end{aligned}$$

for all $n, m \geq n_0$ and $\lambda \in R$ with $\alpha \leq \lambda \leq \beta$.

Proof. Let $r > 0$. Then we can choose $d > 0$ so small that

$$(r + d)(1 - c\delta(\frac{\varepsilon}{r+d})) = r_0 < r,$$

where δ is the modulus of convexity of the norm and $c = \min\{2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta\}$. Let $a > 0$ with $r_0 + 2a < r$. Then we can choose $n_0 \geq 0$ such that

$$\begin{gathered}\|u_n(t) - y\| \geq r - a, \quad \text{for } n \geq n_0, \\ \|u_{m+n}(t) - U(mT + t, t)u_n(t)\| < a, \quad \text{for } n \geq n_0 \quad \text{and } m \geq 0, \\ k_m \leq 2, \quad \text{for } m \geq n_0, \\ k_m\|u_n(t) - y\| \leq r + d, \quad \text{for } n, m \geq n_0.\end{gathered}$$

Suppose that

$$\|U(mT+t,t)(\lambda u_n(t)+(1-\lambda)y)-(\lambda U(mT+t,t)u_n(t) \\ +(1-\lambda)y)\| \geq \varepsilon,$$

for some $m, n \geq n_0$ and $\lambda \in R$ with $\alpha \leq \lambda \leq \beta$. Put $u' = (1-\lambda)(U(mT+t,t)z-y)$ and $v' = \lambda(U(mT+t,t)u_n(t)-U(mT+t,t)z)$, where $z = \lambda u_n(t)+(1-\lambda)y$. Then

$$\begin{aligned}\|u'\| &\leq (1-\lambda)k_m\|z-y\| \\ &\leq \lambda(1-\lambda)(r+d),\end{aligned}$$

$$\begin{aligned}\|v'\| &\leq \lambda k_m\|z-u_n(t)\| \\ &\leq \lambda(1-\lambda)(r+d).\end{aligned}$$

We also have that

$$\begin{aligned}\|u'-v'\| &= \|U(mT+t,t)z-(\lambda U(mT+t,t)u_n(t) \\ &\quad +(1-\lambda)y)\| \\ &\geq \varepsilon,\end{aligned}$$

and

$$\lambda u' + (1-\lambda)v' = \lambda(1-\lambda)(U(mT+t,t)u_n(t)-y).$$

So, by using the Lemma in [3], we have

$$\begin{aligned}\lambda(1-\lambda)&\|U(mT+t,t)u_n(t)-y\| \\ &= \|\lambda u' + (1-\lambda)v'\| \\ &\leq \lambda(1-\lambda)(r+d)(1-2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d})) \\ &\leq \lambda(1-\lambda)(r+d)(1-c\delta(\frac{\varepsilon}{r+d})) \\ &= \lambda(1-\lambda)r_0,\end{aligned}$$

and hence

$$\|U(mT+t,t)u_n(t)-y\| \leq r_0.$$

This implies

$$\begin{aligned}\|u_{m+n}(t)-y\| &\le \|u_{m+n}(t)-U(mT+t,t)u_n(t)\| \\ &\qquad + \|U(mT+t,t)u_n(t)-y\| \\ &\le a+r_0 \\ &< r-a.\end{aligned}$$

This contradicts the fact $\|u_n(t)-y\| \ge r-a$ for all $n \ge n_0$. In the case when $r=0$, let $y \in F(U_t)$ and $\lambda \in R$ with $0 \le \lambda \le 1$. From $\limsup_t k_t \le 1$, there exists $t_0 \ge 0$ such that $k_m \le 2$ and $\|u_n(t)-y\| < \varepsilon$ for all $n, m \ge t_0$. Hence, for $n, m \ge t_0$ and $\lambda \in R$ with $0 \le \lambda \le 1$,

$$\begin{aligned}&\|U(mT+t,t)(\lambda u_n(t)+(1-\lambda)y)-(\lambda U(mT+t,t)u_n(t)+(1-\lambda)y)\| \\ &\quad \le \lambda\|U(mT+t,t)(\lambda u_n(t)+(1-\lambda)y)-U(mT+t,t)u_n(t)\| \\ &\qquad + (1-\lambda)\|U(mT+t,t)(\lambda u_n(t)+(1-\lambda)y)-y\| \\ &\quad \le \lambda k_m\|\lambda u_n(t)+(1-\lambda)y-u_n(t)\| \\ &\qquad + (1-\lambda)k_m\|\lambda u_n(t)+(1-\lambda)y-y\| \\ &\quad = 2\lambda(1-\lambda)k_m\|u_n(t)-y\| \\ &\quad < \varepsilon.\end{aligned}$$

So, we obtain the desired result.

For $x, y \in X$, we denote by $[x, y]$ the set $\{\lambda x+(1-\lambda)y : 0 \le \lambda \le 1\}$. The following lemma is proved in [4];

LEMMA 3. *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm and $\{x_\alpha\}$ a bounded net in C. Let $z \in \bigcap_\beta \overline{co}\{x_\alpha : \alpha \ge \beta\}$, $y \in C$, and $\{y_\alpha\}$ a net of elements in C with $y_\alpha \in [y, x_\alpha]$ and $\|y_\alpha - z\| = \min\{\|u-z\| : u \in [y, x_\alpha]\}$. If $y_\alpha \to y$, then $y = z$.*

LEMMA 4. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and let $U = \{U(t,s) : 0 \le s \le t\}$ be a T-periodic Lipschitzian operator constrained in $\{C_t\}$ with $\limsup_t k_t \le 1$. Let $F(U_t) \ne \phi$, $z \in \cap_k \overline{co}\{u_n(t) : n \ge k\} \cap F(U_t)$, $y \in F(U_t)$ and u be an almost semitrajectory of U. Then, for any $\varepsilon > 0$, there is $n_0 \ge 0$ such that*

$$< u_n(t)-y, J(y-z) > \le \varepsilon\|y-z\|$$

for all $n \geq n_0$.

Proof. Let $z \in \cap_k \overline{co}\{u_n(t) : n \geq k\} \cap F(U_t)$, $y \in F(U_t)$ and $\varepsilon > 0$. If $y = z$, this Lemma is obvious. So, let $y \neq z$. For any $n \geq 0$, define a unique element y_n such that $y_n \in [y, u_n(t)]$ and $\|y_n - z\| = \min\{\|u - z\| : u \in [y, u_n(t)]\}$. Then, since $y \neq z$, by Lemma 3, we have $y_n \to y$. Thus, there exists $c > 0$ such that for any $n \geq 0$, there is $n' \geq n$ with $\|y_{n'} - y\| \geq c$. Setting $y_{n'} = a_{n'}u_{n'}(t) + (1 - a_{n'})y$, $0 \leq a_{n'} \leq 1$. We also obtain $c_0 > 0$ so small that $a_{n'} \geq c_0$. In fact, since

$$\begin{aligned} c &\leq \|y_{n'} - y\| \\ &= a_{n'}\|u_{n'} - y\| \\ &\leq a_{n'} \sup_n \|u_n(t) - y\|, \end{aligned}$$

we may put $c_0 = \dfrac{c}{sup_n\|u_n(t) - y\|}$. Since the limit of $\|u_n(t) - y\|$ exists, putting $k = \lim_{n\to\infty} \|u_n(t) - y\|$, we have $k > 0$. If not, we have $u_n(t) \to y$ and $y_n \to y$, which contradicts $y_n \nrightarrow y$.

Let r be a positive number such that $\varepsilon > r$ and $k > 2r$. Choose $a > 0$ so small that

$$(R + a)(1 - \delta(\frac{c_0 r}{R + a})) < R,$$

where δ is the modulus of convexity of the norm and $R = \|z - y\| > 0$. Fix $a' < a$. By Lemma 2, there exists $n_1 \geq 0$ such that

$$\begin{aligned} &\|U(mT + t, t)(c_0 u_n(t) + (1 - c_0)y) - (c_0 U(mT + t, t)u_n(t) \\ &\qquad + (1 - c_0)y)\| < a' \end{aligned} \tag{2}$$

for all $n, m \geq n_1$. Fix $n_2 \geq 0$ with $n_2 \geq n_1$ and $\|u_{m+n_2}(t) - y\| \geq 2r$ and $\|u_{m+n_2}(t) - U(mT + t, t)u_{n_2}(t)\| < r$ for all $m \geq 0$. Furthermore, since $\limsup_t k_t \leq 1$ and $R + a' < R + a$, we can choose n_3 such that $k_s R + a' \leq R + a$ for all $s \geq n_3$. Now, let $n_0 \geq 0$ with $n_0 \geq n_i$, $i = 1, 2, 3$. Fix $n' \geq n_0$. Then since $a_{n'} \geq c_0$, we have

$$\begin{aligned} c_0 u_{n'} + (1 - c_0)y &\in [y, a_{n'}u_{n'}(t) + (1 - a_{n'})y] \\ &= [y, y_{n'}]. \end{aligned}$$

Hence

$$\begin{aligned}\|c_0u_{n'}(t) + (1-c_0)y - z\| &\le \max\{\|z-y\|, \|z-y_{n'}\|\}\\ &= \|z-y\|\\ &= R.\end{aligned}$$

By using (2), we obtain

$$\begin{aligned}&\|c_0U(mT+t,t)u_{n'}(t) + (1-c_0)y - z\|\\ &\quad\le \|U(mT+t,t)(c_0u_{n'}(t)+(1-c_0)y-z\|\\ &\qquad + \|c_0U(mT+t,t)u_{n'}(t)+(1-c_0)y\\ &\qquad - U(mT+t,t)(c_0u_{n'}(t)+(1-c_0)y)\|\\ &\quad\le \|U(mT+t,t)(c_0u_{n'}(t)+(1-c_0)y)-z\|+a'\\ &\quad\le k_m\|c_0u_{n'}(t)+(1-c_0)y-z\|+a'\\ &\quad\le k_mR+a'\\ &\quad\le R+a,\end{aligned}$$

for all $m \ge n_0$.

On the other hand, since $\|y-z\| = R < R+a$ and

$$\begin{aligned}&\|c_0U(mT+t,t)u_{n'}(t)+(1-c_0)y-y\|\\ &\qquad = c_0\|U(mT+t,t)u_{n'}-y\|\\ &\qquad \ge c_0(\|u_{m+n'}(t)-y\|\\ &\qquad\qquad - \|u_{m+n'}(t)-U(mT+t,t)u_{n'}(t)\|)\\ &\qquad \ge c_0r\end{aligned}$$

for all $m \ge n_0$, it follows

$$\begin{aligned}&\|\frac{1}{2}(c_0U(mT+t,t)u_{n'}(t)+(1-c_0)y-z)+\frac{1}{2}(y-z)\|\\ &\qquad = \|\frac{c_0}{2}U(mT+t,t)u_n(t)+(1-\frac{c_0}{2})y-z\|\\ &\qquad \le (R+a)(1-\delta(\frac{c_0r}{R+a}))\\ &\qquad < R\end{aligned}$$

for all $m \geq n_0$. This implies that if $u_m = \frac{c_0}{2} U(mT+t,t) u_{n'}(t) + (1 - \frac{c_0}{2}) y$, then

$$\|u_m + \alpha(y - u_m) - z\| \geq \|y - z\|$$

for all $\alpha \geq 1$. By Theorem 2.5 in [2], we have

$$< u_m + \alpha(y - u_m) - y, J(y - z) > \geq 0$$

and hence

$$< u_m - y, J(y - z) > \leq 0$$

for $m \geq n_0$. Therefore, for $m \geq n_0$,

$$\begin{aligned} &< u_{m+n'}(t) - y, J(y-z) > \\ &\qquad \leq \|u_{m+n'}(t) - U(mT+t,t) u_{n'}(t)\| \|y - z\| \\ &\qquad\quad + < U(mT+t,t) u_{n'}(t) - y, J(y-z) > \\ &\qquad < r\|y - z\| \\ &\qquad < \varepsilon \|y - z\|. \end{aligned}$$

This completes the proof.

3. Asymptotic Behavior

THEOREM 2. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ with $\limsup_t k_t \leq 1$. Let u be an almost semitrajectory of U. If $F(U_t) \neq \phi$, then for any $n \in N$, the set*

$$\bigcap_k \overline{co}\{u_n(t) : n \geq k\} \bigcap F(U_t)$$

consists of at most one point.

Proof. Let $y, z \in \bigcap_k \overline{co}\{u_n(t) : n \geq k\} \bigcap F(U_t)$. Then, since $\frac{y+z}{2} \in F(U_t)$, it follows from Lemma 4 that for every $\varepsilon > 0$, there exists $n_0 \geq 0$ such that

$$\begin{aligned} < u_n(t) - \frac{y+z}{2}, J(\frac{y+z}{2} - z) > &\leq \varepsilon \|\frac{y+z}{2} - z\| \\ &= \frac{\varepsilon}{2}\|y - z\| \end{aligned}$$

for all $n \geq n_o$. Since $y \in \overline{co}\{u_n(t) : n \geq k\}$, we have

$$< y - \frac{y+z}{2}, J(\frac{y+z}{2} - z) > \leq \frac{\varepsilon}{2}\|y - z\|$$

and hence

$$\begin{aligned} < y - z, J(y - z) > &= \|y - z\|^2 \\ &\leq 2\varepsilon\|y - z\|. \end{aligned}$$

Thus $\|y - z\| \leq 2\varepsilon$. Since ε is arbitrary, we have $y = z$.

We denote by $\omega(u_n(t))$ the set of all weak limit points of subnets of the net $\{u_n(t) : n \in N\}$.

THEOREM 3. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ with $\limsup_t k_t \leq 1$. Let u be an almost semitrajectory of U. If $F(U_t) \neq \phi$ and $\omega(u_n(t)) \subset F(U_t)$, then the sequence $\{u_n(t) : n \in N\}$ converges weakly to some $z \in F(U_t)$.*

Proof. Since $F(U_t) \neq \phi$, $\{u_n(t) : n \in N\}$ is bounded. So, the sequence $\{u_n(t)\}$ must contain a subsequence $\{u_{n_i}(t)\}$ of $\{u_n(t)\}$ which converges weakly to some $z \in C_t$. Since $\omega(u_n(t)) \subset F(U_t)$ and $z \in \bigcap_k \overline{co}\{u_n(t) : n \geq k\}$, we obtain

$$z \in \bigcap_k \overline{co}\{u_n(t) : n \geq k\} \bigcap F(U_t).$$

Therefore, it follows from Theorem 2 that $\{u_n(t) : n \in N\}$ converges weakly to $z \in F(U_t)$.

THEOREM 4. *Let X be a uniformly convex Banach space and let $U = \{U(t,s) : 0 \leq s \leq t\}$ be a T-periodic Lipschitzian evolution operator constrained in $\{C_t\}$ with $\limsup_t k_t \leq 1$. Let u be an almost semitrajectory of U. Let P be the metric projection of X onto $F(U_t)$. If $F(U_t) \neq \phi$, then the $\lim_{n\to\infty} u_n(t)$ exists and $\lim_{n\to\infty} Pu_n(t) = z_0$, where z_0 is a unique element of $F(U_t)$ such that*

$$\lim_{n\to\infty} \|u_n(t) - z_0\| = \min\{\lim_{n\to\infty} \|u_n(t) - z\| : z \in F(U_t)\}.$$

Proof. Since $F(U_t) \neq \phi$, we know that $\{u_n(t) : n \in N\}$ is bounded and $\lim_{n\to\infty} \|u_n(t) - z\| = \rho(z)$ exists for each $z \in F(U_t)$. Let $R = \min\{\rho(z) : z \in F(U_t)\}$. Then, since ρ is convex and continuous on $F(U_t)$ and $\rho(z) \to \infty$ as $\|z\| \to \infty$, there exists $z_0 \in F(U_t)$ such that $\rho(z_0) = R$; see [1, p79]. Since P is the metric projection of X onto $F(U_t)$, we have

$$\|u_n(t) - Pu_n(t)\| \leq \|u_n(t) - y\|$$

for all $n \in N$ and $y \in F(U_t)$, and hence

$$\lim_{n\to\infty} \|u_n(t) - Pu_n(t)\| \leq R.$$

Suppose that $\lim_{n\to\infty} \|u_n(t) - Pu_n(t)\| < R$. Then we may choose $\varepsilon > 0$ and $n_0 \geq 0$ such that $\|u_n(t) - Pu_n(t)\| < R - \varepsilon$ for all $n \geq n_0$. Since

$$\begin{aligned}
&\|u_{m+n}(t) - Pu_n(t)\| \\
&\qquad \leq \|u_{m+n}(t) - U((m+n)T + t, nT + t)u_n(t)\| \\
&\qquad\qquad + \|U((m+n)T + t, nT + t)u_n(t) - Pu_n(t)\| \\
&\qquad \leq \phi(n) + k_m\|u_n(t) - Pu_n(t)\|
\end{aligned}$$

and $\lim_{n\to\infty} \phi(n) = 0$ where $\phi(n) = \sup_m \|u_{m+n}(t) - U((m+n)T + t, nT + t)u_n(t)\|$, we can choose $n \geq n_0$ such that

$$\begin{aligned}
\|u_{m+n}(t) - Pu_n(t)\| &\leq \frac{\varepsilon}{2} + k_m\|u_n(t) - Pu_n(t)\| \\
&\leq \frac{\varepsilon}{2} + k_m(R - \varepsilon).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\lim_{m\to\infty} \|u_m(t) - Pu_n(t)\| &= \inf_m \sup_{m\leq r} \|u_r(t) - Pu_n(t)\| \\
&\leq \frac{\varepsilon}{2} + (\limsup_m k_m)(R - \varepsilon) \\
&\leq R - \frac{\varepsilon}{2} \\
&< R.
\end{aligned}$$

This is a contradiction. So, we conclude that

$$\lim_{n\to\infty} \|u_n(t) - Pu_n(t)\| = R.$$

We claim that $\lim_{n\to\infty} Pu_n(t) = z_0$. If not, then there exists $\varepsilon > 0$ such that $\|Pu_{n'}(t) - z_0\| \geq \varepsilon$ for some $n' \geq n$. Choose $a > 0$ so small that

$$(R+a)(1-\delta(\frac{\varepsilon}{R+a})) = R_1 < R,$$

where δ is the modulus of convexity of the norm of X. We also have

$$\|u_{n'}(t) - Pu_{n'}(t)\| \leq R + a$$

and

$$\|u_{n'}(t) - z_0\| \leq R + a$$

for all large enough n'. Therefore we have

$$\begin{aligned}\|u_{n'} - \frac{Pu_{n'}(t) + z_0}{2}\| &\leq (R+a)(1-\delta(\frac{\varepsilon}{R+a})) \\ &= R_1.\end{aligned}$$

Since the points $\omega_{n'}(t) = \frac{Pu_{n'}(t)+z_0}{2}$ belong to $F(U_t)$, as in the above,

$$\|u_{m+n'}(t) - \omega_{n'}(t)\| \leq \phi(n') + k_m\|u_{n'}(t) - \omega_{n'}(t)\|$$

for all $m \geq 0$. Since $\lim_{n\to\infty} \phi(n') = 0$, there is n_0 such that

$$\begin{aligned}\|u_{m+n'}(t) - \omega_{n'}(t)\| &\leq k_m\|u_{n'}(t) - \omega_{n'}(t)\| + \frac{R-R_1}{2} \\ &\leq k_m R_1 + \frac{R-R_1}{2}\end{aligned}$$

for all $n \geq n_0$ and hence

$$\begin{aligned}\lim_{m\to\infty} \|u_m(t) - \omega_{n'}(t)\| &= \inf_m \sup_{m\leq\tau} \|u_\tau(t) - \omega_{n'}(t)\| \\ &\leq (\limsup_m k_m)R_1 + \frac{R-R_1}{2} \\ &< R.\end{aligned}$$

This is a contradiction. Therefore $\lim_{n\to\infty} Pu_n(t) = z_0$. Consequently, it follows that the element $z_0 \in F(U_t)$ with $\rho(z_0) = \min\{\rho(z) : z \in F(U_t)\}$ is unique. The proof is complete.

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Department of Mathematics
Dong Eui University
Pusan 614–714, Korea

Department of Mathematics
Pusan National University
Pusan 609–735, Korea