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A NOTE ON DECOMPOSITION SPACE OF MAPPINGS

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1. Introduction

For topological spaces X and Y, let $f : X \to Y$ be any map. We define the relation $K(f)$ in X by $x \sim x'$ if $f(x) = f(x')$. This is clearly an equivalent relation in X and therefore we have the quotient mapping $Q : X \to X/K(f)$. $X/K(f)$ is called the decomposition space of f. From Theorem 7.2 [2, Page 130] if f is continuous open surjection then $X/K(f)$ is homeomorphic to Y. Starting with separation properties of the decomposition space, we have mentioned several other topological informations regarding the same linking these with known results available in the literature. We have discussed some results in [5] and [6].

2. To begin with, we investigate some seperation properties of the decomposition spaces. Let us illustrate by an example that underlying spaces of a mapping may satisfy separation properties upto normality i.e T_4 axioms where as the decomposition space of the mapping under consideration, need not be even T_0:

EXAMPLE 2.1. Let R be the set of real numbers equipped with standard topology and $f : R \to R$ be defined as

$$f(x) = \begin{cases} 1 & \text{, if } x \text{ is rational,} \\ 0 & \text{, if } x \text{ is irrational.} \end{cases}$$

$R/K(f)$ is indiscrete which is not even T_0 where as R satisfies all the separation axioms T_i, $i = 0, 1, 2, 3, 4$. However in the positive direction, we prove the following results.

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Proposition 2.1. *Let $f : X \to Y$ be the mapping with closed graph. Then $X/K(f)$ is a T_1 space.*

Proof. f has closed graph. i.e., $G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ and hence from [3] point inverses are closed in X. i.e., $\{x' \in X : f(x') = f(x)\} = Q^{-1}(Q(x))$ is closed in X. i.e., each point of decomposition space $X/K(f)$ is closed. i.e., $X/K(f)$ is a T_1-space.

While dealing with T_2 seperation property of $X/K(f)$, we make use of following lemmas.

Lemma 2.1.1. *Let $f : X \to Y$ be the continuous mapping with closed graph. Then the relation $K(f)$ is closed in the product space $X \times X$.*

Proof. To this end it sufficies to show that for any point $(x,y) \notin K(f) \Rightarrow (x,y) \notin \overline{K(f)}$. Now $(x,y) \notin K(f) \Rightarrow f(x) \neq f(y)$. Since graph of f is closed in $X \times Y$, so there exist neighborhoods U of x and V of $f(y)$ respectively such that $f(U) \cap V = \emptyset$. Since f is continuous so we have a neighborhood W of y where $f(W) \subset V$.

$$f(U) \cap V = \emptyset \Rightarrow f(U) \cap f(W) = \emptyset \Rightarrow (U \times W) \cap K(f) = \emptyset,$$

Or $(x,y) \notin \overline{K(f)}$ and hence the Lemma holds.

Lemma 2.1.2. *Let $f : X \to Y$ be an open (closed) continuous mapping. Then the quotient mapping $Q : X \to X/K(f)$ is an open (closed) mapping.*

Proof. From 4.2 of [2, Page 125], it is sufficient to show that for any open set U (closed) in X, $K(f)(U)$ is open (closed) in X. Now

$$\begin{aligned} K(f)(U) &= \cup\{K(f)(u) : u \in U\} \\ &= \{x : f(x) = f(u) \text{ for } u \in U\} \\ &= \{x : x \in f^{-1}(f(U))\} \\ &= f^{-1}(f(U)) \text{ which is open in } X. \end{aligned}$$

since f is continuous and open. Similarly, it can be shown that if $f : X \to Y$ is closed continuous mapping then Q is an closed mapping.

PROPOSITION 2.2. *Let $f : X \to Y$ be the continuous open mapping with closed graph. Then $X/K(f)$ is a T_2-space.*

Proof. From Lemma 2.1.1, the relation $K(f)$ is closed in $X \times X$ and from Lemma 2.1.2, the quotient mapping Q is open and hence the proposition follows directly from Theorem 11 of [1, Page 98].

PROPOSITION 2.3. *Let $f : X \to X$ be the self continuous open surjection. Then G_f is the quotient space $X/K(f)$.*

Proof. From [4] G_f is hormeomorphic to X. For completeness, we prove the same. Oviously, $P^* : G_f \to X$ defined as $P^*(x, f(x)) = x$, for any $x \in X$ is bijective. P^* being the restriction of the projection $P : X \times X \to X$ is continuous. It remains to prove that it is open. To this end let $(x, f(x)) \in G_f$ be any arbitrary point. Then it suffices to show that for any open set $(U\times V)\cap G_f$ containing $(x, f(x))$, $P^*((U\times V)\cap G_f)$ is open. Suppose

$$(t, f(t)) \in ((U \times V) \cap G_f) \Longrightarrow t \in U \text{ and } f(t) \in V.$$

Since f is continuous so there exists a neighborhood U_1 of t such that $f(U_1) \subset V$. Setting $W = U_1 \cap V$. We have $f(W) \subset V$ and $P^*((W \times V) \cap G_f) = W \subset P^*((U_1 \times V) \cap G_f)$ i.e., P^* is open and hence homeomorphic. From Thoerem 7.2 [2] $X/K(f)$ is homeomorphic to X which in turn is heomorphic to G_f.

From Lemma 2.1.2, if $f : X \to Y$ is closed continuous mapping then the quotient mapping $Q : X \to X/K(f)$ is closed and the following properties of X are easily transferred to $X/K(f)$.

(1) X is paracompact $\Rightarrow X/K(f)$ is paracompact.
(2) X is normal $\Rightarrow X/K(f)$ is normal.

If we consider the mapping $f : X \to Y$ with compact fiber then as the definition of the quotient mapping, $Q : X \to X/K(f)$ has its each fiber compact and hence if $f : X \to Y$ is a perfect mapping then by Lemma 2.1.2 $Q : X \to X/K(f)$ is perfect mapping, so by Theorem 5.3 [2, Page 236] we have the following properties;

(1) $X/K(f)$ is paracompact $\Longleftrightarrow X$ is paracompact.
(2) $X/K(f)$ is compact $\Longleftrightarrow X$ is compact.
(3) $X/K(f)$ is countably compact $\Longleftrightarrow X$ is countably compact.

(4) $X/K(f)$ is Lindelof $\Longleftrightarrow X$ id Lindelof.
(5) $X/K(f)$ is locally compact $\Longleftrightarrow X$ is locally compact.

References

1. J. L. Kelly, *General topology*, D. Van Nastrand Compant INC, 1955.
2. J. Dugundgi, *Topology*, Prentice Hall of India Private Limited, 1975
3. R. V. Fuller, *Relations among continuous and various non continuous functions*, Pacific J. Math 30(1968).
4. V. P. Singh, *Function and its graph*, Under commucation.
5. V. P. Singh, G I. Chae and D. N. Misra, *Closed graph and hyper closed functions*, UOU Report 21(1990), 111–113.
6. G. I. Chae, V. P. Singh and D. N. Misra, *Composition of functions with closed graph and its function spaces*, Pusan-Kyŏngnam Math J. 5(1989).

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