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SOME Hp–THEOREMS FOR HYPERSURFACES

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Let $M^n, n \geq 2$, be an orientable compact n-dimensional manifold without boundary and assume $x : M^n \to R^{n+1}$ is an isometric immersion. Sometimes, x will be considered as the position vector of M^n. For a globally defined unit normal vector field ν of M^n, we call $p = < x, \nu >$ a support fuction of M^n. Rotondaro Giovanni [2] proved that if Hp has a constant value n, then M^n is a standard sphere centered at 0.(Here, H is the mean curvature function of M^n.) In this short note, we will prove Giovanni's theorem by some different methods. Some of the calculation in this note was inspired by computation in a paper by Gerhard Huisken[2].

1. Preliminaries

We need some definitions and lemmas. Many of them are due to [5]. $\overline{\nabla}$ denotes covariant differentiation on R^{n+1}, and ∇ denotes covariant differentiation on M^n.

DEFINITIONS.

(1) $h(X,Y) = - < \overline{\nabla}_x Y, \nu >$ for X, Y sections of TM^n. h is the second fundamental form of the immersion. $<,>$ means the usual inner product of R^{n+1}.

(2) For an orthonormal framing $(e_1, \cdots, e_n)$ of TM^n, $H = \sum h(e_i, e_i)$. This definition of H is independent of the framing.

(3) The Coddazi equations, for X, Y, Z sections in TM^n, are

$$(\nabla_x h)(Y, Z) = (\nabla_y h)(X, Z),$$

where

$$(\nabla_x h)(Y, Z) = \nabla_x h(Y, Z) - h(\nabla_x Y, Z) - h(Y, \nabla_x Z).$$

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(4) The Laplacian Δf of a function f on M^n is given by $\Delta f = \sum g^{ij} \nabla_{xi} \nabla_{xj} f$, where $(x_1, \cdots, x_n)$ is a framing of M^n and $(g^{ij}) = (g_{ij})^{-1}$.

(5) The norm of second fundamental form $|A|^2$ is given by

$$|A|^2 = \sum g^{ij} g^{kl} h(x_i, x_k) h(x_j, x_l).$$

LEMMA 1.1. *If $M^n \subset R^{n+1}$ is immersed, then $n|A|^2 \geq H^2$. Equality holds if and only if M^n is a sphere.*

Proof. See [5].

LEMMA 1.2 (HOPF'S MAXIMUM PRINCIPLES). *If a C^2-function f defined on $M^n \subset R^{n+1}$ has a strict maximum (resp. minimum) value at $p \in M^n$, then $(\Delta f)(p) < 0$). (resp. $(\Delta f)(p) > 0$.)*

Proof. See [1].

2. Proofs of Hp-theorem

If $M^n \subset R^{n+1}$ satisfies the equation $Hp = n$, then we may assume $H > 0$ and $p > 0$.

THEOREM 2.1. *If M^n is compact and satisfies the equation $Hp = n$, then M^n is a standard shere centered at 0.*

Proof. We differentiate the equation $p = < x, \nu >$ in an orthonormal frame $e_1, e_2, \cdots, e_n$ on M^n. Then

$$\begin{aligned} \nabla_{e_i} p &= < \overline{\nabla}_{e_i} x, \nu > + < x, \overline{\nabla}_{e_i} \nu > \\ &= < e_i, \nu > + < x, \sum h_{li} e_l > \\ &= \sum < x, e_l > h_{li} \qquad (\text{ where } \quad h_{ij} = h(e_i, e_j)) \end{aligned} \tag{1}$$

$$\begin{aligned} \nabla_{e_j} \nabla_{e_i} p &= \sum < \overline{\nabla}_{e_j} x, e_l > h_{li} + \sum < x, \overline{\nabla}_{e_j} e_l > h_{li} \\ &\quad + \sum < x, e_l > \nabla_j h_{li} \\ &= \sum < e_j, e_l > h_{li} + \sum < x, h_{jl} v > h_{ei} \\ &\quad + \sum < x, e_l > \nabla_l h_{ji} \\ &= h_{ji} + p \sum h_{jl} h_{li} + \sum < x, e_l > \nabla_l h_{ji} \end{aligned} \tag{2}$$

Here we used again $p = <x, \nu>$ and the Coddazi equation. (We assume $\nabla_{e_i} e_j = 0$ for all i, j.)

From (2) we obtain

$$\begin{aligned}\Delta p &= H - p|A|^2 + \sum < x, e_l > \nabla_l H \\ &= H - (n|A|^2/H) + \sum < x, e_l > \nabla_l H \\ &= (H^2 - n|A|^2)/H + \sum < x, e_l > \nabla_l H \\ &\leq \sum < x, e_l > \nabla_l H\end{aligned}$$

Since M^n is compact, p has a minimum at some point $q \in M^n$. And H has a maximum value at q. Applying the Hopf's maximum principles, we conclude that p is constant and $H^2 = n|A|^2$. This implies M^n is a standard sphere centered at 0.

REMARK 1. If $M^n \subset R^{n+1}$ is embedded and satisfies the equation $Hp = n$, then we can directly derive the result by using Ros' inequality[4] $\int n/H \, dA \geq nV$ and the formula $\int p \, dA = nV$.

REMARK 2. If $M^2 \subseteq R^3$ is noncompact and satisfies the equation $Hp = n$, we expect M is cylinder.

REMARK 3. Gerhard Huisken[2] proved that if $M^n \subset R^{n+1}$ satisfies the equation $H = p$, then M^n is a standard sphere with radius $\sqrt{n}$. His computation may be applicable in several directions.

References

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