SOME $H_P$–THEOREMS FOR HYPERSURFACES

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Let $M^n, n \geq 2,$ be an orientable compact $n$–dimensional manifold without boundary and assume $x : M^n \to R^{n+1}$ is an isometric immersion. Sometimes, $x$ will be considered as the position vector of $M^n.$ For a globally defined unit normal vector field $\nu$ of $M^n,$ we call $p = <x, \nu>$ a support function of $M^n.$ Rotondaro Giovanni [2] proved that if $H_p$ has a constant value $n,$ then $M^n$ is a standard sphere centered at 0. (Here, $H$ is the mean curvature function of $M^n.$) In this short note, we will prove Giovanni's theorem by some different methods. Some of the calculation in this note was inspired by computation in a paper by Gerhard Huisken[2].

1. Preliminaries

We need some definitions and lemmas. Many of them are due to [5]. $\nabla$ denotes covariant differentiation on $R^{n+1},$ and $\nabla$ denotes covariant differentiation on $M^n.$

DEFINITIONS.

(1) $h(X, Y) = - <\nabla_X Y, \nu>$ for $X, Y$ sections of $TM^n.$ $h$ is the second fundamental form of the immersion. $<,>$ means the usual inner product of $R^{n+1}.$

(2) For an orthonormal framing $(e_1, \cdots, e_n)$ of $TM^n,$

$H = \sum h(e_i, e_i).$ This definition of $H$ is independent of the framing.

(3) The Coddazi equations, for $X, Y, Z$ sections in $TM^n,$ are

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z),$$

where

$$(\nabla_X h)(Y, Z) \equiv \nabla_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

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(4) The Laplacian $\Delta f$ of a function $f$ on $M^n$ is given by $\Delta f = \sum g^{ij} \nabla_i \nabla_j f$, where $(x_1, \cdots, x_n)$ is a framing of $M^n$ and $(g^{ij}) = (g_{ij})^{-1}$.

(5) The norm of second fundamental form $|A|^2$ is given by

$$|A|^2 = \sum g^{ij} g^{kl} h(x_i, x_k) h(x_j, x_l).$$

**Lemma 1.1.** If $M^n \subset R^{n+1}$ is immersed, then $n|A|^2 \geq H^2$. Equality holds if and only if $M^n$ is a sphere.

*Proof.* See [5].

**Lemma 1.2 (Hopf's Maximum Principles).** If a $C^2$-function $f$ defined on $M^n \subset R^{n+1}$ has a strict maximum (resp. minimum) value at $p \in M^n$, then $(\Delta f)(p) < 0$. (resp. $(\Delta f)(p) > 0$.)

*Proof.* See [1].

2. Proofs of $H_P$-theorem

If $M^n \subset R^{n+1}$ satisfies the equation $H_P = n$, then we may assume $H > 0$ and $p > 0$.

**Theorem 2.1.** If $M^n$ is compact and satisfies the equation $H_P = n$, then $M^n$ is a standard sphere centered at $0$.

*Proof.* We differentiate the equation $p = \langle x, \nu \rangle$ in an orthonormal frame $e_1, e_2, \cdots, e_n$ on $M^n$. Then

$$\nabla_{e_i} p = \langle \nabla e_i, x, \nu \rangle + \langle x, \nabla e_i, \nu \rangle = \langle e_i, \nu \rangle + \langle x, \sum h_{i1} e_l \rangle = \sum \langle x, e_l \rangle > h_{i1} \quad (\text{where} \quad h_{ij} = h(e_i, e_j))$$

$$\nabla e_i \nabla e_j p = \sum \langle \nabla e_i, x, e_l \rangle > h_{i1} + \sum \langle x, \nabla e_j, e_l \rangle > h_{i1}$$

$$+ \sum \langle x, e_l \rangle > \nabla_j h_{i1}$$

$$= \sum \langle e_j, e_l \rangle > h_{i1} + \sum \langle x, h_{jl} \nu \rangle > h_{i1}$$

$$+ \sum \langle x, e_l \rangle > \nabla_l h_{ji}$$

$$= h_{ji} + p \sum h_{jl} h_{i1} + \sum \langle x, e_l \rangle > \nabla_l h_{ji}$$
Here we used again $p = \langle x, \nu \rangle$ and the Coddazi equation. (We assume $\nabla e_i, e_j = 0$ for all $i, j$.)

From (2) we obtain

$$\Delta p = H - p|A|^2 + \sum <x, e_i > \nabla_i H = H - (n|A|^2/H) + \sum <x, e_i > \nabla_i H = (H^2 - n|A|^2)/H + \sum <x, e_i > \nabla_i H \leq \sum <x, e_i > \nabla_i H$$

Since $M^n$ is compact, $p$ has a minimum at some point $q \in M^n$. And $H$ has a maximum value at $q$. Applying the Hopf's maximum principles, we conclude that $p$ is constant and $H^2 = n|A|^2$. This implies $M^n$ is a standard sphere centered at 0.

**Remark 1.** If $M^n \subseteq R^{n+1}$ is embedded and satisfies the equation $Hp = n$, then we can directly derive the result by using Ros' inequality [4] $\int n/H \ dA \geq nV$ and the formula $\int p \ dA = nV$.

**Remark 2.** If $M^2 \subseteq R^3$ is noncompact and satisfies the equation $Hp = n$, we expect $M$ is cylinder.

**Remark 3.** Gerhard Huisken [2] proved that if $M^n \subseteq R^{n+1}$ satisfies the equation $H = p$, then $M^n$ is a standard sphere with radius $\sqrt{n}$. His computation may be applicable in several directions.

**References**

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