ON EMBEDDED SURFACES WITH CONSTANT NONZERO MEAN CURVATURE

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1. Introduction

The mean curvature function $H$ on an oriented surface $S$ in $R^3$ is defined at a point $p$ in $S$ to be $H(p) = \lambda_1(p) + \lambda_2(p)$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the principal curvatures of $S$ at $p$. When $H$ is constant, $S$ is called a surface of constant mean curvature. In this paper, if $S$ is a surface of constant mean curvature $H$, we call $S$ an MCH-surface. We can (and will) assume $H > 0$.

We consider properly embedded MCH-annuli $A$, which are homeomorphic to the punctured unit disc $D\setminus O$ in $R^2$. Let $F : D\setminus O \to A \subset R^3$ be a homeomorphism. Then $f$ will be a proper map and $F(y) \to \infty$ as $y \to 0$. Due to W. Meeks III [1], every properly embedded MCH annulus $A$ is cylindrically bounded, i.e., $A$ stays a bounded distance from one half infinite straight line. Recently, N.J. Korevaar, R. Kusner and B. Solomon proved that every properly embedded MCH-annulus is asymptotic to a Delaunay surface [2]. They also proved that if $\sum$ is a complete properly embedded MCH-surface and has two annular end, then it is a Delaunay surface.

Modifying the method of three authors, we obtained some different results about properly embedded MCH-annuli. Also, we proved that if $S \subset R^3$ is a compact MCH-graph with $\partial S \subset x^3 = 0$ and if $S$ has a point $p$ such that $x^3(P) = 2H^{-1}$, then $S$ is a hemisphere.

We need some notations and definitions. Many of them are due to three authors.

\begin{equation}
(1.1). \text{ For } 0 < R < \infty, P \in R^3 \text{ and given a unit vector } v \text{ the disc with center } P \text{ and normal } v, \text{ is defined by } D_{v,R}(P) = \{ y \in R^3 : \}
\end{equation}
\[ |y - P| \leq R, \ (y - P) \cdot v = 0 \}. \] The solid half cylinder generated by \( D_{v,R}(p) \) and \( v \) is
\[ C_{v,R}^+(p) = \{ y + xv : y \in D_{v,R}(p), \ x \geq 0 \}. \]

Due to W. Meeks III, if \( A \subset R^3 \) is a properly embedded MCH-annulus, then there exists \( C_{v,R}^+(P) \) such that \( A \subset C_{v,R}^+(P) \). In this case, We call \( v \) an axis vector of \( A \).

2. Compact MCH-graphs

In this section, we will prove that a compact MCH-graph with some property must be a hemisphere.

**Proposition 2.1.** Proposition suppose \( S \subset R^3 \) is a compact MCH-graph with \( \partial S \subset \{ x^3 = 0 \} \). Then \( |x^3(S)| \leq 2H^{-1} \). Furthermore, if \( S \) has a point \( p \) such that \( x^3(p) = 2H^{-1} \) or \(-2H^{-1}\), then \( S \) must be a hemisphere.

**Proof.** We may assume \( x^3(S) \geq 0 \). By the Cauchy-Schwarz inequality, the second fundamental form \( A \) and the mean curvature \( H \) satisfies \( 2|A|^2 - H^2 \geq 0 \). On a graph, the (upward) unit normal \( v \) satisfies \( v^3 \geq 0 \). Combining these inequalities with the equations \( \Delta x^3 = -Hv^3 \) and \( \Delta v^3 = -|A|^2v^3 \) yields the differential inequality \( \Delta (Hx^3 - 2v^3) \geq 0 \) on \( S \). Since \( Hx^3 - 2v^3 \leq 0 \) on \( \partial S \), the maximum principle implies the same inequality on \( S \). The first result follows since \( v^3 \leq |v| = 1 \). Suppose \( x^3(p) = 2H^{-1} \) at some point \( p \in S \). Then \( Hx^3 - 2v^3 \) has an interior maximum at \( p \). The maximum principle implies \( Hx^3 - 2v^3 \) must be constant and \( \Delta (Hx^3 - 2v^3) = (2|A|^2 - H^2)v^3 \) is constantly zero. By continuity, we may conclude that \( 2|A|^2 - H^2 = 0 \). Hence \( S \) is a hemisphere with radius \( H \).

**Remarks.**

1. The first part of Proposition 2.1 are firstly overserved by Serrin [3].
2. For the known examples, if \( S \) is an MCH-graph over a connected closed (not necessarily compact) domain in \( \{ x^3 = 0 \} \) with \( \partial S \subset \{ x^3 = 0 \} \), we expect \( S \) has the property mentioned in Proposition 2.1.
Constant nonzero mean curvature

Corollary 2.2. Let $S$ be a compact MCH-graph with $\partial S \subset \{x^3 = 0\}$. If $S$ is not a hemisphere, then $|x^3(S)| < 2H^{-1}$.

3. Properly embedded MCH-annulus

To prove our results, we need some argument which is similar to three authors'. Let $A$ be a properly embedded MCH-annulus and let $A \subset C^{+}_{a,R}(q)$. We may assume $q = 0$. The axis vector $a$ is parallel to positive $x_1$-axis.

Fix a plane $\Pi \subset R^3$ with unit normal $v$, which is below annulus $A$ and is parallel to the axis vector $a$. Let $L$ be the perpendicular line given by $L = \{tv : t \in R\}$. For $t \in R$ and $p \in \Pi$ define the $\Pi$-parallel plane $\Pi_t$, and the $\Pi$-perpendicular line $L_p$ by

$$\Pi_t = \Pi + tv, \quad L_p = p + L. \tag{3.1}$$

For a point $p \in \Pi$, consider the line $L_p(3.1)$. Let $p_1 = p + t_1v$ be the first point in $L_p \cap A$ as $t$ decreases from $\infty$. If the intersection is transverse and if $L_p$ meets $A$ at $p_2 = p + t_2v$ secondly, (if $L_p$ meets $A$ at $p_1$ tangently, let $p_2 = p_1$) then $p$ is in the domain of Alexandrov function $\alpha_1$ defined by

$$\alpha_1(p) = (t_1 + t_2)/2. \tag{3.2}$$

If $\alpha_1$ has an interior local maximum at $p \in \Pi$, then one can show the plane $\Pi_{\alpha_1(p)}$ is a plane of symmetry for $A$ [2, Lemma 2.6]. Three authors observed that $\alpha_1$ is upper-semicontinuous. Now, we state three authors’ crucial lemma. They proved the following lemma by using cylindrical boundedness of $A$, Alexandrov reflection technique and upper-semicontinuity of $\alpha_1$.

Lemma 3.1 [2]. Let $A \subset C^{+}_{a,R}(0)$. Define the related Alexandrov function $\alpha$ on $A$

$$\alpha(x) = \max_{p \in \Pi} \alpha_1(p). \tag{3.3}$$

Then $\alpha$ is not increasing. i.e., either $\alpha(x)$ is strictly decreasing, or else $A$ has a plane of reflection symmetry parallel to $\Pi$.

By simple application of above lemma, we obtain the following result.
PROPOSITION 3.2. Let $A$ be a properly embedded MCH-annulus and let $A$ be contained in $C_{a,R}^+(0)$ and $\partial A \subset D_{a,R}(0)$. If $\partial A$ has a line of reflection symmetry, and the portion of $\partial A$ above this line is a graph, then $A$ has a plane of symmetry parallel to $a$ and to this line.

Proof. Consider some plane $\Pi$ which lies below $A$ and is parallel to $a$ and the line of symmetry. The symmetry of $\partial A$ implies that $\alpha_1(p)$ is constant for all $p \in \Pi$ with $L_p \cap \partial A \neq \emptyset$. This constant value is equivalent to $\alpha(0)$. If $A$ has not a plane of symmetry parallel to $\Pi$, then $\alpha_1(q) < \alpha(0)$ for all $q$ (at which $\alpha_1$ can be defined) with $q \cdot a > 0$. Consider another plane $\Pi$ which lies above annulus $A$ and is parallel to $\Pi$. Then the function $\alpha$ relative to $\Pi$ has the property $\alpha(0) < \alpha_1(q)$ for all $q \in \Pi$ (at which $\alpha_1$ can be defined) with $q \cdot a > 0$. This is contradiction to Lemma 3.3. Hence $A$ has a plane of symmetry $\Pi_\perp$ parallel to $a$ and the line of symmetry.

COROLLARY 3.3. Let $A$ be a properly embedded MCH-annulus contained in $C_{a,R}^+(0)$. If some plane $a \perp$ which orthogonal to the axis vector $a$ makes a circle by intersecting the annulus $A$, then $A$ is a Delaunay surface.

Proof. If $a \perp \cap A$ bounds a compact component of $A$, then we can show that this component is a piece of sphere by using Alexandrov reflection technique. By annaliticity of MCH-surface, $A$ must be a piece of sphere. This is impossible. Hence we may assume $a \perp \cap A$ seperates $A$ into a compact annulus and an infinite annulus. Consider the infinite part. This annulus has symmetry planes parallel to every plane containing $a$ by Proposition 3.4. But the center of mass of any cross-section of $\Sigma$ perpendicular to $a$ must be contained in each symmetry plane. Hence all symmetry planes intersect in a line parallel to $a$, and this annulus has rotational symmetry about this line.

References
Constant nonzero mean curvature

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