Pusan-Kyŏngnam Math. J. 7(1991), No. 1, pp. 23–27

ON EMBEDDED SURFACES WITH CONSTANT NONZERO MEAN CURVATURE

CHANG-RIM JANG AND SEONG-KOWAN HONG

1. Introduction

The mean curvature function H on an oriented surface S in R^3 is defined at a point p in S to be $H(p) = \lambda_1(p) + \lambda_2(p)$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the principal curvatures of S at p. When H is constant, S is called a surface of constant mean curvature. In this paper, if S is a surface of constant mean curvature H, we call S an MCH-surface. We can (and will) assume $H > 0$.

We consider properly embedded MCH-annulli A, which are homeomorphic to the punctured unit disc $D\backslash O$ in R^2. Let $F : D\backslash O \to A \subset R^3$ be a homeomorphism. Then f will be a proper map and $F(y) \to \infty$ as $y \to 0$. Due to W. Meeks III [1], every properly embedded MCH annulus A is cylindrically bounded, i.e., A stays a bounded distance from one half infinite straight line. Recently, N.J. Korevarr, R. Kusner and B. Solomon proved that every properly embedded MCH- annulus is asymtotic to a Delaunay surface [2]. They also proved that if $\sum$ is a complete properly embedded MCH-surface and has two annular end, then it is a Delaunay surface.

Modifying the method of three authors, we obtained some different results about properly embedded MCH-annulli. Also, we proved that if $S \subset R^3$ is a compact MCH-graph with $\partial S \subset x^3 = 0$ and if S has a point p such that $x^3(P) = 2H^{-1}$, then S is a hemisphere.

We need some notations and definitions. Many of them are due to three authors.

(1.1). *For* $0 < R < \infty, P \in R^3$ *and given a unit vector* v *the disc with center* P *and normal* v, *is defined by* $D_{v,R}(P) = \{ y \in R^3 :$

Received January 15, 1991.

This work was supproted by a research grant of the chief director of U.O.U

$|y - P| \leq R,\ (y - P) \cdot v = 0 \}$. *The solid half cylinder generated by* $D_{v,R}(p)$ *and* v *is*

$$C^{+}_{v,R}(p) = \{ y + xv : y \in D_{v,R}(p),\ x \geq 0 \}.$$

Due to W. Meeks III, if $A \subset R^3$ *is a properly embedded MCH- annulus. then there exists* $C^{+}_{v,R}(P)$ *such that* $A \subset C^{+}_{v,R}(P)$. *In this case, We call* v *an axis vector of* A.

2. Compcat MCH-graphs

In this section, we will prove that a compact MCH-graph with some property must be a hemisphere.

PROPOSITION 2.1. *Proposition suppose* $S \subset R^3$ *is a compact MCH-graph with* $\partial S \subset \{ x^3 = 0 \}$. *Then* $|x^3(S)| \leq 2H^{-1}$. *Futhermore, if* S *has a point* p *such that* $x^3(p) = 2H^{-1}$ *or* $-2H^{-1}$, *then* S *must be a hemisphere.*

Proof. We may assume $x^3(S) \geq 0$. By the Cauchy-Schwarz inequaity, the second fundamental form A and the mean ourvature H satisfies $2|A|^2 - H^2 \geq 0$. On a graph, the (upward) unit normal v satisfies $v^3 \geq 0$. Combining these inequalities with the equations $\Delta x^3 = -Hv^3$ and $\Delta v^3 = -|A|^2 v^3$ yields the differential inequality $\Delta(Hx^3 - 2v^3) \geq 0$ on S. Since $Hx^3 - 2v^3 \leq 0$ on ∂S, the maximum principle implies the same inequality on S. The first result follows since $v^3 \leq |v| = 1$. Suppose $x^3(p) = 2H^{-1}$ at some point $p \in S$. Then $Hx^3 - 2v^3$ has an interior maximum at p. The maximum principle implies $Hx^3 - -2v^3$ must be constant and $\Delta(Hx^3 - 2v^3) = (2|A|^2 - H^2)v^3$ is constantly zero. By continuity, we may conclude that $2|A|^2 - H^2 = 0$. Hence S is a hemisphere with radius H.

REMARKS.

1. The first part of Proposition 2.1 are firstly overserved by Serrin [3].
2. For the known examples, if S is an MCH-graph over a connected closed (not necessarly compact) domain in $\{ x^3 = 0 \}$ with $\partial S \subset \{ x^3 = 0 \}$, we expect S has the property mentioned in Proposition 2.1.

COROLLARY 2.2. *Let S be a compact MCH-graph with $\partial S \subset \{ x^3 = 0 \}$. If S is not a hemisphere, then $|x^3(S)| < 2H^{-1}$.*

3. Properly embedded MCH-annulus

To prove our results, we need some argument which is similar to three authors'. Let A be a properly embedded MCH-annulus and let $A \subset C^+_{a,R}(q)$. We may assume $q = 0$. The axis vector a is parallel to positive x_1-axis.

Fix a plane $\Pi \subset R^3$ with unit normal v, which is below annulus A and is parallel to the axis vector a. Let L be the perpendicular line given by $L = \{ tv : t \in R \}$. For $t \in R$ and $p \in \Pi$ define the Π-parallel plane Π_t, and the Π-perpendicular line L_p by

$$\Pi_t = \Pi + tv, \qquad \mathrm{L}_p = p + L. \tag{3.1}$$

For a point $p \in \Pi$, consider the line L_p(3.1). Let $p_1 = p + t_1 v$ be the first point in $L_p \cap A$ as t decreases from ∞. If the intersection is transverse and if L_p meets A at $p_2 = p + t_2 v$ secondly, (if L_p meets A at p_1 tangently, let $p_2 = p_1$) then p is in the domain of Alexnadrov function α_1 defined by

$$\alpha_1(p) = (t_1 + t_2)/2. \tag{3.2}$$

If α_1 has an interior local maximum at $p \in \Pi$, then one can show the plane $\Pi_{\alpha_1(p)}$ is a plane of symmetry for A[2, Lemma 2.6]. Three authors observed that α_1 is upper-semicontinuous. Now, we state three authors' crucial lemma. They proved the following lemma by using cylindrical boundedness of A, Alexandrov reflection technique and upper-semicontinuity of α_1.

LEMMA 3.1 [2]. *Let $A \subset C^+_{a,R}(0)$. Define the related Alexandrov function α on A*

$$\alpha(x) = \max_{\substack{p \in \Pi \\ p \cdot a = x \geq 0}} \alpha_1(p). \tag{3.3}$$

Thenα is not increasing. i.e., either $\alpha(x)$ is strictly decreasing, or else A has a plane of reflection symmetry parallel to Π.

By simple application of above lemma, we obtaind the following result.

PROPOSITION 3.2. *Let A be a properly embedded MCH-annulus and let A be contained in $C^+_{a,R}(0)$ and $\partial A \subset D_{a,R}(0)$. If ∂A has a line of reflection symmetry, and the portion of ∂A above this line is a graph, then A has a plane of symmetry parallel to a and to this line.*

Proof. Consider some plane Π which lies below A and is parallel to a and the line of symmetry. The symmetry of ∂A implies that $\alpha_1(p)$ is constant for all $p \in \Pi$ with $L_p \cap \partial A \neq \emptyset$. This constant value is equivalent to $\alpha(0)$. If A has not a plane of symmetry parallel to Π, then $\alpha_1(q) < \alpha(0)$ for all q (at which α_1 can be defined) with $q \cdot a > 0$. Consider another plane Π which lies above annulus A and is parallel to Π. Then the function α relative to Π has the property $\alpha(0) < \alpha_1(q)$ for all $q \in \Pi$ (at which α_1 can be defined) with $q \cdot a > 0$. This is contradiction to Lemma 3.3. Hence A has a plane of symmetry Π_z parallel to a and the line of symmetry.

COROLLARY 3.3. *Let A be a properly embedded MCH-annulus contained in $C^+_{a,R}(0)$. If some plane $a\perp$ which orthogonalto the axis vector a makes a circle by intersecting the annulus A, then A is a Delauny surface.*

Proof. If $a\perp \cap A$ bounds a compact component of A, then we can show that this component is a piece of sphere by using Alexandrov reflection technique. By annaliticity of MCH-surface, A must be a piece of sphere. This is impossible. Hence we may assume $a\perp \cap A$ seperates A into a compact annulus and an infinite annulus. Consider the infinite part. This annulus has symmetry planes parallel to every plane containing a by Proposition 3.4. But the center of mass of any cross-section of Σ perpendicular to a must be contained in each symmetry plane. Hence all symmetry planes intersect in a line parallel to a, and this annulus has rotational symmetry about this line.

References

1. W. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, J Diff. Geometry **27**(1988), 539–552.
2. N. J. Korevarr ,R. Kusner B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Diff. Geometry **30**(1989), 465–503.
3. J. Serrin, *On surfaces of constant mean curvature which span a given space curve*, Math. Z. **112**(1969), 77–88.

Department of Mathematics
Ulsan University
Ulsan 680–749, Korea

Department of Mathematic Education
Pusan National University
Pusan 609–390, Korea