Pusan-Kyŏngnam Math. J. 7(1991), No. 1, pp. 29–36

MAX–MIN CONTROLLABILITY FOR TIME DELAY SYSTEM

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1. Introduction

For linear time-delay systems in the Banach spaces, the concept of controllability with constraint has been studied Chan and Li [2] and Park, Nakagiri and Yamamoto [4].

In this paper we study max-min controllability problems for a linear time-delay system in a Banach space. These are problems in game theory, where in order to obtain a desired state, two persons (called playes) can move respective controls in a linear time-delay system; a forcing function and an initial function correspond to two player's controls.

Let X and U be a reflexive Banach spaces over C or R, with norms $\|\cdot\|$ and $\|\cdot\|_U$ respectively.

We consider an abstract control system (1) on X with time-delays;

$$(1)\quad \begin{cases} \dfrac{dx(t)}{dt} = A_0x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(t+s) + B(t)u(t) \quad \text{a.e.} \quad t > 0 \\ x(0) = g^0, \quad x(s) = g^1(s) \qquad \text{a.e.} \quad s \in [-h, 0), \end{cases}$$

where $g = (g^0, g^1) \in X \times L_q([-h,0]; X)$, $u \in L_p^{loc}(R^+; U)$, $p, q \in (1,\infty)$, $\{B(t); t \geq 0\} \subset \mathcal{L}(U, X)$ is a bounded operators from U into X, A_0 generates a C_0-semigroup $\{T(t); t \geq 0\}$ on X and η is a Stieltjes measure given by

$$(2)\quad \eta(s) = -\sum_{r=1}^{m} \chi_{(-\infty,-h_r]}(s)A_r - \int_s^0 A_I(\xi)\,d\xi, \; s \in [-h, 0].$$

In (2), ξ_E denotes the characteristic function of E and it is assumed that $0 < h_1 < \cdots < h_m \equiv h$, $A_r \in \mathcal{L}(X)$ $(r = 1, \cdots, m)$ and $A_I(\cdot) \in$

Received February 20, 1991.

$L_1([-h,0];\mathcal{L}(X))$. Here and henceforth $\mathcal{L}(U,X)$ denotes the set of all bounded linear operators on U into X and also $\mathcal{L}(X)=\mathcal{L}(X,X)$ is defined similary. Then the delayed term in (1) is written by

$$\sum_{r=1}^{m} A_r x(t-h_r) + \int_{-h}^{0} A_I(s)x(t+s)\,ds.$$

Let $W(t)$ be the fundamental solution of (1), which is a unique of the equation

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)W(\xi+s)\,ds, & t \geq 0, \\ 0, \qquad t<0. \end{cases}$$

Then $W(t) \in \mathcal{L}(X)$ for each $t \geq 0$ and $W(t)$ is strongly continuous in R^+ (e.g. Nakagiri [3]).

If the condition

$$(3) \qquad A_I(\cdot) \in L_{q'}([-h,0];\mathcal{L}(X)), \quad 1/q + 1/q' = 1$$

is satisfied, then for each $t \geq 0$, the operator valued function $U_t(\cdot)$ given by

$$(4) \qquad U_t(s) = \int_{-h}^{s} W(t-s+\xi)\,d\eta(\xi) \quad \text{a.e. } s \in [-h,0]$$

belongs to $L_{q'}([-h,0];\mathcal{L}(X))$. This follows from the Hausdorff-Young inequality. Hence the function

$$(5) \qquad x(t;g,u) = \begin{cases} W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)\,ds \\ \qquad + \int_0^t W(t-s)B(s)u(s)\,ds, \ t \geq 0 \\ g^1(t) \quad \text{a.e. } t \in [-h,0) \end{cases}$$

is well-defined and is an element of $C(R^+;X)$. Moreover it is proved in [3] that under the condition (3), the function $x(t) = x(t;g,u)$ is a unique solution of the integrated form of (1) by $T(t)$, i.e.,

$$(6) \qquad \begin{aligned} x(t) =& T(t)g^0 + \int_0^t T(t-s)B(s)u(s)ds \\ &+ \int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)x(s+\xi)ds, \quad t \geq 0. \end{aligned}$$

In this sense, this function $x(t)$ is called the mild solution of (1). In the system (1), $u(t)$ and $g^1(s)$ are called a forcing function control and initial function control, respectively. Here we note that $g^0 \equiv x(0)$ is not considered as a control. we will study the max-min controllability by means of mild solution.

The purpose of this paper is to prove the max-min controllability results for the abstract control system (1).

2. Max-Min Controllability

For each $t \geq 0, \delta > 0, \rho > 0$ and $p, q \in (1, \infty)$, we define the constraint sets

$$U_p^\delta = \{u \in L_p([0,t_1];U); \|u\|_p = (\int_0^{t_1} \|u(s)\|_U^p \, ds)^{1/p} \leq \delta\}$$

and

$$G_q^\rho = \{g^1 \in L_q([-h,0];X); \|g^1\|_q = (\int_{-h}^0 \|g^1(s)\|^q \, ds)^{1/q} \leq \rho\}$$

For convenience, we denote the above linear differential game problem by the notation

$$(g^0, U_p^\delta(T), G_q^\rho(T), T = [0, t_1]).$$

DEFINITION 2.1. The system $(g^0, U_p^\delta(T), G_q^\rho(T), T)$ is said to be max-min controllable if for each initial function control $g^1 \in G_q^\rho(T)$, there exists a forcing function control $u \in U_p^\delta(T)$ such that $x(t_1, (g^0, g^1), u) = 0$.

For each $t > 0$ we define two operators $\mathcal{B}_{t_1}; L_p([0,t_1];U) \longrightarrow X$ and $\mathcal{C}_{t_1}; L_q([-h,0];X) \longrightarrow X$ by

$$\mathcal{B}_{t_1} u = \int_0^{t_1} W(t_1 - s)B(s)u(s) \, ds,$$

and

$$\mathcal{C}_{t_1} g^1 = \int_{-h}^0 U_{t_1}(s) g^1(s) \, ds,$$

respectively.

Put

$$R = W(t_1)g^0 + \mathcal{B}_{t_1}(U_p^\delta).$$

REMARK 2.1. The system $(g^0, U_p^\delta(T), G_q^\rho(T), T)$ is max-min controllable iff, for each initial function control $g^1 \in G_q^\rho(T)$, there exists some forcing function control $u \in U_p^\delta(T)$ such that the corresponding trajectory

$$x(t_1;(g^0,g^1);u) = W(t_1)g^0 + \mathcal{C}_{t_1}g^1 + \mathcal{B}_{t_1}u = 0$$

or

$$\mathcal{C}_{t_1}(-g^1) = W(t_1)g^0 + \mathcal{B}_{t_1}u \in R.$$

Thus, the system $(g^0, U_p^\delta(T), G_q^\rho(T), T)$ is max-min controllable iff

$$\mathcal{C}_{t_1}(G_q^\rho(T)) \subset R$$

or

$$\{\mathcal{C}_{t_1}g^1\} \cap R \neq \emptyset \quad \text{for all } \; g^1 \in G_q^\rho(T).$$

To see that these conditions hold, we need the following Lemmas.

LEMMA 2.1. ([4]) *If Y and Z are closed convex sets of X^* with one being compact, then a necessary and sufficient condition that $Y \cap Z \neq \emptyset$ is that, for all $x^* \in X^*$,* we have

$$\inf_{y \in Y} < y, x^* > \le \sup_{z \in Z} < z, x^* > .$$

LEMMA 2.2. ([4]) *We assume that $T(t)$ is compact for all $t > 0$. Then the operators $\mathcal{B}_t$ and $\mathcal{C}_t$ are continuous linear compact operators.*

LEMMA 2.3. ([4]) *The sets $\mathcal{B}_t(U_p^\delta)$, $\mathcal{C}_t(G_q^\rho)$, R are compact and convex.*

THEOREM 2.1. *The system*

$$(g^0,\, U_p^\delta(T),\, G_q^\rho(T),\, T = [0, t_1])$$

is max-min controllable if and only if

$$\tag{7} \begin{aligned} | < W(t_1)g^0, x^* > | \le \delta\Big(\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'}\, ds\Big)^{1/p'} \\ - \rho\Big(\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'}\, ds\Big)^{1/q'} \end{aligned}$$

for each $x^* \in X^*$. *Where* $1/p + 1/q = 1$, $1/p' + 1/q' = 1$ *and the superscript indicates the adjoint.*

Proof. From the Remark, the above system is max-min controllable iff

$$\{\mathcal{C}_{t_1} g^1\} \cap R \neq \emptyset \text{ for all } g^1 \in G_q^\rho(T).$$

Hence Lemma 2.1 and 2.3, it is equivalent to that for any $x^* \in X^*$, we have

$$\inf_{y \in R} < y, x^* > \leq < \mathcal{C}_{t_1} g^1, x^* > \tag{8}$$

for each $g^1 \in G_q^\rho(T)$ or

$$\inf_{y \in R} < y, x^* > \leq \inf_{g^1 \in G_q^\rho(T)} < \mathcal{C}_{t_1} g^1, x^* > .$$

By symmetry of $U_p^\delta(T)$, we have

$$\begin{aligned} \inf_{y \in R} < y, x^* > &= < W(t_1) g^0, x^* > + \inf_{u \in U_p^\delta(T)} < \mathcal{B}_{t_1} u, x^* > \\ &= < W(t_1) g^0, x^* > - \sup_{u \in U_p^\delta(T)} < \mathcal{B}_{t_1} u, x^* >, \end{aligned} \tag{9}$$

and

(10)

$$\begin{aligned} \sup_{u \in U_p^\delta(T)} < \mathcal{B}_{t_1} u, x^* > &= \sup_{\|u\|_p \leq \delta} \int_0^{t_1} < B^*(s) W^*(t_1 - s) x^*, u(s) > ds \\ &= \delta \sup_{\|u\|_p \leq 1} \int_0^{t_1} < B^*(s) W^*(t_1 - s) x^*, u(s) > ds \\ &= \delta (\int_0^{t_1} \|B^*(s) W^*(t_1 - s) x^*\|_{U^*}^{p'} ds)^{1/p'} \end{aligned}$$

while, by the symmetry of $G_q^\rho(T)$, we have

$$\begin{aligned} \inf_{g^1 \in G_q^\rho(T)} < \mathcal{C}_{t_1} g^1, x^* > &= - \sup_{g^1 \in G_q^\rho(T)} < \mathcal{C}_{t_1} g^1, x^* > \\ &= - \sup_{\|g^1\|_q \leq \rho} \int_{-h}^0 < U_{t_1}^*(s) x^*, g^1(s) > ds \\ &= -\rho (\int_{-h}^0 \|U_{t_1}^*(s) x^*\|_*^{q'} ds)^{1/q'}. \end{aligned} \tag{11}$$

Consequently, by (8), (9), (10) and (11), we have

$$
\begin{aligned}
(12) \qquad < W(t_1)g^0, x^* > \leq \delta\Big(\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\Big)^{1/p'} \\
- \rho\Big(\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'} ds\Big)^{1/q'}.
\end{aligned}
$$

Replacing x^* by $-x^*$ in (12),

$$
\begin{aligned}
| < W(t_1)g^0, x^* > | \leq \delta\Big(\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\Big)^{1/p'} \\
- \rho\Big(\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'} ds\Big)^{1/q'}
\end{aligned}
$$

We are going to consider whether there exists a minimal time interval which preserves the max-min controllability of the system.

THEOREM 2.2. *If the system $(g^0, U_p^\delta(T), G_q^\rho(T), T = [0, t_1])$ is max-min controllable, then there exists a minimal time interval $T_{\hat{t}} = [0, \hat{t}]$ such that the system $(g^0, U_p^\delta(T_{\hat{t}}), G_q^\rho(T_{\hat{t}}), T_{\hat{t}})$ is max-min controllable.*

Proof. Let

$$
\begin{aligned}
(13) \qquad H = \{t \in [0, t_1];\ \text{the system } (g^1, U_p^\delta(T_t), G_q^\rho(T_t), T_t = [0, t_1]) \\
\text{is max-min controllable}\}.
\end{aligned}
$$

Since $t_1 \in H$, $H \neq \emptyset$. Let $\hat{t} = \inf H$, we need to prove that $\hat{t} \in H$. Suppose the contrary, $\hat{t} \in H$; then, by Theorem 2.1, there exists $x^* \in X^*$ such that

$$
\begin{aligned}
(14) \qquad | < W(\hat{t}_1)g^0, \hat{x}^* > | > \delta\Big(\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t} - s)\hat{x}^*\|_{U^*}^{p'} ds\Big)^{1/p'} \\
- \rho\Big(\int_{-h}^{0} \|U_{\hat{t}}^*(s)\hat{x}^*\|_*^{q'} ds\Big)^{1/q'}.
\end{aligned}
$$

By definition of $\hat{t}$, we can choose a time sequence $\{t_n\} \subset H$ such that $\lim_{n\to\infty} t_n = \hat{t}$; and so, for all n, we have

$$
\begin{aligned}
(15) \qquad | < W(t_n)g^0, \hat{x}^* > | \leq \delta\Big(\int_0^{t_n} \|B^*(s)W^*(t_n - s)\hat{x}^*\|_{U^*}^{p'} ds\Big)^{1/p'} \\
- \rho\Big(\int_{-h}^{0} \|U_{t_n}^*(s)\hat{x}^*\|_*^{q'} ds\Big)^{1/q'},
\end{aligned}
$$

which are continuous in the term t_n, thus, passing to the limit as $n \to \infty$, we have

$$(16)\qquad \begin{aligned} |< W(\hat{t})g^0, \hat{x}^* >| \le &\delta(\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t}-s)\hat{x}^*\|_{U^*}^{p'} ds)^{1/p'} \\ &- \rho(\int_{-h}^{0} \|U_{\hat{t}}^*(s)\hat{x}^*\|_*^{q'} ds)^{1/q'} \end{aligned}$$

which contradicts (14), and so the proof is complete.

THEOREM 2.3. *If $T_{\hat{t}} = [0, \hat{t}]$ is the minimal time interval over which the system (g^0, $U_p^\delta(T_{\hat{t}})$, $G_q^\rho(T_{\hat{t}})$, $T_{\hat{t}}$) is max-min controllable, then there exists a $\hat{x}^* \in X^*$ with $\|\hat{x}^*\|_2 = 1$ such that*

$$\begin{aligned} |< W(\hat{t})g^0, \hat{x}^* >| \le &\delta(\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t}-s)\hat{x}^*\|_{U^*}^{p} ds)^{1/p} \\ &- \rho(\int_{-h}^{0} \|U_{\hat{t}}^*(s)\hat{x}^*\|_*^{q} ds)^{1/q} \end{aligned}$$

Proof. Since the system is max-min controllable, then by Theorem 2.1, for all $x^* \in X^*$,

$$(17)\qquad \begin{aligned} |< W(\hat{t})g^0, X^* >| \le &\delta(\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t}-s)x^*\|_{U^*}^{p'} ds)^{1/p'} \\ &- \rho(\int_{-h}^{0} \|U_{\hat{t}}^*(s)x^*\|_*^{q'} ds)^{1/q'}. \end{aligned}$$

Choose a time sequence $\{t_n\}$ such that $t_n < \hat{t}$ with $t_n \longrightarrow \hat{t}$. By definition of $\hat{t} = \min H$ and Theorem 2.2, for each n, there exist $0 \neq x_n^* \in X^*$ such that

$$(18)\qquad \begin{aligned} |< W(t_n)g^0, x_n^* >| > &\delta[\int_0^{t_n} \|B^*(s)W^*(t_n-s)x_n^*\|_{U^*}^{p'} ds]^{1/p'} \\ &- \rho[\int_{-h}^{0} \|U_{t_n}^*(s)x_n^*\|_*^{q'} ds]^{1/q'}. \end{aligned}$$

Without loss of generality, we can assume that $\|x_n^*\|_2 = 1$; otherwise, we can divide (18) by $\|x_n^*\|_2 \neq 0$, and (18) still holds. By reflexiveness of X^*, there exists a convergent subsequence $\{x_{n_k}^*\} \subset \{x_n^*\}$ such that $\lim_{k\to\infty} x_{n_k}^* = \hat{x}^*$, so that $\|\hat{x}^*\|_2 = 1$. Since

$$| < W(t_{n_k})g^0, x_{n_k}^* > | > \delta[\int_0^{t_{n_k}} \|B^*(s)W^*(t_{n_k} - s)x_{n_k}^*\|_{U^*}^{p'} ds]^{1/p'} - \rho[\int_{-h}^0 \|U_{t_{n_k}}^*(s)x_{n_k}^*\|_*^{q'} ds]^{1/q'}$$

are continuous t_{n_k}, $x_{n_k}^*$, thus, passing to the limit as $k \to \infty$, we have

$$| < W(\hat{t})g^0, \hat{x}^* > | \geq \delta[\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t} - s)\hat{x}^*\|_{U^*}^{p'} ds]^{1/p'} - \rho[\int_{-h}^0 \|U_{\hat{t}}^*(s)\hat{x}^*\|_*^{q'} ds]^{1/q'}. \tag{19}$$

Comparing (18) for $x^* = \hat{x}^*$ with (19) shows that the equality of (17) must hold for $x^* = \hat{x}^*$.

References

1. W. L. Chan and C. W. Li, *Controllability, observability and duality in infinite dimensional linear systems with multiple norm-bounded controllers*, Int. J. Control, **33**(1981), 1039–1058.
2. W. L. Chan and C. W. Li, *Max-Min controllability in pursuit games with norm-bounded controls*, JOTA **37**(1982), 89–113
3. S. Nakagiri, *optimal control of linear retarded systems in Banach spaces*, J. Math. Anal. Appl. **120**(1986), 169–210.
4. J. Y. Park, S Nakagiri, and M. Yamamoto, *Admissible controllability for linear time-delay syatems in Banach spaces-A problem in game theory*, J. Tokyo Math., 89–102.

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