Pusan-Kyŏngnam Math. J. 7(1991), No. 1, pp. 45–52

FUZZY S-CONTINUOUS MAPPINGS

PARK, JIN HAN AND LEE, BU YOUNG

1. Introduction

The concept of fuzzy continuous mapping on fuzzy topological spaces was first introduced by Chang [2]. Yalvac [12] defined the fuzzy irresolute mapping between fuzzy topological spaces and investigated some results using the concepts of fuzzy semi-open and fuzzy semi-closed set in fuzzy topological space(henceforth fts for short) defined by Azad [1]. Also similar forms of fuzzy continuous and fuzzy open mappings on fts's have been considered and further studied by many authors [1, 3, 5, 8 and 12].

In this paper we define fuzzy S-continuous and fuzzy S-open and study some properties of these mappings on fts's.

2. Preliminaries

Fuzzy sets of a non-empty set X will be denoted by the capital letter A, B, U, V. The value of a fuzzy set A at the element x of X will be denoted by $A(x)$, and a fuzzy point will be denoted by p and q. And $p \in A$ either means that fuzzy point p takes its non-zero value in $(0,1)$ at the support x_p and $p(x_p) < A(x_p)$ (see in [10]) or fuzzy point p takes its non-zero value in $(0,1]$ and $p(x_p) \leq A(x_p)$ (see in [6]). If we write $p \in_1 A$ then the definition of fuzzy point-fuzzy elementhood will be the same as Srivastava *et al.* used in [10].

For definitions and results not explained in this paper, the reader were referred to the papers [1, 3, 5 and 10] assuming them to be well known. The words 'fuzzy' and 'neighborhood' will be abbreviated as 'f.' and 'nbd', respectively.

DEFINITION 2.1. Let A and B be f.sets of X and let p be f.point in X. p is said to be quasi-coincident with A, denoted by pqA, if $p(x_p) + A(x_p) > 1$. A is said to be quasi-coincident with B, denoted by AqB, if there exists $x \in X$ such that $A(x) + B(x) > 1$ ([6]).

Received March 10, 1991.

LEMMA 2.1 ([6]). *Let A and B be f.sets of X. $A \subset B$ iff A and B are not quasi-coincident denoted by $A\bar{q}B'$.*

LEMMA 2.2 ([11]). *Let A be a f.set and for $x \in X$, $A(x) = t \neq 0$ $(0 < t < 1)$. If for any λ which satisfies the inequality $0 < \lambda < t$, we choose the f.point p such that $p(x) = 1 - \lambda$, then pqA.*

THEOREM 2.1 ([2, 11]). *Let $f : X \longrightarrow Y$ be a mapping, A and B f.sets of X and Y, respectively. The following statements are true :*

(a) $f(A)' \subset f(A')$, $\quad f^{-1}(B') \doteq f^{-1}(B)'$.
(b) $A \subset f^{-1}(f(A))$, $\quad f(f^{-1}(B)) \subset B$.
(c) *If f is one-to-one then $f^{-1}(f(A)) = A$.*
(d) *If f is onto then $f(f^{-1}(B)) = B$.*
(e) *If f is one-to-one and onto then $f(A)' = f(A')$.*

Let $f : A \longrightarrow B$ be a mapping. Clearly, for every $p \in X$, $f(p)$ is a f.point in Y, and if $supp(p) = x_p$, then $supp(f(p)) = f(x_p)$, $f(p)(f(x_p)) = p(x_p)$. If $p \in Y$ then $f^{-1}(p)$ needs not be a f.point in X. If f is one-to-one and $p \in f(X)$ then $f^{-1}(p)$ is a f.point in X. In this case, if $supp(p) = y_p$, then $supp(f^{-1}(p)) = f^{-1}(y_p)$ and $f^{-1}(p)(f^{-1}(y_p)) = p(y_p)$.

LEMMA 2.3 ([11]). *Let $f : X \longrightarrow Y$ be a mapping and $p \in X$.*

(a) *If for $B \subset Y$ $f(p)qB$ then $pqf^{-1}(B)$.*
(b) *If for $A \subset X$ pqA then $f(p)qf(A)$.*

DEFINITION 2.2. Let A be a f.set of fts X.

(a) A is called a f.semi-open set of X if there exists a f.open set U of X such that $U \subset A \subset \overline{U}$ ([1]).
(b) A is called a f.semi-closed set of X if there exists a f.closed set V of X such that $V^o \subset A \subset V$ ([1]).
(c) A is called a f.semi-nbd of a f.point p if there exists a f.semi-open set V such that $p \in V \subset A$ ([5]).

A f.set A is f.semi-open iff A' is f.semi-closed ([1]).

DEFINITION 2.3. Let $A \subset X$ be a f.set. Then

$$\underline{A} = \cap\{\, B | A \subset B,\ B \text{ is f.semi-closed } \}$$

and

$$A_o = \cup\{\, B | B \subset A,\ B \text{ is f.semi- open } \}$$

are said to be f.semi-closure and f.semi-interior of A, respectively ([2]).

Obviously, $A \subset \underline{A} \subset \overline{A}$ and $A^o \subset A_o \subset A$ ([12]).

3. Fuzzy S-continuous mapping

Let f be a mapping from a fts X to another fts Y.

DEFINITION 3.1.

(a) f is called a f.continuous mapping if $f^{-1}(B)$ is a f.open set of X for each f.open set B of Y ([2]).
(b) f is called a f.irresolute mapping if $f^{-1}(B)$ is a f.semi-open set of X for each f.semi-open set of Y ([12]).

f.continuous mapping and f.irresolute mapping are independent concepts.

DEFINITION 3.2. f is said to be f.S-continuous mapping if $f^{-1}(B)$ is a f.open set of X for each f.semi-open set B of Y.

Clearly, f.S-continuous implies f.continuous, and f.S-continuous implies f.irresolute. But that the converse need not be true is shown the following Example 3.1.

EXAMPLE 3.1. Let A_1, A_2, B_1 and B_2 be f.sets of the unit closed interval I in R defined as follows :

$$A_1(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{3} \\ 0.3, & \frac{1}{3} < x \le 1 \end{cases} \qquad A_2(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{3} \\ 0.3, & \frac{1}{3} < x \le \frac{2}{3} \\ 0.2, & \frac{2}{3} < x \le 1 \end{cases}$$

$$B_1(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{3} \\ 0.3, & \frac{1}{3} < x \le \frac{2}{3} \\ 0.6, & \frac{2}{3} < x \le 1 \end{cases} \qquad B_2(x) = \begin{cases} 0, & 0 \le x \le \frac{2}{3} \\ 0.4, & \frac{1}{3} < x \le \frac{2}{3} \\ 0.2, & \frac{2}{3} < x \le 1. \end{cases}$$

(a) We consider f.topologies $\tau_1 = \{I, \emptyset, A_1\}$ and $\tau_2 = \{I, \emptyset, B_1\}$. Let $f : (I, \tau_1) \longrightarrow (I, \tau_2)$ be the mapping defined by

$$f(x) = \begin{cases} x, & 0 \le x \le \dfrac{2}{3} \\ x - \dfrac{1}{3}, & \dfrac{2}{3} < x \le 1. \end{cases}$$

Then f is f.continuous but not f.S-continuous.

Let B be the f.set of I defined by

$$B(x) = \begin{cases} 0.1, & 0 \le x \le \dfrac{1}{3} \\ 0.9, & \dfrac{1}{3} < x \le \dfrac{2}{3} \\ 0.8, & \dfrac{2}{3} < x \le 1. \end{cases}$$

Since $B_1 \subset B \subset \overline{B}_1$, B is a f.semi-open set. But $f^{-1}(B)$ is not a f.open set. Clearly, f is f.continuous.

(b) We consider f.topologies $\tau_1 = \{I, \emptyset, A_2\}$ and $\tau_2 = \{I, \emptyset, B_2\}$.

If $1_I : (I, \tau_1) \longrightarrow (I, \tau_2)$ is identity mapping, then 1_I is f.irresolute. If B is the f.set of I as in (a), $1_I^{-1}(B) = B$ is a f.semi-open but not f.open set. Thus, 1_I is not f.S-continuous.

THEOREM 3.1. *Let $f : X \longrightarrow Y$. The following are equivalent:*

(a) *f is a f.S-continuous.*

(b) *For every f.semi-closed set B in Y, $f^{-1}(B)$ is a f.closed set in X.*

(c) *For every f.set A in X, $f(\overline{A}) \subset \underline{f(A)}$.*

(d) *For every f.set B in Y, $\overline{f^{-1}(B)} \subset f^{-1}(\underline{B})$.*

(e) *For every $p \in X$ and each f.semi-open set B in Y containing $f(p)$, there exists a f.open set A in X such that $p \in A \subset f^{-1}(B)$.*

(f) *For every $p \in X$ and each f.semi-open set B in Y satisfying $f(p) q B$, there exists a f.open set A in X such that $p q A$ and $A \subset f^{-1}(B)$.*

Proof. (a)$\Rightarrow$(b) can be easily seen.

(b)$\Rightarrow$(c): Let A be a f.set in X. Then $\underline{f(A)}$ is a f.semi-closed set in Y. Since $f^{-1}(\underline{f(A)})$ is a f.closed set, $f^{-1}(\underline{f(A)}) = \overline{f^{-1}(\underline{f(A)})}$. By Theorem 2.1, $f(\overline{A}) \subset \underline{f(A)}$.

(c)$\Rightarrow$(d): Let B be a f.set in Y. By hypothesis and Theorem 2.1, $\overline{f^{-1}(B)} \subset f^{-1}(\underline{B})$.

(d)$\Rightarrow$(a): Let B be a f.semi-open set in Y. By hypothesis and Theorem 2.1,

$$\overline{f^{-1}(B')} \subset f^{-1}(\underline{B'}) = f^{-1}(B').$$

Since $\overline{f^{-1}(B')} \subset f^{-1}(B')$ and $f^{-1}(B') \subset \overline{f^{-1}(B')}$, $\overline{f^{-1}(B')} = f^{-1}(B')$. Hence $f^{-1}(B)$ is a f.open set, that is, f is a f.S-continuous.

(a)$\Rightarrow$(e): Let $p \in X$ and B be any f.semi-open set in Y such that $f(p) \in B$. Since f is f.S-continuous, $f^{-1}(B)$ is a f.open set. Therefore, $p \in f^{-1}(B) = A \subset f^{-1}(B)$.

(e)$\Rightarrow$(a): Let $B \subset Y$ be a f.semi-open set and $p \in f^{-1}(B)$ be any f.point. From hypothesis, there exists a f.open set A in X such that $p \in A \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is a f.open set.

(a)$\Rightarrow$(f): Let $p \in X$ and B be any f.semi-open set such that $f(p)\mathrm{q}B$. Clearly, $f^{-1}(B)$ is a f.open set. By Lemma 2.3, $p\mathrm{q}f^{-1}(B) = A \subset f^{-1}(B)$.

(f)$\Rightarrow$(a): Let $B \subset Y$ be any f.semi-open set and $p \in_1 f^{-1}(B)$. Then $f(p) \in_1 B$. By Lemma 2.2, choose the f.point p' as $p'(x_p) = 1 - p(x_p)$. For this p', we have $f(p')\mathrm{q}B$. From hypothesis, there exists a f.open set A such that $p'\mathrm{q}A \subset f^{-1}(B)$. Since $p'\mathrm{q}A$,

$$p'(x_p) + A(x_p) = 1 - p(x_p) + A(x_p) > 1.$$

Thus $A(x_p) > p(x_p)$. That is, $p \in_1 A$. Hence $p \in_1 A \subset f^{-1}(B)$ and so $f^{-1}(B)$ is a f.open set.

THEOREM 3.2. *$f : X \to Y$ is a f.S-continuous iff for every f.set B in Y, $f^{-1}(B_o) \subset (f^{-1}(B))^o$.*

Proof. Let $B \subset Y$. B_o is a f.semi-open set in Y. Clearly, $f^{-1}(B_o)$ is a f.open set and

$$f^{-1}(B_o) \subset (f^{-1}(B_o))^o \subset (f^{-1}(B))^o.$$

Conversely, let B be any f.semi-open set in Y. Then $B_o = B$ and so

$$f^{-1}(B) = f^{-1}(B_o) \subset (f^{-1}(B))^o.$$

Hence $f^{-1}(B) = (f^{-1}(B))^o$. This shows that $f^{-1}(B)$ is a f.open set.

THEOREM 3.3. *Let $f : X \longrightarrow Y$ be bijection. f is a f.S-continuous iff for every f.set A in X, $(f(A))_o \subset f(A^o)$.*

Proof. Let $A \subset X$ be a f.set. $f^{-1}((f(A))_o)$ is a f.open set in X. By Theorem 3.2 and f is onto,

$$f^{-1}((f(A))_o) = (f^{-1}((f(A))_o))^o \subset (f^{-1}(f(A)))^o = A^o.$$

Since f is onto,

$$(f(A))_o = f(f^{-1}((f(A))_o)) \subset f(A^o).$$

Conversely, let $B \subset Y$ be any f.semi-open set. Immediately, $B = B_o$. From hypothesis,

$$f((f^{-1}(B))^o) \supset (f(f^{-1}(B)))_o = B_o = B.$$

This implies that $f^{-1}(f((f^{-1}(B))^o)) \supset f^{-1}(B)$. Since f is one-to-one, $f^{-1}(B) \subset (f^{-1}(B))^o$. Hence $f^{-1}(B) = (f^{-1}(B))_o$, that is, $f^{-1}(B)$ is a f.open set.

THEOREM 3.4. *Let X and Y be fts's such that X is product related to Y and let $f : X \longrightarrow Y$ be a mapping. Then if the graph $g : X \longrightarrow X \times Y$ of f is f.S-continuous, then f is also f.S-continuous.*

Proof. Let B be a f.semi-open set of Y. Then we have $f^{-1}(B) = 1 \cap f^{-1}(B) = g^{-1}(1 \times B)$. Since g is a f.S-continuous and $1 \times B$ is a f.semi-open set of $X \times Y$, $f^{-1}(B)$ is a f.open set of X. Hence f is a f.S-continuous.

THEOREM 3.5. *Let $f : X \longrightarrow Y$ be one-to-one and f.S-continuous. If Y is f.semi-T_i, then X is f.T_i ($i = 0, 1, 2$).*

Proof. We give a proof for $i = 1$ only; the other cases being similar, are omitted. Let p and q be two distinct f.points in X.

When $x_p \neq x_q$, we have $f(x_p) \neq f(x_q)$, and by the f.semi-T_1 property of Y, $f(p)$ and $f(q)$ have f.semi-open semi-nbds U and V such that $f(p)\bar{\text{q}}V$ and $f(q)\bar{\text{q}}U$, respectively. By hypothesis and Lemma 2.3, $f^{-1}(U)$ and $f^{-1}(V)$ are f.open nbds of p and q such that $p\bar{\text{q}}f^{-1}(V)$ and $q\bar{\text{q}}f^{-1}(U)$, respectively.

When $x_p = x_q$ and $p(x_p) < q(x_q)$(say), then $f(x_p) = f(x_q)$. Y being f.semi-T_1, there is a f.semi-open semi-q-nbd V of $f(q)$ such that $f(p)\bar{\text{q}}V$. By hypothesis and Lemma 2.3, $f^{-1}(V)$ is a f.q-nbd of q in X such that $p\bar{\text{q}}f^{-1}(V)$. Hence X is f.T_1.

DEFINITION 3.3. Let $f : X \longrightarrow Y$ be a mapping between two fts's. f is said to be a f.S-open mapping if $f(A)$ is a f.open set in Y, for each f.semi-open set A in X.

Obviously, f.S-open implies f.open, and f.S-open implies f.presemi-open.

THEOREM 3.6. *Let $f : X \longrightarrow Y$ be bijection. If f is a f.S-open mapping, then $f^{-1}(\overline{B}) \subset \underline{f^{-1}(B)}$ for each f.set B of Y.*

Proof. Let $B \subset Y$. Clearly, $(\underline{f^{-1}(B)})'$ is f.semi-open set in X. Since f is f.S-open and bijection,

$$f((\underline{f^{-1}(B)})') = (f(\underline{f^{-1}(B)}))'$$

is f.open set in Y. This implies that $f(\underline{f^{-1}(B)})$ is a f.closed set in Y. Thus,

$$\overline{B} = \overline{f(f^{-1}(B))} \subset \overline{f(\underline{f^{-1}(B)})} = f(\underline{f^{-1}(B)}).$$

Since f is one-to-one,

$$f^{-1}(\overline{B}) \subset f^{-1}(f(\underline{f^{-1}(B)})) = \underline{f^{-1}(B)}.$$

THEOREM 3.7. *Let X, Y and Z be fts's and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$. If f and g are f.S-continuous (or f.S-open) mappings, then $g \circ f$ is too.*

Proof. Since

$$(g \circ f)(A) = g(f(A))$$

and

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

for $A \subset X$ and $B \subset Z$, it is clear.

References

1. K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82**(1981), 14–32.
2. C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24**(1968), 182–190.
3. S. Ganguly and S. Saha, *Separation axioms and T_i-fuzzy continuity*, Fuzzy Sets and Systems **16**(1985), 265–275.
4. S. Ganguly and S. Saha, *A note on semi-open sets in fuzzy topological spaces*, Fuzzy Sets and Systems **18**(1986), 83–96.
5. B. Ghosh, *Semi-continuous and semi-closed mappings and semi-connected in fuzzy setting*, Fuzzy Sets and Systems **35**(1990), 345–355.
6. P. P. Ming and L. Y. Ming, *Fuzzy topology,I, Neighborhood structure of fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76**(1980), 571–599.
7. M. N. Mukherjee and S. P. Sinha, *Irresolute and almost open functions between fuzzy topological spaces*, Fuzzy Sets and Systems **29**(1989), 381–388.
8. M. M. MUkherjee and S. P. Sinha, *On some weaker forms of fuzzy topological spaces*, Fuzzy Sets and Systems **32**(1989), 103–114.
9. A. R. Singal and D. S. Yadav, *S-continuous functions*, Ganita Sandesh **2**(1988), 82–86.
10. R. Srivastava, S. N. Lal and A. K. Srivastava, *Fuzzy Hausdorff topological spaces*, J. Math. Anal. Appl. **81**(1981), 497–506.
11. T. H. Yalvac, *Fuzzy sets and functions on fuzzy topological spaces*, J. Math. Anal. Appl. **126**(1987), 409–423.
12. T. H. Yalvac, *Semi-interior and semi-closure of a fuzzy set*, J. Math. Anal. Appl. **132**(1988), 356–364.

Department of Mathematics
Dong-A university
Pusan 604–714, Korea