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REPRESENTATION ON A HILBERT B-MODULE

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1. Introduction

Each cyclic $*$-representation gives rise to a state on a C^*-algebra. And it turns out that each state generates a cyclic $*$-representation that reproduces it ([2, p261],[3]). But, if we replace a state by a completely positive map, what happens? This paper is an investigation of a $*$-representation (on a Hilbert B- module) generated by a completely positive map.

In §2, we introduce some definitions and their properties which will be needed in next section.

In §3, we show that a completely positive map gives rise to a pre-Hilbert B-module in much the same way that a state gives rise to a pre-Hilbert space. The properties of pre-Hilbert B-module generated by a completely positive map are described and this section contains main theorems([Theorem 3.4], [Theorem 3.5]).

2. Properties of B-valued inner product

DEFINITION 2.1. Let B be a C^*-algebra.
A *pre-Hilbert B-module* is a right B-module X equipped with a conjugate bilinear map $[\,,\,] : X \times X \to B$ satisfying:

(a) $[x,x] \geq 0 \quad \forall x \in X$;
(b) $[x,x] = 0 \quad$ only if $x = 0$;
(c) $[x,y] = [y,x]^* \quad$ for $x, y \in X$;
(d) $[x \cdot b, y] = [x,y]b \quad$ for $x, y \in X, \quad b \in B$.

The map $[\,,\,]$ will be called a *B-valued inner product* on X.
The following simple facts are obvious:

1^o $[x, y \cdot b] = b^*[x,y] \quad x, y \in X,\ b \in B$.
2^o if B has 1, X is unital (i.e., $x \cdot 1 = x,\ x \in X$).

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3° defining $\|\cdot\|_X$ on X by $\|x\|_X = \|[x,x]\|^{1/2}$, $\|\cdot\|_X$ becomes a norm on X.

4° $\|[x,y]\| \le \|x\|_X \, \|y\|_X$, $\|x\cdot b\|_X \le \|x\|_X \, \|b\|$ $x, y \in X,\ b \in B$.

DEFINITION 2.2. A pre-Hilbert B-module X which is complete with respect to $\|\cdot\|_X$ will be called a *Hilbert B-module*.

For a pre-Hilbert B-module X, we let $A(X)$ denote the set of operators $T \in B(X)$ for which there is an operator $T^* \in B(X)$ such that $[Tx, y] = [x, T^*y]$ for $x, y \in X$. And for $x, y \in X$, define $x \otimes y : X \to X$ by $x \otimes y(w) = x \cdot [w, y]$.

The following simple facts are obvious:

5° $A(X)$ is a $*$-algebra with involution $T \to T^*$.

6° if $T \in A(X)$, then $T(x\cdot b) = (Tx)\cdot b$ for $x \in X,\ b \in B$ (i.e. T is a module map).

LEMMA 2.3.

(1) $x \otimes y \in A(X)$ *with* $(x\otimes y)^* = y \otimes x$.

(2) $T(x \otimes y) = Tx \otimes y, \quad T \in A(X)$.

(3) $\{x \otimes y : x, y \in X\}$ *spans a two-sided ideal for* $A(X)$.

Proof. (1) For all $w_1, w_2 \in X$,

$$\begin{aligned}[][w_1, (x\otimes y)w_2] &= [w_1, x\cdot[w_2, y]] = [w_2, y]^*\,[w_1, x] \\ &= [y, w_2]\,[w_1, x] = [y\cdot[w_1, x], w_2] \\ &= [(y\otimes x)w_1, w_2]\end{aligned}$$

For all $w \in X$, $b \in B$,

$$(x\otimes y)(w\cdot b) = x\cdot[w\cdot b, y] = x\cdot[w, y]b = (x\otimes y(w))\cdot b.$$

(2)$T(x\otimes y)w = T(x\cdot[w,y]) = (Tx)\cdot[w,y]$ (since T is a module map)

$$= (Tx\otimes y)w.$$

(3) By (1),(2), it is clear.

We write X' for the set of B-module maps from X to B which are bounded with respect to $\|\cdot\|_X$. We make X' into a right B-module by defining

$$(\lambda\tau)x = \bar{\lambda}\tau(x) \text{ and } (\tau\cdot b)(x) = b^*\tau(x),$$

for $\lambda \in C,\ \tau \in X',\ x \in X,\ b \in B$.

Each $x \in X$ gives rise to a map $\hat{x} \in X'$ defined $\hat{x}(y) = [y, x]$ for $y \in X$.

We will call X *self-dual* if $\hat{X} = X'$.

THEOREM 2.4. *Let X be a pre-Hilbert B-module.*

(1) *The B-valued inner product extends to $X' \times X'$ in such a way as to make X' into a Hilbert B-module.*
(2) *Each $T \in A(X)$ extends to a unique $\tilde{T} \in A(X)$. Moreover, the map $T \to \tilde{T}$ is a $*$-isomorphism of $A(X)$ into $A(X')$.*

Proof. [7]

Let B be a von-Neumann algebra of operators on a Hilbert space H, and let $X \otimes H$ be the algebraic tensor product of X with H.
Define $<,>: X \otimes H \times X \otimes H \to C$ by $< x \otimes \xi, y \otimes \eta >= ([x,y]\xi, \eta)$.
Let $Z = \{w \in X \otimes H :< w, w >= 0\}$, so Z is a subspace of $X \otimes H$ and $K_o = X \otimes H/Z$ is a pre-Hilbert space with inner product$(w_1 + Z, w_2 + Z) =< w_1, w_2 >$.
Let K be the Hilbert space completion of K_o. For $T \in A(X)$, define a linear map $\theta_o(T) : X \otimes H \to X \otimes H$ by $\theta_o(T)(x \otimes \xi) = Tx \otimes \xi$.
It is shown in 5.3 of [9] that $\theta_o(T)$ induces a bounded linear map $\theta(T) : K \to K$ satisfying $\theta(T)(x \otimes \xi + Z) = Tx \otimes \xi + Z$.

LEMMA 2.5. *If A is a self-adjoint operator on H and $(Ah, h) = 0$ for all h, then $A = 0$.*

Proof. [3]

THEOREM 2.6. *The map θ is a faithful $*$-representation of $A(X)$ on K.*

Proof. By the above statements, it is clear that θ is a homomorphism. Also, for each $x, y \in X, \quad \xi, \eta \in H$,

$$
\begin{aligned}
(\theta(T^*)(x \otimes \xi + Z), y \otimes \eta + Z) &= (T^*x \otimes \xi + Z, y \otimes \eta + Z) \\
&=< T^*x \otimes \xi, y \otimes \eta >= ([T^*x, y]\xi, \eta) \\
&= ([x, Ty]\xi, \eta) =< x \otimes \xi, Ty \otimes \eta > \\
&= (x \otimes \xi + Z, Ty \otimes \eta + Z) \\
&= (x \otimes \xi + Z, \theta(T)(y \otimes \eta + Z)) \\
&= (\theta(T)^*(x \otimes \xi + Z), y \otimes \eta + Z).
\end{aligned}
$$

Let $T \in \mathrm{Ker}\theta$. Then $0 =< Tx \otimes \xi, Tx \otimes \xi >= ([Tx, Tx]\xi, \xi)$. Since $[Tx, Tx]$ is self-adjoint, by Lemma 2.5, $T = 0$.

3. Representation on a Hilbert B-module

DEFINITION 3.1. Let B be a C^*-algebra, A a $*$-algebra and $\phi : A \to B$ a linear map. We call ϕ *positive* if $\phi(a^*a) \geq 0, \quad a \in A$.

For $n = 1, 2, \cdots$, ϕ induces a map ϕ_n from algebra A of $n \times n$ matrices with entries in A (made into a $*$-algebra by setting $[a_{ij}]^* = [a^*_{ij}]$ for matrices $[a_{ij}] \in A_{(n)}$) into the corresponding C^*-algebra B defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$; we asy that ϕ is *completely positive* if each of the induced map ϕ_n is positive.

According to [10, p194], a linear map $\phi : A \to B$ is completely positive iff $\sum_{ij} b_i^* \phi(a_i^* a_j) b \geq 0$ for $a_1, \cdots, a_n \in A, \quad b_1, \cdots, b_n \in B$. Let ϕ be completely positive and suppose in addition that $\phi(a^*) = \phi(a)^*$ for $a \in A$. The map ϕ gives rise to a pre-Hilbert B-module as follows : Consider the algebric tensor product $A \otimes B$, which becomes a right B-module when we set $(a \otimes b) \cdot \beta = a \otimes b\beta$ for $b, \beta \in B, \ a \in A$.

$$\text{Define } \ll, \gg : (A \otimes B) \times (A \otimes B) \to B$$
$$\left(\sum_{j=1}^{n} a_j \otimes b_j, \sum_{i=1}^{m} \alpha_i \otimes \beta_i \right) \rightsquigarrow \sum_{i,j} \beta_i^* \phi(\alpha_i^* a_j) b_j$$

for $a_1, \cdots, a_n, \ \alpha_1, \cdots, \alpha_m \in A, \quad b_1, \cdots, b_n, \ \beta_1, \cdots, \beta_m \in B$.

$\ll, \gg$ is clearly well-defined and conjugate-bilinear. Since ϕ is completely positive, for all $x \in A \otimes B, \quad \ll x, x \gg \geq 0$. Since ϕ is $*$-map, $\ll x, y \gg = \ll y, x \gg$ and $\ll x \cdot b, y \gg = \ll x, y \gg b$ for $x, y \in A \otimes B$ and $b \in B$.

Put $N = \{x \in A \otimes B : \ll x, x \gg = 0\}$. Then N is a submodule of $A \otimes B$ and $X_o = A \otimes B / N$ is a pre-Hilbert B-module with B-valued inner product $[x + N, y + N] = \ll x, y \gg$ for $x, y \in A \otimes B$.

THEOREM 3.2. *Let A be a U^*-algebra with 1, B a C^*- algebra with 1, and $\phi : A \to B$ a completely positive map. Then*

(1) *there is a Hilbert B-module X, a $*$-representation π of A on X, and an element $e \in X$ such that $\phi(a) = [\pi(a)e, e]$ for $a \in A$.*
(2) *the set $\{\pi(a)(e \cdot b) : a \in A, \ b \in B\}$ spans a dense subset of X.*

Proof. [7],[10].

In particular, note that $\pi(a)(x+N) = a \cdot x + N \quad \forall x \in A \otimes B$ and $\pi(a) \in A(X)$(i.e., $\pi(a)$ is a B-module map), X a completion of X_o, also $e = 1 \otimes 1 + N$.

Let A be a U^*-algebra with 1, and B a W^*-algebra. If X, π and e are as in Theorem 3.2, we may define a $*$-representation $\tilde{\pi}$ of A on the self-dual Hilbert B-module X' by $\tilde{\pi}(a) = \pi(a)^\sim \in A(X')$ for $a \in A$ (see 2.4).

Suppose $\psi : A \to B$ is another completely positive map. We write $\psi \leq \phi$ if $\phi - \psi$ is compeletely positive and let $[0, \phi]$ denote the set of compeletely positive maps from A into B which are $\leq \phi$.

For $T \in A(X')$, define $\phi_T : A \to B$ by $\phi_T(a) = [T\tilde{\pi}(a)\hat{e}, \hat{e}]$. Notice that $\phi_I = \phi$ and that the map $T \rightsquigarrow \phi_T$ is a linear map of $A(X')$ into the space of linear transformations of A into B.

THEOREM 3.3. *Under the above circumstance,*

(1) *for each $T \in \tilde{\pi}(A)'$ with $0 \leq T \leq I'_X$, the formula $\phi_T(a) = [T\tilde{\pi}(a)\hat{e}, \hat{e}]$ defines a completely positive map such that $\phi_T \leq \phi$.*
(2) *the correspondence $T \rightsquigarrow \phi_T$ described in (1) is a bijection of $\{T \in \tilde{\pi}(A)' : 0 \leq X \leq I_{X'}\}$ onto $[0, \phi]$.*
(3) *the correspondence preserves convex combinations, where $\tilde{\pi}(A)'$ denotes the commutant of $\tilde{\pi}(A)$ in $A(X')$.*

Proof. [2],[7].

Let A be a U^*-algebra with 1, and B a W^*-algebra with 1. We denote the set of completely positive maps $\phi : A \to B$ such that $\phi(1) = 1$ by $\sum(A, B, 1)$.

Note that $\sum(A, B, 1)$ is a convex subset of the space of linear maps from A into B.

THEOREM 3.4. *Under the above circumstance, tha following conditions on $\phi \in \sum(A, B, 1)$ are equivalent:*

(1) *ϕ is an extremal point of $\sum(A, B, 1)$;*
(2) *the map $T \rightsquigarrow [T\hat{e}, \hat{e}]$ of $A(X')$ into B is injective on $\tilde{\pi}(A)'$;*
(3) *If ψ is any completely positive map on A such that $\psi \leq \phi$, then $\psi = \alpha\phi$ with $0 \leq \alpha \leq 1$.*

Proof. (1)$\Longleftrightarrow$(3) This follows immediately from (68.24) in [2].

(2)$\Longrightarrow$(1) Suppose that the map is injective and let $\phi = t\phi_1 + (1-t)\phi_2$, $\phi_1, \phi_2 \in \sum(A,B,1)$ $(0 < t < 1)$. then $t\phi_1 \leq \phi$. i.e., $t\phi_1(a) \in [0,\phi]$.
By 3.3, there are $T \in \tilde{\pi}(A)'$, $0 \leq T \leq I_{X'}$ such that $t\ \phi_1(a) = [T\tilde{\pi}(a)\hat{e}, \hat{e}]$ $\forall\ a \in A$.
Setting $a = 1$, $t\phi_1(1) = [T\hat{e}, \hat{e}]$. By tha way, since $t\phi_1(1) = t \cdot 1$, $t\phi_1(1) = [T\hat{e}, \hat{e}] = t$. Therefore $[(T - tI)\hat{e}, \hat{e}] = 0$. By the hypothesis, $T = tI$. Also,

$$t\phi_1(a) = [tI\tilde{\pi}(a)\hat{e}, \hat{e}] = t[\tilde{\pi}(a)\hat{e}, \hat{e}] = t = t\phi_I(a) = t\phi(a).$$

Thus, $t\phi_1 = t\phi$ and $\phi_1 = \phi_2 = \phi$.

(1)$\Longrightarrow$(2) Suppose that $\phi \in \sum(A,B,1)$ is an extremal point.
Take $T \in \tilde{\pi}(A)'$ such that $\mu(T) = [T\hat{e}, \hat{e}] = 0$. i.e.,

$$\begin{aligned} \mu : \tilde{\pi}(A)' \subset A(X') &\longrightarrow B \\ T \quad &\rightsquigarrow \quad \mu(T) = [T\hat{e}, \hat{e}]. \end{aligned}$$

Choose $s, t > 0$ such that $\frac{1}{4}I_{X'} \leq sT + tI_{X'} \leq \frac{3}{4}I_{X'}$ and set $F = sT = tI_{X'}$. Then, since $\mu(\frac{1}{4}I_{X'}) \leq \mu(F) \leq \mu(\frac{3}{4}I_{X'})$, it follows that $\frac{1}{4} \leq t \leq \frac{3}{4}$.
Define $\phi_1(a) = [F\tilde{\pi}(a)\hat{e}, \hat{e}]$, $\phi_2(a) = [(I - F)\tilde{\pi}(a)\hat{e}, \hat{e}]$. Since $0 \leq F \leq I_{X'}$, By 3.3, ϕ_1, ϕ_2 are completely positive. Also $\phi_1(1) = t \cdot 1$, $\phi_2(1) = (1-t)1$, $(\phi_1 + \phi_2)(a) = \phi_I(a) = \phi(a)$. Since $t^{-1}\phi_1$, $(1-t)^{-1}\phi_2$ belong to $\sum(A,B,1)$, from extremality of ϕ, $t^{-1}\phi_1 = (1-t)^{-1}\phi_2 = \phi$.
In particular, $[F\tilde{\pi}(a)\hat{e}, \hat{e}] = \phi_1(a) = t[\tilde{\pi}(a)\hat{e}, \hat{e}]$, $\forall\ a \in A$. Thus $F = tI_{X'}$, and so $sT = 0$.
Therefore $T = 0$ and μ is injective on $\tilde{\pi}(A)'$.

THEOREM 3.5. *If π is a $*$-representation of A on a Hilbert B-module Y and $\phi(a) = [\pi(a)e, e]$ and if π_ϕ is constructed as in 3.2, then*

(1) *there exists an isometric mapping U from X into Y.*
(2) *$U\pi_\phi(a)$ and $\pi(a)U$ agree on X.*

Proof. (1) By theorem 3.2, $\pi_\phi(A)(e_\phi \cdot B)$ and $\pi(A)(e \cdot B)$ are dense subspaces of X, Y, respectively. Now define $U_o\pi_\phi(a)\ (e_\phi \cdot b) = \pi(a)$

$(e_\phi \cdot b) \qquad b \in B.$

$$\begin{aligned}
\|\pi_\phi(a)(e_\phi \cdot b)\|^2 &= \|[\pi_\phi(a)(e_\phi \cdot b),\ \pi_\phi(a)(e_\phi \cdot b)]\| \\
&= \|[\pi_\phi(a^*a)(e_\phi \cdot b),\ e_\phi \cdot b]\| \\
&= \|[(\pi_\phi(a^*a)e_\phi) \cdot b,\ e_\phi \cdot b]\| \\
&\qquad \text{(since } \pi_\phi(a^*a) \text{ is module map)} \\
&= \|b^*[\pi_\phi(a^*a)e_\phi,\ e_\phi]b\| \quad \text{(by } 1^\circ\text{)} \\
&= \|b^*[\pi(a^*a)e, e]b\| \quad \text{(by the hypothesis)} \\
&= \|[\pi(a^*a)(e \cdot b),\ e \cdot b]\| \\
&= \|[\pi(a)(e \cdot b),\ \pi(a)(e \cdot b)]\| \\
&= \|\pi(a)(e \cdot b)\|^2.
\end{aligned}$$

Thus U_o is well-defined and isometric on $X_o(= \pi_\phi(A)(e_\phi \cdot B))$. Therefore U_o extends to an isometric mapping U of X into Y.

(2) By definition and continuity of U_o, it is clear.

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