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ON JOINT HYPONORMALITY OF COMMUTING n–TUPLES OF OPERATORS

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1. Introduction

Throughout this paper, H will always denote a Hilbert space, $B(H)$ will denote the algebra of bounded linear operators on H, and $B(H^n)$, the set of a commuting n-tuple of operators in $B(H)$, where H^n denotes the orthogonal direct sum of H with itself n times. For $S, T \in B(H)$ we let $[S,T] = ST - TS$; $[S,T]$ is the commutator of S and T. Given an n-tuple $T = (T_1, T_2, \ldots T_n)$ of operaters on H, we let $\mathcal{M}_{n\times n}(T) = ([T_j^*, T_i])$ denote the self-commutator of T, define by

$$\mathcal{M}_{n\times n}(T) = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \ldots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \ldots & [T_n^*, T_2] \\ \multicolumn{4}{c}{\dotfill} \\ [T_1^*, T_n] & [T_2^*, T_n] & \ldots & [T_n^*, T_n] \end{pmatrix}$$

The notions of jointly hyponormal has been considered by A. Athavale, J. Conway and W. Szymanski, R. E. Curto, P. Xia, and other as follows: $T(\in B(H^n))$ is jointly hyponormal if $\mathcal{M}_{n\times n}(T) = ([T_j^*, T_i])$ is positive semi-definite, equivalently, if $\sum_{i,j=1}^{n} ([T_j^*, T_i])x_j, x_i)_H \geq 0$ for any $x_1, x_2, \ldots, x_n$ in H. But the notion of jointly hyponormal which has been considered by M. Chō and A. T. Dash is not equivalent to the notions of jointly hyponormal such as previously stated. M. Chō and M. T. Dash say that T is jointly hyponormal if $[T_i^*, T_i] \geq 0$ for $i = 1, 2, \ldots n$. Then we shall call it (C.D) jointly hyponormal. $T(\in B(H^n))$ will be called weakly jointly hyponormal if $\{\sum_{i=1}^{n} \alpha_i T_i$: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}^n\}$ consists entirely of hyponormal operators

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[4], T is jointly normal if T is commuting and each T_i is normal operator [5]. In particular, $T = (T_1, T_2, \ldots, T_n)$ is weakly doubly commuting n-tuples of operators if, for $i, j = 1, \ldots, n$ $\quad T_jT_i = T_iT_j$ and $T_i^*T_j = T_jT_i^*$ for $i \neq j$, [7]. We say that a point $z = (z_1, z_2, \ldots, z_n)$ of C^n is in the joint approximate point spectrum $\sigma_a(T)$ of T if there exists of a sequence $\{x_n\}$ of unit vectors in H such that $\|(z_k - T_k)x_n\| \to 0 \quad (n \to \infty), \quad k = 1, \ldots, n$. A point $z = (z_1, \ldots, z_n)$ will be said to be in the joint approximate compression spectrum $\sigma_l(T)$ of T if there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|(z_k - T_k)^*x_n\| \to 0 \quad (n \to \infty), \quad k = 1, \ldots, n$. There are several definitions of the joint spectrum. J. L. Taylor [8] has defined $T \in B(H)$ to be non-singular if sequence (*) is exact ; i.e. $\mathrm{im}\delta_T^p = \ker\delta_T^{p-1}$ for all $p = 1, \ldots, n+1$. And He has defined the joint spectrum$S_P(T)$ of T, to be the set of the point $z = (z_1, \ldots, z_n)$ such that $z - T = (z_1 - T_1, \ldots, z_n - T_n)$ is singular. On the other hand, A. T. Dash [6] has defined the joint spectrum $\sigma(T)$ of T as follows : a point $z = (z_1, \ldots, z_n)$ is in $\sigma(T)$ if and only if for all $B_1, \ldots, B_n$ in T'' $\quad \sum_{i=1}^{n} B_i(z_i - T_i) \neq I$, where I denotes the identity operator and T'', double commutant algebra of the set $\{T_1, \ldots, T_n\}$ in $B(H)$. It is well known that $\sigma_a(T)$, $\sigma(T)$ and $S_p(T)$ are non-empty compact sets, and that $S_p(T) \subset \sigma(T)$ (see [8, Lemma 1]). Further, it is evident that $\sigma_a(T) \subset \sigma(T) \subset \sigma(T_1) \times \cdots \times \sigma(T_n)$ and $\sigma_a(T) \cup \sigma_l(T) \subset \sigma(T)$ [6, 8].Let $T = (T_1, \ldots, T_n)$ be a weakly doubly commuting n-tuple of operators. Then we have the followings :

jointly hyponormal

$\Leftrightarrow$ (C,D) jointly hyponormal

$\Leftrightarrow$ weakly jointly hyponormal.

Moreover, we show properties of jointly hyponormal and investigate some characterizations of n-tuple of operators with Cartesian decomposition.

2. Properties of jointly hyponormal

We give properties and relations of jointly hyponormal, (C,D) jointly hyponormal and weakly jointly hyponormal. Further, we have Remarks to be made regarding jointly hyponormal.

LEMMA 2.1. [5] *Let* $T = (T_1, \ldots, T_2)$ *be a commuting* n*- tuple* *operators on H. Consider the following three statements :*

(1) *T is jointly normal.*

(2) *T is jointly hyponormal.*

(3) *T is weakly jointly normal.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

R. Curto, P. Muhly and J. Xia [5] have an example of a pair $T = (T_1, T_2)$ of commuting operators whichis weakly jointly hyponormal but not jointly hyponormal.

LEMMA 2.2. ([1]) *Let $T = (T_1, \ldots, T_n)$ be a commuting n- tuple of operators. If T is jointly hyponormal, then each T_i is a hyponormal operator on H.*

LEMMA 2.3. ([1]) *Let $T = (T_1, \ldots, T_n)$ be a commuting n- tuple of operators. Then T is jointly hyponormal if and only if*

$$\sum_{i,j=1}^{n} (T_i x_j, T_j x_i)_H \geq \| \sum_{i}^{n} T_i^* x_i \|^2$$

LEMMA 2.4. *$T = (T_1, \ldots, T_n)$ jointly hyponormal if and only if*

$$\sum_{i=1}^{n} ([T_i^*, T_i] x_i, x_i) + 2\mathrm{Re} \sum_{i<j} ([T_j^*, T_i] x_j, x_i) \geq 0 \quad \text{for all } x_1, \ldots, x_n \in H.$$

Proof. Straighforward from Lemma 2.3, in fact, T is jointly hyponormal

$$\Longleftrightarrow 0 \leq \left((\mathcal{M}_{n\times n}(T)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} [T_1^*, T_1] & \ldots & [T_n^*, T_1] \\ [T_2^*, T_1] & \ldots & [T_n^*, T_2] \\ \cdots & \cdots & \cdots \\ [T_n^*, T_1] & \ldots & [T_n^*, T_n] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right)$$

for all $x_i \in H, \quad i = 1 \dots, n.$

$$\Longleftrightarrow \sum_{i=1}^{n} \|T_i x_i\|^2 + 2\mathrm{Re} \sum_{i<j} (T_i x_j, T_j x_i) \geq \| \sum_{i=1}^{n} T_i^* x_i \|^2$$

$$\Longleftrightarrow \sum_{i=1}^{n} ([T_i^*, T_i] x_i, x_i) + 2\mathrm{Re} \sum_{i<j} ([T_j^*, T_i] x_j, x_i) \geq 0$$

R. Curto, P.S. Muhly and J.Xia and A.Athavale showed the following theorem. We prove it by using Lemma 2.4.

THEOREM 2.5. . *Let $T = (T_1, \dots, T_n)$ be jointly hyponormal. Then span $\{T_1, \dots, T_n\}$ is a hyponormal operator on H (i.e., T is weakly jointly hyponormal)*

Proof. If T is jointly hyponormal, then by Lemma 2.4, we have

$$\sum_{i=1}^{n} ([T_i^*, T_i] x_i, x_i) + 2\mathrm{Re} \sum_{i<j} ([T_j^*, T_i] x_j, x_i) \geq 0.$$

Letting $x_j = \bar{k}_j x$ for a fixed vector x in H and $k_j \in C$ for $j = 1, \dots, n$. We have that

$$0 \leq \sum_{i=1}^{n} ([T_i^*, T_i] k_i x_i, k_i x_i) + 2\mathrm{Re} \sum_{i<j} ([T_j^*, T_i] \bar{k}_j x, \bar{k}_i x)$$

$$= ([(\sum_{k=1}^{n} k_i T_i)^*, \sum_{k=1}^{n} k_i T_i] x, x).$$

Thus $((\sum_{=1}^{n} k_i T_i)^*, \sum_{k=1}^{n} k_i T_i)$ is positive semi-definite, and so T is weakly jointly hyponormal.

THEOREM 2.6. *Let $T = (T_1, \dots, T_n)$ be a weakly doubly commuting n-tuple of operators on H. Then the following statements are equivalent :*

(1) *T is jointly hyponormal.*
(2) *T is (C.D)jointly hyponormal.*
(3) *T is weakly jointly hyponormal.*

Proof. It is sufficient to show that the relations (2)⇒(1) and (3)⇔(2) holds.

(2)⇒(1) : If T is (C.D) jointly hyponormal, we have an $n \times n$ matrix $\mathcal{M}_{n\times n}[(T_j^*, T_i)]$ is positive semi-definite, and T is jointly hyponormal.

(3)⇔(2) : T is (C.D) jointly hyponormal if and only if

$$\begin{aligned}&\sum_{i=1}^{n}([(k_iT_i)^*, (k_iT_i)]x, x) + 2\mathrm{Re}\sum_{i<j}([(k_jT_j)^*, (k_iT_i)]x, x)\\ &= \sum_{i=1}^{n}([(k_iT_i)^*, (k_iT_i)]x, x)\end{aligned}$$

is positive semi-definite for all $x \in H$, $k_i \in C$ $\;i = 1, \dots, n$, since $(k_1T_1, \dots, k_nT_n)$ is also (C.D) jointly hyponormal ; that is, T is weakly jointly hyponormal.

We give an example which $T = (T_1, \dots, T_n)$ satisfies the conclusion of Theorem 2.6 even though T is not a weakly doubly commuting n-tuple of operators.

EXAMPLE 2.7. Let$T = (T_1, \dots, T_n)$ be an n-tuple of operators such that $T_i = a + T_{i-1}$ for $a \in C$ and $i = 2, \dots, n$. Then T is jointly hyponormal if and only if T is (C.D) jointly hyponormal if and only if T is weakly jointly hyponormal.

Proof. Let $T_i = a + T_{i-1}$ for $a \in C$ and $i = 2, \dots, n$. Then, it follows from a simple calculation that T is not a weakly doubly commuting n-tuple of operators. If T is (C.D) jointly hyponormal, then we know an $n \times n$ matrix $\mathcal{M}_{n\times n}(T)$ is positive semi- definite since T_1 is a hyponormal operator. Thus T is jointly hyponormal. Also, if T is jointly hyponormal, then each T_i is a hyponormal operator, and T is (C.D) jointly hyponormal. To complete the proof it is sufficient to show that, if T is weakly jointly hyponormal, then T is jointly hyponormal. Suppose that T is weakly jointly hyponormal. Then, it is clear that $\sum_{i=1}^{n} \alpha_i T_i$ is hyponormal operator for $\alpha_i \in C$, and $i = 1, \dots, n$, that is

$$(\sum_{i=1}^{n} \alpha_i T_i) + (\alpha_1 + 2\alpha_2 + \cdots + (n-1))\, a$$

is a hyponormal operator. Thus, T_1 is a hyponormal operator, and hence each T_i is a hyponormal operator and $\mathcal{M}_{n\times n}(T)$ is positive semi-definite. Therefore T is jointly hyponormal.

The following facts are obvious by concepts of jointly hyponormality or Lemma 2.4.

REMARK 1. If $T = (T_1, \ldots, T_n)$ is jointly hyponormal, then so are both $(k_1 T_1, \ldots, k_n T_n)$ and $(T_1 - k_1 I, \ldots, T_n - k_n I)$ for $k_1, \ldots, k_n$ in C.

REMARK 2. If $T = (T_1 \ldots, T_n)$ is jointly hyponormal and N is any normal operator commuting with each T_i, then $(NT_1, \ldots, NT_n)$ is jointly hyponormal.

From some properties of a pair of operators on H, we induce that a commuting n-tuple of operators is jointly hyponormal.

THEOREM 2.8. *Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators. If $(T_i, T_j^* T_j)$ is jointly hyponormal for $i, j = 1, \ldots, n$, then T is jointly hyponormal.*

Proof. It follows from [5] that T_i and $T_j^* T_j$ are hyponormal operators for each $i, j = 1, \ldots, n$, and the inequality

$$|([T_j^* T_j, T_i])y, x)|^2 \leq ([T_i^*, T_i]x, x)([T_j^* T_j, T_j^* T_j]y, y)$$

holds for all $x, y \in H$ and $i, j = 1, \ldots, n$, and so, $([T_j^* T_j, T_i]y, x) = 0$, that is, $([T_j^*, T_i]T_j y, x) = 0$ implies $([T_i^*, T_j]x, T_j y) = 0$ for all $x, y \in H$ and $i, j = 1, \ldots, n$. Thus we have the equality

$$\sum_{j<i}^{n} ([T_i^*, T_j]x, T_j y) = \sum_{j<i}^{n} ([T_i^*, T_j]y_i, y_j) = 0$$

for replacing $T_j y$ by y_j and x by y_i for $y_i, y_j \in H$. It is clear from Lemma 2.4 that T is jointly hyponormal.

From Remark 2, we have a necessary and sufficient condition by a weakly doubly commuting n-tuple of operators.

THEOREM 2.9. *Let $T = (T_1, \ldots, T_n)$ be a weakly doubly commuting n-tuple of operators and let N be a normal operator commuting with each T_i. Then T is jointly hyponormal if and only if $(NT_1, \ldots, NT_n)$ is jointly hyponormal.*

Proof. Suppose T is jointly hyponormal. It is clear from Remark 2 that $(NT_1, \ldots, NT_n)$ is jointly hyponormal. Conversely, suppose $(NT_1, \ldots, NT_n)$ is jointly hyponormal. By Lemma 2.4 we have

$$\sum_{i=1}^{n}([(NT_i)^*, NT_i]x_i, x_i) + 2\mathrm{Re}\sum_{i<j}([(NT_j)^*, NT_i]x_j, x_i) \geq 0$$

for all $x_i, x_j \in H$. By the assumption and Fuglede theorem,

$$([(NT_i)^*, NT_i]x_i, x_i) = (N^*[T_i^*, T_i]Nx_i, x_i) = ([T_i^*, T_i]y_i, y_i) \geq 0$$

for replacing Nx_i by y_i, $y_i \in H$, $i = 1, \ldots, n$. Therefore each T_i is a hyponormal operator, and T is jointly hyponormal.

3. Some characterization of n-tuple of operators with Cartesian decomposition

For $T \in B(H)$, the real part of T, denote by A, is defined to be $\dfrac{T + T^*}{2}$ and the imaginary part of T, denote by B, is defined to be $\dfrac{T - T^*}{2i}$. It is easy to check that $T = A + iB$, that is, T has Cartesian decomposition and T is self-adjoint if and only if $B = 0$.

The following results was proved by Che-Kao Fong and V. I. Istrăţescu [2].

LEMMA 3.1. [2] *If $T \in B(H)$ and there exists a scalar $\alpha < 1$ such that $T^*T \leq A^2 + \alpha B^2$, then T is self-adjoint.*

We generalize this result to weakly doubly commuting n-tuple of operators on H. At first, we give the following Lemma.

LEMMA 3.2. *Let $T = (T_1, \ldots, T_n)$ be a weakly doubly commuting n-tuple of operators with a Cartesian decomposition $T_j = A_j + iB_j$ for $j = 1, \ldots, n$. Where we note $T^* = (T_1^*, \ldots, T_n^*)$ and $A = (A_1, \ldots, A_n)$,* and $B = (B_1, \ldots, B_n)$. *Then* T^* *is jointly hyponormal if and only if* $T_j^* T_j \leq A_j^2 + B_j^2$ *for* $j = 1, \ldots, n$.

Proof. Suppose that $T_j^* T_j \leq A_j^2 + B_j^2$ for $j = 1, \ldots, n$, then we have $T_j^* T_j = A_j^2 + B_j^2 + i(A_j B_j - B_j A_j) \leq A_j^2 + B_j^2$ for $j = 1, \ldots, n$. And hence $i(A_j B_j - B_j A_j) \leq 0$ for $j = 1, \ldots, n$. Since $T_j T_j^* - T_j^* T_j = 2i(B_j A_j - A_j B_j) \geq 0$ for $j = 1, \ldots, n$, each T_j^* is a hyponormal operator. It is clear that an $n \times n$ matrix $\mathcal{M}_{n \times n}(T^*)$ is positive semi-definite. Therefore T^* is jointly hyponormal. Conversely, suppose T^* is jointly hyponormal. Then, by Lemma 2.4,

$$0 \leq \sum_{j=1}^{n} ([T_j, T_j^*] x_j, x_j) + 2\mathrm{Re} \sum_{j<k} ([T_k, T_j^*] x_k, x_j) = \sum_{j=1}^{n} ([T_j, T_j^*] x_j, x_j)$$

for all $x_j, x_k \in H$. Thus, for each j, $[T_j, T_j^*] \geq 0$ implies that $2i(B_j A_j - A_j B_j) \geq 0$, and

$$T_j^* T_j = A_j^2 + B_j^2 + i(A_j B_j - B_j A_j \geq A_j^2 + B_j^2,$$

for $j = 1, \ldots, n$.

THEOREM 3.3. *Let* $T = (T_1, \ldots, T_n)$ *be a weakly doubly commuting* n*-tuple of operators with a Cartesian decomposition* $T_j = A_j + iB_j$ *for* $j = 1, \ldots, n$, *where* $A = (A_1, \ldots, A_n)$ *and* $B = (B_1, \ldots, B_n)$, *and we note* $T^* = (T_1^*, \ldots, T_n^*)$. *Then each* T_j *is self-adjoint if and only if there exists a scalar* $\alpha < 1$ *such that* $T_j^* T_j \leq A_j^2 + \alpha B_j^2$ *for* $j = 1, \ldots, n$.

Proof. Suppose that there exists a scalar $\alpha < 1$ such that $T_j^* T_j \leq A_j^2 + \alpha B_j^2$ for $j = 1, \ldots, n$. It follows from Lemma 3.2 that T^* is jointly hyponormal. We must show that the Taylor's joint spectrum $S_p(T)$ of T is included om R^n. Since T^* is weakly doubly commuting n-tuple of hyponormal operators by joint hyponormality of T^* and the assumption, it follows from [3] that $S_p(T^*)$ equals to the joint approximate compression spectrum$\sigma_l(T^*)$ of T^*, that is, $S_p(T^*) = \sigma_l(T^*)$. Let $\bar{\lambda} \in S_p(T^*)$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n$. Then there exists a sequence $\{x_n\}$ of unit vector in H such that

$$\|(T_j^* - \bar{\lambda}_j)^* x_n\| = \|(T_j - \lambda_j) x_n\| \to 0 \text{for} j = 1, \ldots, n, \quad (n \to \infty).$$

From the assumption we have

$$(T_j^*T_jx_n, x_n) \leq ((A_j^2 + iB_j^2)x_n, x_n)$$
and
$$\|T_jx_n\|^2 \leq \|A_jx_n\|^2 + \alpha\|B_jx_n\|^2$$

for $j = 1, \dots, n$. Since, for each $j = 1, \dots, n$,

$$|\lambda_j|^2 = \|\lambda_jx_n\|^2 = \lim_{n\to\infty} \|T_jx_n\|^2 \leq |\mathrm{Re}\lambda_j|^2 + \alpha|\mathrm{Im}\lambda_j|^2,$$

and $\alpha < 1$, it is clear that $\mathrm{Im}\lambda_j = 0$ for $j = 1, \dots, n$, and $S_p(T^*) = S_p(T)$. This implies $S_p(T)$ is real. Therefore each T_j is self-adjoint.

Conversely, suppose that each T_j is self-adjoint for $j = 1, \dots, n$. Then it is obvious that T is jointly hyponormal and T^* is jointly hyponormal. By Lemma 3.2 we have $T_j^*T_j \leq A_j^2 + B_j^2$ for $j = 1, \dots, n$. Then there exists a scalar $\alpha < 1$ such that, for each j, the inequality $T_j^*T_j \leq A_j^2 + \alpha B_j^2$ is equivalent to $i(A_jB_j - B_jA_j) \leq (\alpha - 1)B_j^2$. The proof is completed.

By Lemma3.2 and Theorem 3.3, we have the following.

COROLLARY 3.4. *Let $T = (T_1, \dots, T_n)$ be a weakly doubly commuting n-tuple of operators with a Cartesian decomposition $T_j = A_j + iB_j$ for $j = 1, \dots, n$, where $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$. Then T is jointly hyponormal if and only if $T_j^*T_j \geq A_j^2 + B_j^2$ for $j = 1, \dots, n$.*

COROLLARY 3.5. *Let $T = (T_1, \dots, T_n)$ be a weakly doubly commuting n-tuple of operators with a Cartesian decomposition $T_j = A_j + iB_j$ for $j = 1, \dots, n$, where $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$. Then each T_j is self-adjoint if and only if there exists a scalar $\alpha < 1$ such that $T_j^*T_j \geq A_j^2 + \alpha B_j^2$ for $j = 1, \dots, n$.*

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