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REGULAR NEAR-RINGS AND π-REGULAR NEAR-RINGS

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1. Introduction

The concept of a regular near-ring was introduced in 1968 by J.C. Beidlemān [1] and several elementary properties of such near-rings were developed. Later Steve Ligh [7] was the first to give some structure theory for regular near-rings. In 1972, H. E. Heatherly [6] established the structure theory for some types of regular near-rings. Here we will investigate some properties of regular near-ring. And we will introduce reflexive inverse of a near-ring element and give the relating properties. In [10], it is proved that if a ring R with identity contains no non-zero nilpotent elements, then R is regular if and only if every principal left ideal is the left annihilator of an element of R. We will prove that if a zero-symmetric near-ring N with identity contains no non-zero nilpotent elements, then N is regular if and only if every principal left N-subgroup is the right annihilator of an element of N.

In 1979, A.K.Goal and S.C.Choudhary [4] got many interesting properties of π-regular near-rings. We now give some characterizations of π-regular near-ring. An element a in N is *regular* if $a = axa$ has a solution in N and any such solution x is called a *generalized inverse* of a. An element a in N will be called *unit regular* if N has an identity and a has an invertible generalized inverse. A *reflexive inverse* of a in N is a near-ring element x such that $a = axa$ and $x = xax$. Corresponding concepts in ring theory are given in [3],[5] and [11]. Every regular element contains a reflexive inverse for $a = axa$ implies that $y = xax$ is a reflexive inverse of a. The near-ring N is *regular* if each of its element is regular element. A near-ring N is *right strongly regular(left strongly regular)* if for all a in N, there exists x in N with $a = a^2x(a = xa^2)$[9].

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2. Regular near-rings

THEOREM 2.1. *For a non-zero distributive regular element a of a near-ring N, the following are equivalent:*

(1) *a has a unique generalized inverse.*
(2) *a is neither a right nor a left divisor of zero.*
(3) *N has an identity and a is a unit element.*

Proof. (1) $\to$ (2). If a is the unique generalized inverse and if $ab = 0$ or $ba = 0$, then $a(x+b)a = a$. By uniqueness, $x + b = x$ whence $b = 0$.

(2) $\to$ (3). Suppose that a is neither a right nor a left divisor of zero. Choose an element x with $a = axa$. For any b in N, we have $a(b - xab) = 0 = (b - bax)a$ and therefore, $xab = b = bax$. Thus xa is a left identity and xa is a right identity for N. Hence $e = ax = xa$ is the identity for N and a is clearly a unit element.

(3) $\to$ (1). If N has the identity e and a is a unit element, then $a = axa$ implies $ax = e = xa$, so $x = a^{-1}$.

COROLLARY 2.2. *A non-zero distributive near-ring N with identity is a near-field if and only if each non-zero element of N has a unique generalized inverse.*

THEOREM 2.3. *If a is a regular element of the distributive near-ring N, then the following are equivalent:*

(1) *a has a unique reflexive inverse.*
(2) *There is an element x in N such that $a = axa$ and both ax and xa are central idempotents.*
(3) *If $a = aya$, then $ay = ya$.*
(4) *If $a = aya = axa$, then $ay = ax = xa = ya$.*

Proof. (1) $\to$ (2). Let x be the unique element of N for which $a = axa$ and $x = xax$. For any y in N, the elements $x + y - xay$ and $x + y - yax$ are generalized inverses of a and hence

$$\begin{aligned} x &= (x+y-xay)a(x+y-xay) = x + yax - xayax \\ &= (x+y-yax)a(x+y-yax) = x + xay - xayax. \end{aligned}$$

Therefore $yax = xay$ for every y in N. Letting y be ax and xa successively, we have $ax = (ax)^2 = xa^2x = xa$, since $(ax)ax = (xa)ax$ implies $ax = xa^2x$ and $(xa)ax = (xa)xa$ implies $xa^2x = xa$.

$(2) \to (3)$. Choose an element x with $a = axa$ and both ax and xa in the center. Hence if $a = aya$, then $ay = (ax)ay = ayax = (aya)x = ax = xa = x(aya) = (ya)(xa) = ya$.

$(3) \to (4)$. If $a = aya = axa$, then by (3), $ay = ya = y(axa) = (ya)(xa) = (ay)(ax) = ax = xa$.

$(4) \to (1)$. If y and x are reflexive inverses of a, then $y = y(ay) = yax = (ya)x = xax = x$.

Now we consider the following condition [8] on a near-ring N:

(C) $aN = aNa$ for each a in N.

LEMMA 2.4. *([8]) Let N be a zero-symmetric near-ring with (C). Then $ab = 0$ implies $ba = 0$ and $axb = 0$ for any a, b, x in N.*

LEMMA 2.5. *([6]) Let N be a zero-symmetric near-ring without non-zero nilpotent elements. Then $ab = 0$ implies $ba = 0$ and $axb = 0$ for any a, b, x in N.*

THEOREM 2.6. *The center of a regular near-ring is also regular. Moreover the center of a left(right) strongly regular near-ring is left(right) strongly regular.*

Proof. Let N be a regular near-ring and let $a \in Cent(N)$, the center of N. Then $a = axa$ for some x in N. This implies $a = axa = axaxa$. It is sufficient to show that $xax \in Cent(N)$. If $y \in N$, we have $(ax)y = a(xy) = xya = xyaxa = a(xy)xa = axayx = ayx = y(ax)$ and so $ax \in Cent(N)$. Now $(xax)y = x(ax)y = xyax = axyx = y(ax)x = y(xax)$ and therefore,$xax \in Cent(N)$. It is easy to see that $Cent(N)$ is in fact left strongly regular as well as right strongly regular.

LEMMA 2.7. *Let N be a near-ring with identity. If every principal left N-subgroup generated by a is the right annihilator of an element of N, then every non-zero divisor is a regular element.*

Proof. If a is a non-zero divisor of N, let b be an element of N such that $Na = Ann_r(b)$,then $ba = 0$ implies $b = 0$ and therefore $Na = N$ which implies that a has a left inverse. Thus we have $a = aba$, where b is the left inverse of a.

THEOREM 2.8. *Let N be a zero-symmetric near-ring with identity 1 and without non-zero nilpotent elements. Then the following are equivalent:*

(1) *N is regular*
(2) *Every principal left N-subgroup generated by a is the right annihilator of an element of N*

Proof. If N is regular,for any a in N, there exists b in N such that $a = aba$. Since $e = ba$ is an idempotent and $Na = Ne$, then Ne is the right annihilator $Ann_r(1-e)$ of $1-e$. For any $x = ne \in Ne$, $(1-e)x = (1-e)ne = ne - ene = 0$. And if $y \in Ann_r(1-e)$,then $(1-e)y = 0$ and so $y - ey = 0$. Then $y = ey = ye \in Ne$. Thus we have that (1) implies (2). Conversely, assume that (2) holds. We first note that since N contains no non- zero nilpotent element, if $ab = 0$ for a, b in N,then $(ba)^2 = b(ab)a = b0a = 0$ implies $ba = 0$. Thus $Ann_l(a) = Ann_r(a)$ for every a in N. Let a be not zero in N. If a is a non-zero divisor , by the Lemma 2.7, a is a regular element. If a is a zero divisor, let $Na = Ann_r(b)$, then b is non-zero and $ab = ba = 0$. Let $c = a + b$. Suppose $cx = (a+b)x = 0$ for some x in N. Then $ax = -bx \in Ann_r(b) \cap Ann_r(a)$. If $y \in Ann_r(b) \cap Ann_r(a)$, then $y = za$ for some z in N since $Na = Ann_r(b)$ and $aza = ay = 0$ which implies $(za)^2 = z(aza) = 0$. Since N contains no non-zero nilpotent elements, $y = za = 0$. Then $ax = -bx = 0$ which implies $x \in Ann_r(b) \cap Ann_r(a) = \{0\}$. Thus $c = a+b$ is a non-zero divisor. Now we have that $ca = (a+b)a = a^2$ and by the Lemma 2.7, $a = ad^2$ where d is the left inverse of c. Then $(a-ada)^2 = a(a-ada) - ada(a-ada) = 0$, since $(a-ada)a = a^2 - ada^2 = 0$ which implies $a(a-ada) = 0$. By the hypothesis, $a = ada$ which will prove that (2) implies (1).

THEOREM 2.9. *Let N be a zero-symmetric near-ring and N has no non-zero nilpotent elements. Then N satisfies the condition (C) if and only if N is regular.*

Proof. Suppose that N satisifies the condition (C). For each a in N, there is an element b in N such that $a^2 = aba$. Hence $(a-ab)a = 0$ implies $a(a-ab) = 0$ and $(a-ab)^2 = 0$, since $(a-ab)^2 = a(a-ab) - ab(a-ab) = 0$. Thus by the hypothesis $a = ab$. Since $aN = aNa$, it follows that $a = ab = axa$ for some x in N and so N is regular near-ring. Conversely, assume that N is regular. Then for each a in N, $a = axa$

and xa is an idempotent for some x in N. Let $e = e^2$ be in N. Then for each a in N, $(ea - eae)ea = 0$ implies $ea(ea - eae) = eae(ea - eae) = 0$. Thus $(ea - eae)^2 = 0$ and so $ea = eae$. Now let y be any element element in N. Then $ay = (axa)y = a(xa)y = axayxa = a(xayx)a$. Thus $aN = aNa$, i.e., N satisifies the condition (C).

From the Theorem 2.8 and Theorem 2.9, we have that

COROLLARY 2.10. *Let N be a zero-symmetric near-ring with identity and without non-zero nilpotent elements. Then the following are equivalent:*

(1) *N is regular.*
(2) *N satisifies the condition (C).*
(3) *Every principal left N- subgroup generated by a is the right annihilator of an element of N.*

3. π-Regular near-rings

DEFINITION 3.1. ([4]) An element a of a near-ring N is said to be *π -regular element* if there exists x in N and an integer n such that $a^n = a^n x a^n$.

If every element of N are π-regular element, we say that N is a *π-regular near-ring*. Clearly every regular near-ring is π-regular but not conversely.

EXAMPLE 3.2. ([4]) Let $N = \{0, a, b, c\}$ with addition and multiplication be defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

*	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	a	b	c

It can be seen that $(N,+,.)$ is a π-regular but N is not regular, as the element a is not a regular element.

LEMMA 3.3. *Let N be a near-ring with identity and without non-zero nilpotent element. If every principal left N-subgroup generated by a^n for an element a in N and for some integer n is the right annihilator of an element of N, every non-zero divisor element is a π-regular element.*

Proof. If a is a non-zero divisor, let x be an element of N such that $Na^n = Ann_r(x)$. Then $a^n \in Ann_r(x)$ and so $xa^n = 0$ implies $x = 0$ and therefore $Na^n = N$ which implies a^n has a left inverse. Thus we have $a^n = a^nxa^n$, where x is the left inverse of a^n.

THEOREM 3.4. *Let N be a zero-symmetric near-ring with identity and without non-zero nilpotent elements. Then N is π-regular if and only if every principal left N-subgroup generated by a^n for an element a in N and for some positive integer n is the right annihilator of an element of N.*

Proof. If N is π-regular, for any a in N, there exists x in N such that $a^n = a^nxa^n$ for some integer n. Since $e = xa^n$ is an idempotent $Na^n = Ne$ and Ne is the right annihilator $Ann_r(1-e)$ of $1-e = 1-xa^n$. Conversely, assume that every principal left N-subgroup generated by a^n is the right annihilator of an element of N. Since N contains no non-zero nilpotent elements,$Ann_l(a) = Ann_r(a)$ for every a in N. Let a be non-zero in N. If a is a non-zero divisor,by the Lemma 3.3, a is π-regular element. If a is zero divisor, let $Na^n = Ann_r(b)$, then b is non zero and $a^nb = ba^n = 0$. Let $c = a^n+b$. Suppose that $cx = (a^n+b)x = 0$ for some x in N. Then $a^nx = -bx \in Ann_r(b) \cap Ann_r(a^n)$. If

$y \in Ann_r(b) \cap Ann_r(a^n)$, then $y = za^n$ for some z in N, since $Na^n = Ann_r(b)$ and $a^n za^n = a^n y = 0$. Then we have $(za^n)^2 = z(a^n za^n) = 0$. Since N contains no non-zero nilpotent element, $y = za^n = 0$. Then $a^n x = -bx = 0$ which implies $x \in Ann_r(b) \cap Ann_r(a^n) = \{0\}$. Thus $c = a^n + b$ is a non-zero divisor. Now we have $ca^n = (a^n + b)\, a^n = (a^n)^2$ and by the Lemma 3.3, $a^n = d(a^n)^2$ where d is the left inverse of c. Then $(a^n - a^n da^n)^2 = a^n(a^n - a^n da^n) - a^n da^n(a^n - a^n da^n) = 0$ because $(a^n - a^n da^n)a^n = (a^n)^2 - a^n d(a^n)^2 = 0$ which implies $a^n(a^n - a^n da^n) = 0$. By the hypothesis, $a^n = a^n da^n$ and so N is π-regular near-ring.

THEOREM 3.5. *Let N be a zero-symmetric near-ring without non-zero nilpotent elements. Then $a^n N = a^n N a^n$ for each a in N and some integer n if and only if N is π-regular.*

Proof. Suppose that for each a in N and some integer n,$a^n N = a^n N a^n$. For each a in N,there is an element x in N such that $(a^n)^2 = a^n x a^n$. Hence $(a^n - a^n x)a^n = 0$ implies $a^n(a^n - a^n x) = 0$ and so $(a^n - a^n x)^2 = 0$. By the hypothesis $a^n = a^n x$. Since $a^n N = a^n N a^n$, it follows that $a^n = a^n x = a^n y a^n$ for some y in N and N is π-regular near-ring . Conversely, if N is π-regular then for each a in N,$a^n = a^n x a^n$ and xa^N is an idempotent for some x in N and integer n. Let $e = e^2$. Then for each a in N and integer n,$(ea^n - ea^n e)ea^n = 0$ implies $ea^n(ea^n - ea^n) = ea^n e(ea^n - ea^n e) = 0$. Thus $(ea^n - ea^n e)^2 = 0$ and so $ea^n = ea^n e$. Now let y be any element in N. Then $a^n y = (a^n x a^n)y = a^n(xa^n)y = a^n x a^n y x a^n = a^n(xa^n yx)a^n$. Therefore we have $a^n N = a^n N a^n$.

From the Theorem 3.4 and Theorem 3.5, we have that

COROLLARY 3.6. *Let N be a zero-symmetric near-ring with identity and without non-zero nilpotent elements. Then the following are equivalent:*

(1) *N is π-regular.*
(2) *$a^n N = a^n N a^n$ for each a in N and some integer n.*
(3) *Every principal left N-subgroup generated by a^n for some element a in N is the right annihilator of an element of N.*

THEOREM 3.7. *Let N be a near-ring in which if $ab = 0$, then $ba = 0$ and $axb = 0$ for any a, b, x in N. If $a^n N = a^n N a^n$ for* *each a in N and some integer n, then (1) for any a in N, there is an element y such that $a^n = a^nya^n$ is nilpotent (2) $ea = eae$ for any a and $e = e^2$ in N.*

Proof. Suppose that for each a in N and some integer n, $a^nN = a^nNa^n$. Then we have $(a^n)^2 = a^nxa^n$ for some x in N and $(a^n - a^nx)a^n = 0$. By the hypothesis $a^n(a^n - a^nx) = 0$ and $a^nx(a^n - a^nx) = 0$ and so $(a^n - a^nxz)^2 = 0$. That is, $a^n - a^nx$ is nilpotent. Now $a^nx = a^nya^n$ for some y in N and $a^n - a^nya^n = a^n - a^nx$. Thus $a^n - a^nya^n$ is nilpotent. So (1) holds. Since for $e = e^2$ in N and for any integer n, $e^n = e$, we have $eN = eNe$ and so $ea = ex_1e$ and $ex_1 = ex_2e$ for some x_1, x_2 in N. Then $ea = ex_2e = ex_1$ and thus $ea = ex_1e = eae$.

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