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ISOMORPHISMS OF $\mathcal{A}_{2n}^{(2)}$

HA, DAE YEON AND KIM, JONG GEOUN

1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was begun by W. B. Arveson [1] in 1974. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the sequence of "tridiagonal" algebras, discovered by Gilfeather and Larson [8]. These algebras possess many surprising properties related to isomorphisms and cohomology, and are not yet well understood.

Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{f_1, f_2, \cdots, f_{2n}\}$. Then a member of the tridiagonal algebra on $\mathcal{H}$ has the form

$$\begin{pmatrix} * & * & & & & & * \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \ddots & \\ & & & & & \cdot & * \\ & & & & & & * \end{pmatrix}$$

with respect to the basis $\{f_1, f_2, \cdots, f_{2n}\}$, where all non-starred entries are zero. If we write the given basis in the order

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$\{f_1, f_3, f_5, \cdots, f_{2n-1}, f_2, f_4, \cdots, f_{2n}\}$, then the above matrix looks like this

$$\begin{pmatrix}
* & & & & * & & & & * \\
& * & & & * & * & & & \\
& & \ddots & & & * & & & \\
& & & * & & & \ddots & & \\
& & & & * & & & * & * \\
& & & & & * & & & \\
& & & & & & \ddots & & \\
& & & & & & & \ddots & \\
& & & & & & & & *
\end{pmatrix}$$

where all non-starred entries are zero. Let $\mathcal{H}$ be a complex Hilbert space with an orthonomal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let $\mathcal{A}_{2n}^{(2)}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$, such that an operator A is in $\mathcal{A}_{2n}^{(2)}$ if and only if all but the $(i,i)-$, $(i, n+i)-$, $(k, n+k-1)-$, and $(1, 2n)-$component are zero
$(i = 1, 2, \cdots, n;\ k = 2, 3, \cdots, n;\ n \geq 2)$.

In this paper, we shall show necessary and sufficient condition in which isomorphisms of $\mathcal{A}_{2n}^{(2)}$ are spatially implemented.

First we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$.

If $\mathcal{A}$ is a vector space over C and if $\mathcal{A}$ is closed under the composition of maps, then $\mathcal{A}$ is called an algebra. $\mathcal{A}$ is called a self-adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$. Otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra.

If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, $Alg\mathcal{L}$ denotes the algebra of all operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I.

Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections which leave invariant each operator in $\mathcal{A}$.

An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is called a CSL-algebra.

If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$.

2. Isomorphisms of $\mathcal{A}_{2n}^{(2)}$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$.

Let $\mathcal{H}$ be a $2n$-dimensional Hilbert space with a fixed basis $[e_1, e_2, \cdots, e_{2n}]$ and let $\mathcal{L}$ be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_2], \cdots, [e_n], [e_1, e_2, e_{n+1}], [e_2, e_3, e_{n+2}], \cdots, [e_{n-1}, e_n, e_{2n-1}], [e_1, e_n, e_{2n}]\}$. Then $\mathcal{A}_{2n}^{(2)} = Alg\mathcal{L}$.

We will introduce a theorem in order that automorphisms of $\mathcal{A}_{2n}^{(n)}$ need not be spatially implemented.

THEOREM 1. ([9]) *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and let $\mathcal{L}_1$ be completely distributive. Let $\rho : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ be an algebraic isomorphism. The followings are equivalent :*

i) *ρ is quasi-spatial, implemented by a closed, injective linear transformation $T : \mathcal{H}_1 \to \mathcal{H}_2$ whose range and domain are dense.*

ii) *ρ preserves the rank of every finite-rank operator ; that is, $rank(\rho(R)) = rankR$ for all finite-rank R.*

Let $\varphi : \mathcal{A}_6^{(2)} \to \mathcal{A}_6^{(2)}$ be defined by

$$\begin{pmatrix} a_1 & 0 & 0 & a_2 & 0 & a_3 \\ 0 & a_4 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & a_7 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} a_1 & 0 & 0 & a_2 & 0 & -a_3 \\ 0 & a_4 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & a_7 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} \end{pmatrix}$$

It is easy to check that φ is an isomorphism. However, the rank of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is 2, whereas the rank of $\varphi(A)$ is 3. Hence φ is not spatially implemented by Theorem 1.

Let i and j be positive integers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other components are 0.

THEOREM 2. *Let $\varphi : \mathcal{A}_{2n}^{(2)} \to \mathcal{A}_{2n}^{(2)}$ be an isomorphism such that $\varphi(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$. Then there exist nonzero complex numbers α_{ij} such that $\varphi(E_{ij}) = \alpha_{ij} E_{ij}$ for all E_{ij} in $\mathcal{A}_{2n}^{(2)}$.*

Proof. Since $\varphi(E_{ii}) = E_{ii}$ and φ is an isomorphism, we have $\varphi(E_{ii}^{\perp}) = E_{ii}^{\perp}$ for all $i = 1, 2, \cdots, 2n$. Since

$$E_{ij} = E_{jj}^{\perp}\, E_{ij}\, E_{jj} \qquad \text{and } E_{ij} = E_{ii}\, E_{ij}\, E_{ii}^{\perp} \qquad (j = 1, 2, \cdots, 2n),$$

$$(*) \qquad \begin{aligned} \varphi(E_{ij}) &= \varphi(E_{jj}^{\perp})\, \varphi(E_{ij})\, \varphi(E_{jj}) \\ &= E_{jj}^{\perp}\, \varphi(E_{ij})\, E_{jj} \quad \text{and} \\ \varphi(E_{ij}) &= E_{ii}\, \varphi(E_{ij})\, E_{ii}^{\perp}. \end{aligned}$$

Comparing the components of the first equation of $(*)$ with that of the second equation of $(*)$, we have $\varphi(E_{ij}) = \alpha_{ij} E_{ij}$ for all E_{ij} in $\mathcal{A}_{2n}^{(2)}$. Since $E_{ij} \neq 0, \alpha_{ij} \neq 0$.

THEOREM 3. *Let $\varphi : \mathcal{A}_{2n}^{(2)} \to \mathcal{A}_{2n}^{(2)}$ be an isomorphism such that $\varphi(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$ and let $\varphi(E_{ij}) = \alpha_{ij}E_{ij}$, $\alpha_{ij} \neq 0$, for all E_{ij} in $\mathcal{A}_{2n}^{(2)}$. Then $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}_{2n}^{(2)}$ and for some $(2n, 2n)$-diagonal invertible operator T if and only if*

$$\alpha_{1,2n}\,\alpha_{n,2n-1}\,\alpha_{n-1,2n-2}\cdots\alpha_{3,n+2}\,\alpha_{2,n+1} = \alpha_{n,2n}\,\alpha_{n-1,2n-1}\cdots\alpha_{2,n+2}\,\alpha_{1,n+1}$$

Proof. ($\Rightarrow$) Let $A = (a_{ij})$ be in $\mathcal{A}_{2n}^{(2)}$. Then $\varphi(A) = (\alpha_{ij}\,a_{ij})$. Let $T = (t_{ii})$ be a $(2n, 2n)$-diagonal matrix such that $t_{ii} \neq 0$ for all $i = 1, 2, \cdots, 2n$. If $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}_{2n}^{(2)}$, then $TAT^{-1} = (t_{ii}\,\alpha_{ij}\,t_{jj}^{-1})$. So the following linear system for unknown variables t_{ii} $(i = 1, 2, \cdots, 2n)$;

$$\begin{array}{ll}
\alpha_{1,n+1} = t_{11}\,t_{n+1,n+1}^{-1}, & \alpha_{1,2n} = t_{11}\,t_{2n,2n}^{-1} \\
\alpha_{2,n+1} = t_{22}\,t_{n+1,n+1}^{-1}, & \alpha_{2,n+2} = t_{22}\,t_{n+2,n+2}^{1} \\
\alpha_{3,n+2} = t_{33}\,t_{n+2,n+2}^{-1}, & \alpha_{3,n+3} = t_{33}\,t_{n+3,n+3}^{-1}, \\
\cdots & \cdots \\
\alpha_{k,n+1} = t_{kk}\,t_{n+1,n+1}^{-1}, & \alpha_{k,n+k} = t_{kk}\,t_{n+k,n+k}^{-1} \\
\cdots & \cdots \\
\alpha_{n,2n-1} = t_{nn}\,t_{2n-1,2n-1}^{-1}, & \alpha_{n,2n} = t_{nn}\,t_{2n,2n}^{-1}
\end{array}$$

has solutions, Put $t_{11} = 1$. Then from the above relations $t_{2n,2n} = \alpha_{1,2n}^{-1}$ and also,

$$\begin{array}{l}
t_{n+1,n+1} = \alpha_{1,n+1}^{-1}, \qquad t_{2n,2n} = \alpha_{1,2n}^{-1}, \\
t_{22} = \alpha_{2,n+1}\,\alpha_{1,n+1}^{-1}, \qquad t_{n+2,n+2} = \alpha_{2,n+2}^{-1}\,\alpha_{2,n+1}\,\alpha_{1,n+1}^{-1}, \\
t_{33} = \alpha_{3,n+2}\,\alpha_{2,n+2}^{-1}\,\alpha_{2,n+1}\,\alpha_{1,n+1}^{-1}, \\
t_{n+3,n+3} = \alpha_{3,n+3}^{-1}\,\alpha_{3,n+2}\,\alpha_{2,n+2}^{-1}\,\alpha_{2,n+1}\,\alpha_{1,n+1}^{-1}, \\
\cdots
\end{array}$$

$$\begin{array}{l}
t_{kk} = \alpha_{k,n+k-1}\,\alpha_{k-1,n+k-1}^{-1}\cdots\alpha_{1,n+1}^{-1}, \\
t_{n+k,n+k} = \alpha_{k,n+k}^{-1}t_{kk} = \alpha_{k,n+k}^{-1}\,\alpha_{k,n+k-1}\cdots\alpha_{2,n+2}^{-1}\,\alpha_{2,n+1}\,\alpha_{1,n+1}^{-1}, \\
\cdots
\end{array}$$

$$t_{nn} = \alpha_{n,2n-1}\,\alpha^{-1}_{n-1,2n-1}\,\alpha_{n-1,2n}\cdots\alpha^{-1}_{2,n+2}\,\alpha_{2,n+1}\,\alpha^{-1}_{1,n+1},$$
$$t_{2n,2n} = \alpha^{-1}_{n,2n}\,\alpha_{n,2n-1}\,\alpha^{-1}_{n-1,2n-1}\cdots\alpha_{2,n+1}\,\alpha^{-1}_{1,n+1},$$

i.e.,

$$t_{kk} = \Big(\prod_{i=0}^{k-2}\alpha_{k-i,n+k-1-i}\Big)\times\Big(\prod_{j=1}^{k-2}\alpha_{k-j,n+k-j}\Big)^{-1} \qquad \text{and}$$
$$t_{n+k,n+k} = \Big(\prod_{i=0}^{k-2}\alpha_{k-i,n+k-1-i}\Big)\times\Big(\prod_{j=1}^{k-1}\alpha_{k-j,n+k-j}\Big)^{-1}.$$

Thus $t_{2n,2n} = \alpha^{-1}_{n,2n}\,\alpha_{n,2n-1}\,\alpha^{-1}_{n-1,2n-1}\cdots\alpha_{2,n+1}\,\alpha^{-1}_{1,n+1} = \alpha^{-1}_{1,2n}$, i.e., $\alpha_{1,2n}\,\alpha_{n,2n-1}\cdots\alpha_{3,n+2}\,\alpha_{2,n+1} = \alpha_{n\,2n}\,\alpha_{n-1,2n-1}\cdots\alpha_{2,n+2}\,\alpha_{1,n+1}$.

($\Leftarrow$) Let $A = (a_{ij})$ be in $\mathcal{A}^{(2)}_{2n}$ and let $T = (t_{kk})$ be a $(2n,2n)-$diagonal matrix such that $t_{kk} \neq 0$ for all $k = 1,2,\cdots,2n$. If $\alpha_{1,2n}\,\alpha_{n,2n-1}\cdots\alpha_{3,n+2}\cdots\alpha_{2,n+1} = \alpha_{n,2n}\,\alpha_{n-1,2n-1}\cdots\alpha_{2,n+2}\,\alpha_{1,n+1}$, then since $\varphi(A) = (\alpha_{ij}\,a_{ij})$, $\quad \alpha_{ij} \neq 0 \quad (1 \le i,j \le 2n)$,

$$\alpha^{-1}_{n,2n}\,\alpha_{n,2n-1}\alpha^{-1}_{n-1,2n-1}\cdots\alpha_{2,n+1}\,\alpha^{-1}_{1,n+1} = \alpha^{-1}_{1,2n}.$$

Put $t_{n+k,n+k} = \alpha^{-1}_{k,n+k}\,\alpha_{k,n+k-1}\cdots\alpha^{-1}_{2,n+2}\,\alpha_{2,n+1}\,\alpha^{-1}_{1,n+1}$

$$t_{kk} = \alpha_{k,n+k}t_{n+k,n+k} = \alpha_{k,n+k-1}\,\alpha^{-1}_{k-1,n+k-1}\cdots\alpha^{-1}_{1,n+1}.$$

Then $TAT^{-1} = (t_{ii}\,a_{ij}\,t^{-1}_{jj}) = (\alpha_{ij}\,a_{ij})$ for all $A = (a_{ij})$ in $\mathcal{A}^{(2)}_{2n}$. Hence there exists a $(2n,2n)-$diagonal invertible operator T such that $\varphi(A) = TAT^{-1}$.

THEOREM 4 (GILFEATHER AND MOORE [9]). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and suppose that $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is an algebraic isomorphism. Let $\mathcal{M}$ be a maximal abelian self-adjoint subalgebra (masa) contained in $Alg\mathcal{L}_1$. Then there exist a bounded invertible operator $Y : \mathcal{H}_1 \to \mathcal{H}_2$ and an automorphism $\rho : Alg\mathcal{L}_1 \to Alg\mathcal{L}_1$ such that*

(i) *$\rho(M) = M$ for all M in $\mathcal{M}$*
(ii) *$\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $Alg\mathcal{L}_1$.*

THEOREM 5. *Let $\varphi : \mathcal{A}_{2n}^{(2)} \to \mathcal{A}_{2n}^{(2)}$ be an isomorphism and let in Theorem 4, $\rho(E_{ij}) = \alpha_{ij}$,*

$$\alpha_{1,2n}\,\alpha_{n,2n-1}\,\alpha_{n-1,2n-2}\cdots\alpha_{3,n+2}\,\alpha_{2,n+1} = \alpha_{n,2n}\,\alpha_{n-1,2n-1}\cdots\alpha_{2,n+2}\,\alpha_{1,n+1}.$$

Then there exists an invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}_{2n}^{(2)}$, i.e. *φ is spatially implemented.*

Proof. Since $(\mathcal{A}_{2n}^{(2)}) \cap (\mathcal{A}_{2n}^{(2)})^*$ is a masa of $\mathcal{A}_{2n}^{(2)}$ and E_{ii} is in $(\mathcal{A}_{2n}^{(2)}) \cap (\mathcal{A}_{2n}^{(2)})^*$ for all $i = 1, 2, \cdots, 2n$, by Theorem 4 there exist an invertible operator Y in $\mathcal{B}(\mathcal{H})$ and an isomorphism $\rho : \mathcal{A}_{2n}^{(2)} \to \mathcal{A}_{2n}^{(2)}$ such that $\rho(E_{ii}) = E_{ii}$ and $\varphi(A) = Y\rho(A)Y^{-1}$ for all A in $\mathcal{A}_{2n}^{(2)}$ and for all $i = 1, 2, \cdots, 2n$. By Theorem 3 $\rho(A) = SAS^{-1}$ for some invertible diagonal operator S and for all A in $\mathcal{A}_{2n}^{(2)}$. Hence $\varphi(A) = (YS)A(S^{-1}Y^{-1})$. Let $T = YS$. Then $\varphi(A) = TAT^{-1}$ for all A in $\mathcal{A}_{2n}^{(2)}$.

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Basic Studies
Ulsan Junior College
Ulsan 680–735, Korea

Basic studies
Pusan Junior College
Pusan 616–092, Korea