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ALMOST INVERTIBILITY MODULO CLOSED IDEALS

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1. Introduction

The concept of "almost invertibility modulo closed ideals" in $L(X, Y)$ was introduced, implicitly, by Robin Harte [1,Theorem 3.9.5]. The aims of this note are to make the formal definition of "almost left and right invertible modulo closed ideals" operators and show that they form open sets.

In what follows, suppose X and Y are non-zero normed spaces over a field K, write $L(X, Y)$ for the set of all bounded linear operators from X to Y, write I for the identity operator, and abbreviate $L(X, X)$ to $L(X)$. If A is a closed ideal of $L(X)$ then $L(X)/A$ denotes the quotient algebra of $L(X)$ modulo A. Recall, by [1] or [2], that $T \in L(X, Y)$ is said to be *almost open* if there is $k > 0$ for which

$$y \in cl\{Tx : \|x\| \leq k\|y\|\} \quad \text{for all } y \in Y,$$

is said to be *almost left invertible* if there is (T_n') in $L(Y, X)$ for which

$$\|I - T_n'T\| \longrightarrow 0 \text{ with } \sup_n \|T_n'\| < \infty,$$

and is said to be *almost right invertible* if there is (T_n'') in $L(Y, X)$ for which

$$\|I - TT_n''\| \longrightarrow 0 \text{ with } \sup_n \|T_n''\| < \infty,$$

and is said to be *dense* if T has a dense range. Also recall, by [1, Theorems 3.4.2; 3.4.3], that

$$T \text{ almost open} \implies T \text{ dense}$$

and

(1.1) $\{T \in L(X, Y) : T \text{ is almost open}\}$ is an open set.

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2. Almost invertibility modulo closed ideals

we begin with:

DEFINITION 1. If A is a colsed ideal of $L(X)$ and B is a closed ideal of $L(Y)$, and if $T \in L(X,Y)$ then T is said to be *almost left invertible modulo* A if there is (T'_n) in $L(Y,X)$ for which

$$\|I - T'_n T + A\| \longrightarrow 0 \quad \text{with} \quad \sup{}_n \|T'_n\| < \infty, \tag{2.1}$$

and is said to be *almost right invertible modulo* B if there is (T''_n) in $L(Y,X)$ for which

$$\|I - TT''_n + B\| \longrightarrow 0 \quad \text{with} \quad \sup{}_n \|T''_n\| < \infty. \tag{2.2}$$

For example, if $A = \{0\}$ then (2.1) reduces to almost left invertibility, while if $A = L(X)$ then (2.1) holds for all $T \in L(X,Y)$.

If the product of two operators is almost left (or right) invertible modulo closed ideals then so is one factor:

THEORM 1. *If X, Y, and Z are normed spaces and $A \subseteq L(X), B \subseteq L(Y)$, and $D \subseteq L(Z)$ are closed ideals, then, if $T \in L(X,Y)$ and $S \in L(Y,Z)$:*

(2.3)

$$ST \text{ almost left invertible mod } A \Longrightarrow T \text{ almost left invertible mod } A$$

and

(2.4)

$$ST \text{ almost right invertible mod } D \Longrightarrow S \text{ almost right invertible mod } D.$$

If A, B and D satisfy

$$L(Y,X) \bullet B \bullet L(X,Y) \subseteq A \quad \text{and} \quad L(Y,Z) \bullet B \bullet L(Z,Y) \subseteq D \tag{2.5}$$

then

$$\begin{aligned} &S \text{ almost left invertible mod } B,\ T \text{ almost left invertible mod } A \\ &\Longrightarrow ST \text{ almost left invertible mod } A \end{aligned} \tag{2.6}$$

and

$$S \text{ almost right invertible mod } D,\ T \text{ almost right invertible mod } B$$
$$(2.7) \qquad \Longrightarrow ST \text{ almost right invertible mod } D.$$

Proof. If $\|I - U_nST + A\| \longrightarrow 0$ with $U_n \in L(Z, X)$ then (2.1) holds with $T_n{}' = U_nS$, giving (2.3), and similarly (2.4). Towards the implication (2.6) suppose that

$$\|I - S_n{}'S + B\| \longrightarrow 0 \text{ with } \sup_n \|S_n{}'\| < \infty$$

and

$$\|I - T_n{}'T + A\| \longrightarrow 0 \text{ with } \sup_n \|T_n{}'\| = k' < \infty.$$

Observe that, using (2.5),

$$I - T_n{}'S_n{}'ST + A = T_n{}'(I - S_n{}'S + B)T + (I - T_n{}'T + A);$$

thus it follows

$$\begin{aligned}\|I - T_n{}'S_n{}'ST + A\| &\le \|T_n{}'(I - S_n{}'S + B)T\| + \|I - T_n{}'T + A\| \\ &\le k'\|T\|\ \|I - S_n{}'S + B\| + \|I - T_n{}'T + A\| \longrightarrow 0.\end{aligned}$$

The argument for (2.7) is similar.

If A is a closed ideal of $L(X)$ and B is a closed ideal of $L(Y)$, and if $T \in L(X.Y)$ then we shall write

$$R_T/A : T' \mapsto T'T + A \qquad \text{from } L(Y, X) \text{ to } L(X)/A$$

and

$$L_T/B : T' \mapsto TT' + B \qquad \text{from } L(Y, X) \text{ to } L(Y)/B.$$

As we might expect, R_T/A and L_T/B are also bounded linear operators:

THEORM 2. *If A is closed ideal of $L(X)$ and B is a closed ideal of $L(Y)$, and if $T \in L(X,Y)$ then R_T/A and L_T/B are linear and bounded with $\|R_T/A\| \leq \|T\|$ and $\|L_T/B\| \leq \|T\|$. If also $S \in L(X,Y)$ then, for each $\alpha, \beta \in K$,*

$$R_{\alpha T+\beta S}/A = \alpha R_T/A + \beta R_S/A$$

and

$$L_{\alpha T+\beta S}/B = \alpha L_T/B + \beta L_S/B. \tag{2.8}$$

Proof. For each $T', T'' \in L(Y,X)$ and each $\alpha, \beta \in K$, we have

$$\begin{aligned} R_T/A(\alpha T' + \beta T'') = (\alpha T' + \beta T'')T + A &= \alpha(T'T + A) + \beta(T''T + A) \\ &= \alpha R_T/A(T') + \beta R_T/A(T'') \end{aligned}$$

and

$$\|R_T/A(T')\| = \|T'T + A\| \leq \|T'T\| \leq \|T'\|\|T\|,$$

which say that R_T/A is linear and bounded.

Towards the first part of (2.8), if $T' \in L(Y,X)$ is arbitrary then

$$\begin{aligned} R_{\alpha T+\beta S}/A(T') = T'(\alpha T + \beta S) + A &= \alpha(T'T + A) + \beta(T'S + A) \\ &= \alpha R_T/A(T') + \beta R_S/A(T'). \end{aligned}$$

The argument for L_T/B is the same.

Now we can connect almost invertibility modulo closed ideals to composition operators modulo closed ideals :

THEOREM 3. *If A is a closed ideal of $L(X)$ and B is a closed ideal of $L(Y)$, and if $T \in L(X,Y)$ then*

$$\begin{aligned} R_T/A \text{ dense } &\Longrightarrow T \text{ almost left invertible mod } A \\ &\Longrightarrow R_T/A \text{ almost open} \end{aligned} \tag{2.9}$$

and

$$L_T/B \text{ dense } \implies T \text{ almost right invertible mod } B$$
(2.10)
$$\implies L_T/B \text{ almost open.}$$

Proof. If R_T/A is dense then there is $T_0' \in L(Y, X)$ for which

$$\|I - T_0'T + A\| = \|I + A - R_T/A(T_0')\| < 1.$$

Now with

$$U = I - T_0'T \text{ and } T_n' = (I + U + \cdots + U^n)T_0' \text{ for each } n \in N$$

we have

$$\|I - T_n'T + A\| = \|U^{n+1} + A\| \leq \|U + A\|^{n+1} \longrightarrow 0, \qquad \text{and}$$

$$\|T_n'\| \leq \frac{\|T_0'\|}{(1 - \|U\|)}$$

giving the first implication of (2.9). Towards the second implication of (2.9), suppose that $\|I - T_n'T + A\| \longrightarrow 0$ and $\sup \|T_n'\| = k' < \infty$. If A is closed and $0 < t < 1$ then by the Riesz lemma, for each $U \in L(X)$, we can choose $U' \in U + A$ with $\|U'\| \leq (1/t)\|U + A\|$ (see [1,Theorem 1.5.2]). Further, if A is an ideal of $L(X)$ then

$$U' - U \in A \implies (U' - U)T_n'T \in A \implies U'T_n'T - UT_n'T \in A,$$

thus we have

$$\begin{aligned}\|U + A - R_T/A(U'T_n')\| &= \| U - U'T_n'T + A\| \\ &\leq \| U - UT_n'T + A\| + \|UT_n'T - U'T_n'T + A\| \\ &\leq \| U\|\|I - T_n'T + A\| \longrightarrow 0.\end{aligned}$$

and

$$\|U'T_n'\| \leq (1/t)\|T_n'\|\|U + A\| \leq k\|U + A\| \quad \text{with } k = (1/t)k',$$

which says that R_T/A is almost open. The argument for (2.10) is similar.

Using Theorem 3, we can conclude the following:

THEOREM 4. *If A is a colsed ideal of $L(X)$ and B is a closed ideal of $L(Y)$, then*

(2.11)
$$\{T \in L(X,Y) : T \text{ is almost left invertible mod } A\} \text{ is an open set}$$

and

(2.12)
$$\{T \in L(X,Y) : T \text{ is almost right invertible mod } B\} \text{ is an open set.}$$

Proof. By Theorem 2 the mappings $T \mapsto R_T/A$ and $T \mapsto L_T/B$ are both continuous. Thus, in view of Theorem 3 and (1.1), the sets in (2.11) and (2.12) are continuous inverse images of open sets.

References

1. R.E Harte, *Invertibility and singularity for bounded linear operators*, Marcel Dekker, New York, 1988.
2. A.Wilansky, *Modern methods in topological vertor spaces*, McGraw-Hill, New York, 1978.

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