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SOME REMARK ON GENERALIZATION OF DELIGNE ESTIMATE

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1. Introduction

Let H be a complex upper half plane and k be a positive integer. Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ ($z \in H$) is a cusp form of weight k for $SL_2(\mathbf{Z})$ which is a normalized eigen form of all the Hecke operators $T(m)$. As is well known ([5], [7], [10]), its Mellin transform

$$\phi_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

has an Euler product (as a Dirichlet series)

$$\phi_f(s) = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}. \tag{1}$$

Moreover the converse is true as well ([5], [12]).

The connection of a cusp form with an Euler product was first mentioned by Ramanujan ([9]). He considered the Fourier coefficient $\tau(n)$ of the function

$$(2\pi)^{-12}\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z} \quad (z \in H)$$

where $\Delta(z)$ is the modular discriminant, and made the conjecture

$$|\tau(p)| \le 2p^{\frac{11}{2}} \quad \text{for all prime } p.$$

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In [8], Petersson generalized it as follows: Let $P(X) = 1 - a(p)X + p^{k-1}X^2$ be the polynomial defined in (1) by setting $p^{-s} = X$. We can then write

$$P(X) = (1 - \alpha_p X)(1 - \beta_p X)$$

with

$$\alpha_p + \beta_p = \alpha(p) \text{ and } \alpha_p\beta_p = p^{k-1}.$$

He conjectured that α_p and β_p are complex conjugates. We can also express it by $|\alpha_p| = |\beta_p| = p^{\frac{k}{2}-\frac{1}{2}}$ or

$$|\alpha(p)| \leq 2p^{\frac{k}{2}-\frac{1}{2}}, \tag{2}$$

which has been shown by Deligne ([1], [2]). For $k \geq 2$ his method extends to the case when Γ is a congruence subgroup of $SL_2(\mathbf{Z})$; for $k = 1$, see Deligne and Serre ([3]). It mat be noted that this conjecture was proved earlier, in the particular case when $k = 2$, by Eichler ([4]) and Igusa ([6]).

Since $a(p)$ are all real numbers by Deligne estimate ([2]), we obtain the following theorem by quite elementary but beautiful method.

THEOREM. *$|a(n)| \leq \sigma_0(n)n^{\frac{k}{2}-\frac{1}{2}}$ for all $n \geq 1$, where $\sigma_0(n)$ is the number of positive divisors of n.*

2. Proof of the Theorem

Let p be a prime. It then follows from the properties of Hecke operators ([5], [7], [10], [11]) that

$$a(p^e) - a(p)a(p^{e-1}) + p^{k-1}a(p^{e-2}) = 0. \tag{3}$$

By (2), we let

$$\cos\vartheta_p = \frac{1}{2}p^{-(\frac{k}{2}-\frac{1}{2})}a(p) \tag{4}$$

and for $e \geq 0$

$$X_e = p^{-(\frac{k}{2}-\frac{1}{2})e} a(p^e). \tag{5}$$

Multiplying (3) by $p^{-(\frac{k}{2}-\frac{1}{2})e}$, we get that

$$\begin{aligned} 0 &= p^{-(\frac{k}{2}-\frac{1}{2})e} a(p^e) - p^{-(\frac{k}{2}-\frac{1}{2})e} a(p) a(p^{e-1}) + p^{k-1} p^{-(\frac{k}{2}-\frac{1}{2})e} a(p^{e-2}) \\ &= p^{-(\frac{k}{2}-\frac{1}{2})e} a(p^e) - 2 \cdot \frac{1}{2} p^{-(\frac{k}{2}-\frac{1}{2})} a(p) \cdot p^{-(\frac{k}{2}-\frac{1}{2})(e-1)} a(p^{e-1}) + \\ &\quad p^{-(\frac{k}{2}-\frac{1}{2})(e-2)} a(p^{e-2}). \end{aligned}$$

By (4) and (5),

$$X_e - 2\cos\theta_p \cdot X_{e-1} + X_{e-2} = 0. \tag{6}$$

We claim that

$$X_e = \frac{\sin(e+1)\theta_p}{\sin\theta_p}. \tag{7}$$

The proof goes by induction on e. If $e = 0$, then

$$X_0 = a(1) = 1 = \frac{\sin\theta_p}{\sin\theta_p}$$

because f is a normalized cusp form.

Let $e \geq 1$ and suppose that it is true for all positive integers $< e$. From (6),

$$\begin{aligned} X_e &= 2\cos\theta_p \frac{\sin e\theta_p}{\sin\theta_p} - \frac{\sin(e-1)\theta_p}{\sin\theta_p} \\ &= \frac{2\cos\theta_p \sin e\theta_p - \sin e\theta_p \cos\theta_p + \cos e\theta_p \sin\theta_p}{\sin\theta_p} \\ &= \frac{\sin(e+1)\theta_p}{\sin\theta_p}. \end{aligned}$$

Thus it follow from (5) and (7) that

$$a(p^e) = p^{-(\frac{k}{2}-\frac{1}{2})e}\frac{\sin(e+1)\theta_p}{\sin\theta_p}. \tag{8}$$

By the fact that the coefficients $a(n)$ are multiplicative ([5], [7], [10], [11]) and (8), for $n = p_1^{e_1}\cdots p_r^{e_r}$ and $e_j \geq 1$

$$\begin{aligned} a(n) &= \prod_{j=1}^{r} a(p_j^{e_j}) = \prod_{j=1}^{r} p_j^{(\frac{k}{2}-\frac{1}{2})e_j}\frac{\sin(e_j+1)\theta_{p_j}}{\sin\theta_{p_j}} \\ &= n^{\frac{k}{2}-\frac{1}{2}}\prod_{j=1}^{r}\frac{\sin(e_j+1)\theta_{p_j}}{\sin\theta_{p_j}}. \end{aligned}$$

On the other hand, we can readily show by induction on e_j that for $j = 1, 2, \ldots, r$

$$\left|\frac{\sin(e_j+1)\theta_{p_j}}{\sin\theta_{p_j}}\right| \leq (e_j+1).$$

Therefore

$$|a(n)| \leq n^{\frac{k}{2}-\frac{1}{2}}\sigma_0(n) \qquad \text{q.e.d.}$$

REMARK. The theorem asserts that $a(n) = O(n^{\frac{k}{2}-\frac{1}{2}+\varepsilon})$ for every $\varepsilon > 0$, from which we conclude that the Dirichlet series $\phi_f(s)$ converges absolutely for $\mathrm{Re}(s) > \frac{k}{2}+1$.

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