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# ON A QUESTION OF KIM CONCERNING CERTAIN GROUP PRESENTATIONS

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Ann-Chi Kim recently posed the question of determining when the group $G(n)$ defined by the presentation

$$\langle x_1, x_2, \cdots, x_n : x_i x_{i+2} = x_{i+1}\,(i \in \mathbb{Z}_n)\rangle$$

is infinite; the purpose of this paper is to answer his question by showing that $G(n)$ is infinite if and only if $n \geq 6$. It was shown in [8] that $G(1)$ is trivial, and that $G(2)$ is isomorphic to the cyclic group $C_3$ of order 3, $G(3)$ to the quaternion group $Q_8$, $G(4)$ to $SL(2,3)$ and $G(5)$ to $SL(2,5)$; on the other hand, $G(6)$ is infinite, as $G(6)/G(6)'$ is infinite. So the onus is on us to show that $G(n)$ is infinite for $n \geq 7$; in fact, $G(n)$ has a free subgroup of rank 2 for $n \geq 7$, unlike $G(6)$, which is metabelian. The case $n = 7$ has been solved independently by David Gill, who used some nifty Tietze transformations to show that there is a subgroup $H$ of index 7 in $G(7)$ and a homomorphism from $H$ onto the modular group.

The groups $G(n)$ bring to mind the **Fibonacci groups** $F(2,n)$ defined by the presentations

$$\langle x_1, x_2, \cdots, x_n : x_i x_{i+1} = x_{i+2}\,(i \in \mathbb{Z}_n)\rangle,$$

which are known to be infinite for $n = 6$ [4] and $n \geq 8$ [2, 9, 10], and not otherwise [2, 3, 4, 5]; see [13] for a recent survey of these groups. Some of the techniques useful in dealing with the Fibonacci groups are appropriate to the groups $G(n)$ also.

Let $A(n)$ be the group $G(n)/G(n)'$; we may use the techniques of [6] and [7] to work out the order of $A(n)$. If $f(x)$ is the polynomial $x^2-x+1$, then $A(n)$ has order $\pm\prod\{f(\zeta) : \zeta^n = 1\}$; so $A(n)$ is infinite if and only

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if $\zeta^2 - \zeta + 1 = 0$ for some $n^{\text{th}}$ root $\zeta$ of unity. Now $\zeta^2 - \zeta + 1 = 0$ is equivalent to $\zeta^2 + 1 = \zeta$, and hence to $\zeta^3 + \zeta^2 + \zeta + 1 = \zeta^2 + \zeta$ with $\zeta \neq -1$. So $A(n)$ is infinite if and only if $\zeta^3 = -1 \neq \zeta$ for some $\zeta$; in other words, we have:

THEOREM A. *$A(n)$ is infinite if and only if 6 divides $n$.*

As with the Fibonacci groups, there is an automorphism of $G(n)$ of order dividing $n$ permuting the $x_i$ in a cycle of length $n$. We may therefore form the semi-direct product of $G(n)$ with a cyclic group $\langle t \rangle$ of order $n$ to get the group $I(n)$ with presentation

$$\left\langle x, t : xt^{-2}xt^2 = t^{-1}xt, t^n = 1 \right\rangle,$$

where $x$ denotes $x_1$ (say). The relation $xt^{-2}xt^2 = t^{-1}xt$ is equivalent to $xt^{-2}xtx^{-1}t = 1$. We introduce a new generator $u := xt^{-1}$, and then delete $x = ut$, to get

$$\left\langle u, t : ut^{-1}utu^{-1}t = t^n = 1 \right\rangle.$$

Now the relation $ut^{-1}utu^{-1}t = 1$ is equivalent $(tu^{-1})^{-1}u(tu^{-1}) = t^{-1}$, and, given $t^n = 1$, we also have that $u^n = 1$. Since the relation $ut^{-1}utu^{-1}t = 1$ is equivalent to $tu^{-1}tut^{-1}u = 1$, we see that there is an automorphism interchanging $u$ and $t$. We form the semi-direct product of $I(n)$ with a cyclic group $\langle a \rangle$ of order 2 to get the group $K(n)$ with presentation

$$\left\langle a, t : a^2 = t^n = atat^{-1}atatat^{-1}at = 1 \right\rangle.$$

We are now in a position to prove the following result:

THEOREM B. *$G(n)$ is infinite if and only if $n \geq 6$.*

Since $G(n)$ is finite for $n \leq 5$ and $G(6)$ is infinite, it is enough to show that $K(n)$, and hence $G(n)$, is infinite for $n \geq 7$. The relation $atat^{-1}atatat^{-1}at = 1$ is equivalent to $(at^{-1}atat)^2 = 1$; our strategy is to find matrices $A$ and $T$ in $SL(2, \mathbb{C})$ such that $A^2 = T^n = (AT^{-1}ATAT)^2 = -I$, but such that $AT$ has infinite order. If we let $\tilde{K}(n)$, denote the group with presentation

$$\left\langle a, t, z : a^2 = t^n = atat^{-1}atatat^{-1}at = z,\ z^2 = 1 \right\rangle,$$

then the subgroup $\langle A, T\rangle$ of $SL(2,\mathbb{C})$ would be an infinite homomorphic image of $\tilde{K}(n)$, so that the subgroup $\langle A, T\rangle$ of $PSL(2,\mathbb{C})$ (with the usual abuse of notation) would be an infinite homomorphic image of $K(n)$.

We make use of two facts here that are of great help when tackling this sort of problem; compare [1] and [12] for example. Firstly, a matrix $M$ in $SL(2,\mathbb{C})$ has finite order $m > 2$ if and only if the trace $\mathrm{Tr}(M)$ of $M$ is of the form $\alpha + \alpha^{-1}$ for some primitive $m^{\text{th}}$ root $\alpha$ of unity. Secondly, we have the identity $\mathrm{Tr}(UV) + \mathrm{Tr}(U^{-1}V) = \mathrm{Tr}(U)\,\mathrm{Tr}(V)$ for any matrices $U$ and $V$ in $SL(2,\mathbb{C})$. Writing $U$ as $WV$, this becomes

$$\mathrm{Tr}(WV^2) + \mathrm{Tr}(W) = \mathrm{Tr}(WV)\,\mathrm{Tr}(V),$$

since $\mathrm{Tr}(V^{-1}W^{-1}V) = \mathrm{Tr}(W^{-1}) = \mathrm{Tr}(W)$. Now put $W = AT^{-1}$ and $V = AT$ to get

$$\mathrm{Tr}(AT^{-1}ATAT) + \mathrm{Tr}(AT^{-1}) = \mathrm{Tr}(AT^{-1}AT)\,\mathrm{Tr}(AT).$$

Let $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix}$, where $\alpha = e^{\pi i/n}$, so that $AT = \begin{pmatrix} \beta & \alpha^{-1} \\ -\alpha & 0 \end{pmatrix}$, $A^2 = T^n = -I$ and $A^{-1} = -A$. Then $\mathrm{Tr}(AT^{-1}) = \mathrm{Tr}(TA^{-1}) = \mathrm{Tr}(-TA) = -\mathrm{Tr}(AT)$, and we have

$$\mathrm{Tr}(AT^{-1}ATAT) = \left[\mathrm{Tr}(AT^{-1}AT) + 1\right]\mathrm{Tr}(AT).$$

Now $\mathrm{Tr}(AT^{-1}AT)+\mathrm{Tr}(TA^{-1}AT) = \mathrm{Tr}(AT^{-1})\,\mathrm{Tr}(AT)$ and $\mathrm{Tr}(AT^{-1}) = -\mathrm{Tr}(AT)$ as above, so that

$$\mathrm{Tr}(AT^{-1}AT) = -\mathrm{Tr}(AT)^2 - \mathrm{Tr}(T^2) = -\mathrm{Tr}(AT)^2 - (\alpha^2 + \alpha^{-2}),$$

and then $\mathrm{Tr}(AT^{-1}ATAT) = [1 - \beta^2 - (\alpha^2 + \alpha^{-2})]\beta$. So we want $1 - \beta^2 - (\alpha^2 + \alpha^{-2}) = 0$, i.e.

$$\beta = \pm\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right)}.$$

Since $n \geq 7$, $\beta$ is non-zero and imaginary. Now, since

$$(AT)^2 = \begin{pmatrix} \beta^2 - 1 & \beta\alpha^{-1} \\ -\alpha\beta & -1 \end{pmatrix},$$

$AT$ does not have order 1 or 2; so, for $AT$ to have finite order, we would need to have $\beta = \gamma + \gamma^{-1}$ for some primitive $m^{\text{th}}$ root $\gamma$ of unity with $m > 2$. But $\gamma + \gamma^{-1}$ would be real, while $\beta$ is imaginary; so $AT$ has infinite order for $n \geq 7$; so $\tilde{K}(n)$, and hence $K(n)$, is infinite.

In fact, we can say more than this. Let

$$M := AT = \begin{pmatrix} \beta & \alpha^{-1} \\ -\alpha & 0 \end{pmatrix}, \qquad N := AT^{-1} = \begin{pmatrix} -\beta & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}.$$

Recall that an element $U$ of $SL(2, \mathbb{C})$ is said to be **elliptic** if $\text{Tr}(U) \in \mathbb{R}$ and $|\text{Tr}(U)| < 2$, and a subgroup $S$ of $SL(2, \mathbb{C})$ is said to be **elliptic** if all elements of $S$, apart from $\pm I$, are elliptic. Since $\beta \notin \mathbb{R}$, we see that $\langle M, N \rangle$ is not elliptic. Also, a subgroup $S$ of $SL(2, \mathbb{C})$ is said to be **elementary** if the commutator of any two elements of infinite order has trace 2, and to be **discrete** if it does not contain any convergent sequence of distinct elements. Now, since $\beta \notin \mathbb{R}$, $M$ and $N$ have infinite order, as was pointed out above, and we may readily check that $N^{-1}M^{-1}NM$ has trace $-2\cos\left(\frac{2\pi}{n}\right)$, and thus $\langle M, N \rangle$ is not elementary. So, by [11], there is a generating pair $\{P, Q\}$ for $\langle M, N \rangle$ such that $\langle P^k, Q^k \rangle$ is a discrete free subgroup of rank 2 for sufficently large $k$. So $\tilde{K}(n)$ has a homomorphic image $\langle A, T \rangle \cong \tilde{K}(n)/\tilde{N}$ with a free subgroup $\left\langle \tilde{c}\tilde{N}, \tilde{d}\tilde{N} \right\rangle$ of rank 2, and then $\tilde{K}(n)$ has a free subgroup $\tilde{H} := \left\langle \tilde{c}, \tilde{d} \right\rangle$ of rank 2. If $c$ and $d$ are elements of $K(n)$ such that $c\langle z \rangle = \tilde{c}$ and $d\langle z \rangle = \tilde{d}$, then $H := \langle c, d \rangle$ is a free subgroup of $K(n)$ of rank 2. Since $G(n)$ has finite index in $K(n)$, $H \cap G(n)$ is of finite index in $H$, and hence is a non-cyclic free subgroup of $G(n)$; so we have:

THEOREM C. *$G(n)$ has a free subgroup of rank 2 for $n \geq 7$.*

We now turn to the case $n = 6$. We know that $G(6)$ is infinite, as $G(6)/G(6)'$ is infinite, but we can say a little more. Let us consider the presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 : x_1x_3 = x_2,\ x_2x_4 = x_3,\ x_3x_5 = x_4,$$
$$x_4x_6 = x_5,\ x_5x_1 = x_6,\ x_6x_2 = x_1 \rangle$$

for $G(6)$. We may eliminate the generator $x_6 = x_5x_1$ to get

$$\langle x_1, x_2, x_3, x_4, x_5 : x_1x_3 = x_2,\ x_2x_4 = x_3,\ x_3x_5 = x_4,\ x_4x_5x_1 = x_5,\ x_5x_1x_2 = x_1\rangle .$$

Now eliminate $x_4 = x_3x_5$ to get

$$\langle x_1, x_2, x_3, x_5 : x_1x_3 = x_2,\ x_2x_3x_5 = x_3,\ x_3{x_5}^2x_1 = x_5,\ x_5x_1x_2 = x_1\rangle .$$

Next, eliminate $x_2 = x_1x_3$ to get

$$\langle x_1, x_3, x_5 : x_1{x_3}^2x_5 = x_3,\ x_3{x_5}^2x_1 = x_5,\ x_5{x_1}^2x_3 = x_1\rangle .$$

We now eliminate $x_5 = x_1{x_3}^{-1}{x_1}^{-2}$ to get

$$\langle x_1, x_3 : x_1{x_3}^2x_1{x_3}^{-1}{x_1}^{-2} = x_3,\ x_3x_1{x_3}^{-1}{x_1}^{-1}{x_3}^{-1}{x_1}^{-1} = x_1{x_3}^{-1}{x_1}^{-2}\rangle .$$

Since we're getting tired of the subscripts, we rewrite $x_1$ as $a$ and $x_3$ as $b$ to get

$$\langle a, b : ab^2ab^{-1}a^{-2}b^{-1} = bab^{-1}a^{-1}b^{-1}aba^{-1} = 1\rangle .$$

Introduce $c := aba^{-1}$; the relation $bab^{-1}a^{-1}b^{-1}aba^{-1} = 1$ is then equivalent to $bc^{-1}b^{-1}c = 1$, and hence to $[b, c] = 1$, and the relation $ab^2ab^{-1}a^{-2}b^{-1} = 1$ to $c^2ac^{-1}a^{-1}b^{-1} = 1$, and hence to $aca^{-1} = b^{-1}c^2$; so we have the presentation

$$\langle a, b, c : aba^{-1} = c,\ aca^{-1} = b^{-1}c^2,\ [b, c] = 1\rangle .$$

We see that $N := \langle b, c\rangle$ is a normal abelian subgroup isomorphic to $C_\infty \times C_\infty$, and that $G(6)/N$ is isomorphic to $C_\infty$; so we have

THEOREM D. *$G(6)$ is an infinite metabelian group.*

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