Bull. Korean Math. Soc. **28**(1991), No. 2, pp. 231–241

ON THE TRANSVERSAL CONFORMAL CURVATURE TENSOR ON HERMITIAN FOLIATIONS

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Recently, many mathematicians([NT], [Ka], [TV], [CW], etc.) studied foliated structures on a smooth manifold with the viewpoint of transversal differential geometry. In this paper, we shall discuss certain hermitian foliations $\mathcal{F}$ on a riemannian manifold with a bundle-like metric, that is, their transversal bundles to $\mathcal{F}$ have hermitian structures. We shall show the following theorem;

THEOREM A. *Let $\mathcal{F}$ be a 1-dimensional regular geodesic kähler foliation of codimension $2n$ $(n > 2)$ on a compact simply-connected riemannian manifold (M, g) with positive sectional curvature. Then we have*

(a) *if B is parallel then the leaf space $M/\mathcal{F}$ is biholomorphic to a complex projective space $\mathbf{C}P^n$,*
(b) *if, in particular, B vanishes everywhere then it is holomorphically isometric to $\mathbf{C}P^n$ with a Fubini-Study metric.*

And applying Theorem A to Sasakian manifolds, we have easily the following theorem which somewhat generalizes [JLOP], in fact, their contact conformal curvature tensor restricted to the transversal bundle may be considered as our tensor B.

THEOREM B. *If the transversal conformal curvature tensor of a $(2n+1)$-dimensional fibred riemannian space M $(n > 2)$ with Sasakian structure vanishes everywhere then the leaf space $M/\mathcal{F}$ is a kähler manifold of constant holomorphic sectional curvature.*

We shall be in C^∞-category and all manifolds are assumed to be paracompact, Hausdorff spaces. And we use the Einstein summation

Received December 14, 1990.

convention and adopt ranges of indices as follows;

$$a, b, c, \cdots = p+1, \cdots, p+n, \quad \bar{a}, \bar{b}, \bar{c}, \cdots = p+n+1, \cdots, p+2n$$
$$\alpha, \beta, \gamma, \cdots = p+1, \cdots, p+2n, \quad i, j, k, \cdots = 1, \cdots, p.$$

1. Hermitian foliations and its transversal conformal curvature tensor

Let $(M, \mathcal{F}, g)$ be an oriented riemannian manifold of dimension $m := p+2n$ with a riemannian foliation $\mathcal{F}$ of dimension p. By means of the riemannian metric g, we can decompose TM as follows;

$$TM = \mathcal{F} \oplus \mathcal{H}, \qquad \mathcal{H} \simeq TM/\mathcal{F}.$$

(1.1) A *hermitian foliation* $\mathcal{F}$ of codimension $2n$ is defined by the following data;

(1.1.1) $\mathcal{H}$ has a hermitian structure (h, J), i.e. J is a complex structure with respect to h satisfying $h(JX, JY) = h(X, Y)$ for $X, Y \in \Gamma(\mathcal{H})$,

(1.1.2) J and h are holonomy invariant, i.e. $L_V J = 0$ and $L_V h = 0$ for all $V \in \Gamma(\mathcal{F})$.

Here and hereafter, $\Gamma(\ )$ denotes the space of sections of $(\ )$ and L_Y the transversal Lie derivative operator for an element Y in the space $\mathcal{B}$ of basic vector fields.

REMARK. Every riemannian foliation $\mathcal{F}$ whose transversal bundle $\mathcal{H}$ is equipped with a holonomy invariant complex structure J induces a holonomy invariant hermitian metric h on $\mathcal{H}$. Indeed, $\mathcal{H}$ admits a holonomy invariant riemannian metric $g_{\mathcal{H}}$. Set for $X, Y \in \Gamma(\mathcal{H})$, $h(X, Y) := g_{\mathcal{H}}(JX, JY) + g_{\mathcal{H}}(X, Y)$. Then clearly $h(JX, JY) = h(X, Y)$. Since $L_V J = 0$ for all $V \in \Gamma(\mathcal{F})$, $(L_V h)(X, Y) = (L_V g_{\mathcal{H}})(JX, JY) + (L_V g_{\mathcal{H}})(X, Y) = 0$. Thus h is the desired one.

Let $\phi(X, Y) := h(X, JY)$ for $X, Y \in \Gamma(\mathcal{H})$. $L_V \phi = 0$ for all $V \in \Gamma(\mathcal{F})$. Thus ϕ is a basic real 2-form on $\mathcal{H}$. The basic forms are defined by

$$\Lambda^*_{\mathcal{B}} := \{\omega \in \Lambda^* M \mid i_V \omega = 0,\ L_V \omega = 0 \quad \text{for all} \quad V \in \Gamma(\mathcal{F})\}.$$

The exterior derivative d restricts to $d_B : \Lambda_B^* \to \Lambda_B^{*+1}$. If, in particular $d_B\phi = 0$, we call $\mathcal{F}$ a *kähler foliation.*

Example of hermitian but not kähler foliations. Let $E := \Gamma_0 \backslash N$ be the Iwasawa manifold with complex structure $\check{J}$ i.e. N is the complex Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & z^1 & z^3 \\ & 1 & z^2 \\ & & 1 \end{pmatrix},$$

and Γ_0 is the subgroup of N of those matrices whose entries are Gauss integers. Letting $z^a := x^a + iy^a$ $(a = 1, 2, 3)$, we have on N a basis $\{\theta^a = \frac{1}{2}(dx^a - idy^a), \bar{\theta}^a = \frac{1}{2}(dx^a + idy^a)\}$ of left invariant 1-forms. Then $\check{h} := \theta^\alpha \bar{\theta}^\alpha$ is a left invariant hermitian metric on N. Let $\Gamma_1 \supset \Gamma_0$ be the subgroup of N of matrices of the form

$$\begin{pmatrix} 1 & x^1 + i(y^1 + sy'^1) & x^3 + sx'^3 + i(y^3 + sy'^3) \\ & 1 & x^2 + iy^2 \\ & & 1 \end{pmatrix},$$

where $s \in Q^c$ and $x^a, x'^a, y^a, y'^a \in Z$. Γ_1 can be considered as a uniform subgroup of $U = (\mathbf{R}^9, \circ)$ whose matricial form is

$$\begin{pmatrix} 1 & x^3 & x^7 & x^9 & x^1 & x^2 & x^6 & x^8 \\ & 1 & -x^5 & x^4 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & x^4 & x^5 \\ & & & & & 1 & -x^5 & x^4 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix},$$

where the group operation $\circ$ is defined by

$$\begin{aligned}(x^1, \dots, x^9) \circ (y^1, \dots, y^9) := (&x^a + y^a, x^6 + y^6 + x^1y^4 - x^2y^5, \\ &x^7 + y^7 - x^3y^5, x^8 + y^8 + x^1y^5 + x^2y^4, x^9 + y^9 + x^3y^4).\end{aligned}$$

And a homomorphism $u : U \to N$ given by

$$(x^1, \dots, x^9) \to (x^1 + i(x^2 + sx^3), x^4 + ix^5, x^6 + sx^7 + i(x^8 + sx^9))$$

is a surjective submersion with connected fibres and a foliation $\tilde{\mathcal{F}}$ by the fibres of u is Γ_1-invariant. Moreover the canonical projection $\tilde{u} : (U, \tilde{\mathcal{F}}) \to M := \Gamma_1 \backslash U$ is a Galois covering mapping, i.e. we have a foliation $\mathcal{F}$ on M of dimension 3 whose leaves $\mathcal{L}$ are given by $\tilde{u}(\tilde{\mathcal{L}}) = \mathcal{L}$, where $\tilde{\mathcal{L}}$ is a leaf of $\tilde{\mathcal{F}}$. Let $\mathcal{H} \simeq TM/\mathcal{F}$ by taking a riemannian metric g on M. Then the maximal integral submanifold of $\mathcal{H}$ corresponds to the hermitian manifold $(E, \check{J}, \check{h})$ of real dimension 6. Thus by the transversal lift satisfying the cocycle condition, we have a hermitian structure (J, h) on $\mathcal{H}$. Therefore, $\mathcal{F}$ is a hermitian foliation, and it can not be made kähler.

Now the complex structure J induces a splitting of the complexified transversal bundle $\mathcal{H}^{\mathbf{C}}$ as the standard way;

$$\mathcal{H}^{\mathbf{C}} := \mathcal{H} \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}.$$

Then we have the usual decomposition of complex differential basic forms;

$$\Lambda_B^r = \sum_{r=s+t} \Lambda_B^{s,t}, \tag{1.2}$$

and so the decomposition of d_B;

$$d_B = \partial_B + \bar{\partial}_B,$$

where $\partial_B : \Lambda_B^{s,t} \to \Lambda_B^{s+1,t}$, $\bar{\partial}_B : \Lambda_B^{s,t} \to \Lambda_B^{s,t+1}$.

Let $(z^1, \dots, z^n)$ be a local transversal coordinate system and $\{dz^\alpha\}$ a local frame of $(\mathcal{H}^{\mathbf{C}})^*$ (:= the dual of $\mathcal{H}^{\mathbf{C}}$). Since $\mathcal{F}$ is bundle-like, we can choose a local unitary moving frame $\{\omega^\alpha\}$ on $(\mathcal{H}^{\mathbf{C}})^*$ such that $\omega^\alpha \in \Lambda_B^1$. Let $h := h_{\alpha\beta}\omega^\alpha\omega^\beta$. Since h is J-invariant, we have $h_{ab} = h_{\bar{a}\bar{b}} = 0$. Thus ϕ is locally written by $\phi = ih_{a\bar{b}}\omega^a \wedge \bar{\omega}^b$.

Consider the exact sequence defining $\mathcal{F}$ of real vector bundles;

$$0 \to \mathcal{F} \to TM \xrightarrow{\mathcal{H}} \mathcal{H} \to 0.$$

Then we have an exact sequence of complex vector bundles with respect to a complex structure J^M on $T^{\mathbf{C}}M$ (:= the complexification of TM) such that $J^M|\mathcal{H}^{\mathbf{C}} = J$;

$$0 \longrightarrow \mathcal{F}^{\mathbf{C}} \longrightarrow T^{\mathbf{C}}M \overset{\mathcal{H}}{\longrightarrow} \mathcal{H}^{\mathbf{C}} \longrightarrow 0.$$

There exists in $(T^{\mathbf{C}}M, g)$ a canonical associated linear connection ∇^c (called the *second complex connection*) uniquely defined by the conditions ([Va]);

(V1) if $X \in \Gamma(\mathcal{F}^{\mathbf{C}})$ (resp. $\Gamma(\mathcal{H}^{\mathbf{C}})$) then $\nabla^c_Y X \in \Gamma(\mathcal{F}^{\mathbf{C}})$ (resp. $\Gamma(\mathcal{H}^{\mathbf{C}})$) for any $Y \in \Gamma(T^{\mathbf{C}}M)$,

(V2) if $X, Y, Z \in \Gamma(\mathcal{F}^{\mathbf{C}})$ (resp. $\Gamma(\mathcal{H}^{\mathbf{C}})$) then $(\nabla^c_Z g)(X, Y) = 0$,

(V3) $\nabla^c_X J^M = 0$ for all $X \in \Gamma(T^{\mathbf{C}}M)$,

(V4) $\tau(X, Y)|_{\mathcal{F}^{\mathbf{C}}(\text{resp. } \mathcal{H}^{\mathbf{C}})} = 0$ if at least one of the arguements in $\Gamma(\mathcal{F}^{\mathbf{C}})$ (resp. $\Gamma(\mathcal{H}^{\mathbf{C}})$),

(V5) if $X, Y \in \Gamma(\mathcal{F}^{\mathbf{C}})$ (resp. $\Gamma(\mathcal{H}^{\mathbf{C}})$) then $\tau(J^M X, Y)|_{\mathcal{F}^{\mathbf{C}}(\text{resp. } \mathcal{H}^{\mathbf{C}})} = \tau(X, J^M Y)|_{\mathcal{F}^{\mathbf{C}}(\text{resp. } \mathcal{H}^{\mathbf{C}})}$, where τ is the torsion tensor of ∇^c given by $\tau(X, Y) := \nabla^c_X Y - \nabla^c_Y X - [X, Y]$ for $X, Y \in \Gamma(T^{\mathbf{C}}M)$.

We may define a partial connection $\widehat{\nabla} : \Gamma(\mathcal{F}) \times \Gamma(\mathcal{H}^{\mathbf{C}}) \to \Gamma(\mathcal{H}^{\mathbf{C}})$ by

$$(1.3) \qquad \widehat{\nabla}_V X := \mathcal{H}[V, X] \quad \text{for} \quad V \in \Gamma(\mathcal{F}),\ X \in \Gamma(\mathcal{H}^{\mathbf{C}}).$$

Note that $\widehat{\nabla}$ is well-defined ([**BB**]). Thus we define an adapted connection ∇ in $\mathcal{H}^{\mathbf{C}}$ for $\mathcal{F}$ by for $X \in \Gamma(\mathcal{H}^{\mathbf{C}})$

$$(1.4) \qquad \nabla_Y X := \begin{cases} \widehat{\nabla}_Y X & \text{for } Y \in \Gamma(\mathcal{F}) \\ \mathcal{H}\nabla^c_Y X & \text{for } Y \in \Gamma(\mathcal{H}^{\mathbf{C}}). \end{cases}$$

Let $\{Z_\alpha\}$ be the local vector fields associated to $\{\omega^\alpha\}$ by h-duality. Then $Z_\alpha \in \mathcal{B}$. Let (ω^α_β) be the connection form of ∇. Then by properties of ∇^c and (1.2), we have

$$(1.5) \qquad \omega^a_b = h^{a\bar{c}}(\partial_{\mathcal{B}})h_{b\bar{c}},\ \omega^a_b = -\bar{\omega}^b_a,\ \omega^a_i = \omega^i_a = 0,\ \omega^i_j = 0,$$

and its torsion tensor τ^∇ satisfies

$$(1.6) \quad \tau^\nabla(Z_\alpha,\ JZ_\beta) = \tau^\nabla(JZ_\alpha,\ Z_\beta), \quad i_V\tau^\nabla = 0 \text{ for all } V \in \Gamma(\mathcal{F}).$$

Thus ∇ plays transversally a role as the hermitian connection on an ordinary hermitian manifold. Let $\Omega^\alpha_\beta := -K^{\nabla}{}^{\alpha}{}_{\beta\gamma\bar\delta}$ be the curvature form. Note that Ω^a_b is a basic 2-form of type (1,1).

We define the *transversal conformal curvature tensor* B to $\mathcal{F}$ by the same way of Kitahara-Matsuo-Pak (**[KMP]**);

$$(1.7)\quad B_{a\bar b c\bar d} := K^{\nabla}{}_{a\bar b c\bar d} + \frac{1}{n}(h_{a\bar b}T^{\nabla}{}_{c\bar d} + S^{\nabla}{}_{a\bar b}h_{c\bar d}) - \frac{nr^{\nabla} + (n^2-2)s^{\nabla}}{2n^2(n^2-1)}h_{a\bar b}h_{c\bar d} + \frac{nr^{\nabla} - s^{\nabla}}{2n(n^2-1)}h_{a\bar d}h_{c\bar b},$$

where $K^{\nabla}{}_{a\bar b c\bar d} = h_{a\bar e}K^{\nabla}{}^{\bar e}{}_{\bar b c\bar d}$ and R^{∇}, S^{∇}, T^{∇} are distinct transversal Ricci curvature tensors locally given by

$$R^{\nabla}{}_{a\bar b} := -h^{c\bar d}K^{\nabla}{}_{a\bar d c\bar b},\ S^{\nabla}{}_{a\bar b} := -h^{c\bar d}K^{\nabla}{}_{a\bar b c\bar d},\ T^{\nabla}{}_{a\bar b} := -h^{c\bar d}K^{\nabla}{}_{c\bar d a\bar b},$$

and r^{∇}, s^{∇}, t^{∇} distinct transversal scalar curvature tensors by

$$r^{\nabla} := 2h^{a\bar b}R^{\nabla}{}_{a\bar b},\ s^{\nabla} := 2h^{a\bar b}S^{\nabla}{}_{a\bar b} = t^{\nabla} := 2h^{a\bar b}T^{\nabla}{}_{a\bar b}.$$

PROPOSITION 1. *A hermitian foliation $\mathcal{F}$ is kähler if and only if ∇ coincides with the transversal Levi-Civita connection D in $\mathcal{H}$, or equivalently $D_X J = 0$ for all $X \in \Gamma(\mathcal{H})$ (for definition of D, see e.g.* [TY], [NT]*).*

Proof. By the uniqueness of D, it suffices to prove that ∇ is torsionfree. We take a local unitary basic frame $\{Z_\alpha\}$. By definition we have $\tau^{\nabla}(Z_\alpha, Z_\beta) = \nabla_{Z_\alpha}Z_\beta - \nabla_{Z_\beta}Z_\alpha = (\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha})Z_\gamma$, where $\Gamma^\alpha_{\beta\gamma} := \omega^\alpha_\beta(Z_\gamma)$. Together with the conjugate relation, it follows by (1.2), (1.6) that ∇ is torsionfree if and only if $\Gamma^a_{bc} = \Gamma^a_{cb}$, $\bar\Gamma^a_{bc} = \bar\Gamma^a_{cb}$. By (1.5), ∇ is torsionfree if and only if $(\partial_B)_b h_{c\bar a} = (\partial_B)_c h_{b\bar a}$, i.e. $\partial_B\phi = 0$ and by taking conjugates $\bar\partial_B\phi = 0$. Thus ∇ is torsionfree if and only if $d_B\phi = 0$. The second assertion follows from the general formula $2h((\nabla_X J)Y, Z) = d_B\phi(X, JY, JZ) - d_B\phi(X, Y, Z)$ for $X, Y, Z \in \Gamma(\mathcal{H})$. The proof is similar to the usual ones in hermitian geometry.

Let $R^D{}_{XYZW} := h(R^D(Z,W)Y,X)$ be the transversal curvature tensor with respect to D and S^D, c^D its respective transversal Ricci, scalar curvature tensors for $X,Y,Z,W \in \Gamma(\mathcal{H})$. B can be expressed in terms of the transversal curvature data with respect to D as follows;

(1.8)

$$\begin{aligned}
B_{XYZW} := R^D{}_{XYZW} &+ \frac{1}{2n}\{h(X,W)S^D(Y,Z) \\
&- h(Y,W)S^D(X,Z) + S^D(X,W)h(Y,Z) - S^D(Y,W)h(X,Z) \\
&- \phi(Y,Z)\rho^D(X,W) + \phi(X,Z)\rho^D(Y,W) - \rho^D(Y,Z)\phi(X,W) \\
&+ \rho^D(X,Z)\phi(Y,W) - 2\rho^D(X,Y)\phi(Z,W) - 2\phi(X,Y)\rho^D(Z,W)\} \\
&+ \frac{(n+2)c^D}{4n^2(n+1)}\{\phi(X,Z)\phi(Y,W) - \phi(Y,Z)\phi(X,W) \\
&\qquad + 2\phi(X,Y)\phi(Z,W)\} \\
&- \frac{(3n+2)c^D}{4n^2(n+1)}\{h(Y,Z)h(X,W) - h(X,Z)h(Y,W)\},
\end{aligned}$$

where $\rho^D(X,Y) := S^D(X,JY)$.

Let $\mathcal{F}$ be a hermitian foliation on (M,g) and $(\mathcal{H},J,h)$ be as in (1.1). We say that a diffeomorphism f on M is *transversally conformal* if at each point $x \in M$ the restriction $f_*|_{\mathcal{H}_x} : \mathcal{H}_x \to T_{f(x)}M$ satisfies the following conditions;

(C1) f_* is transverse to $\mathcal{F}$, i.e. $f_*(\mathcal{H}_x) \oplus \mathcal{F}_{f(x)} = T_{f(x)}M$,

(C2) $\tilde{h}_x := f^*h_{f(x)}$ is a conformal change of h_x, i.e. $\tilde{J}_x = J_x$ and $\tilde{h}_x = e^{2\sigma}h_x$ for some real-valued basic function σ on M.

$(\mathcal{H},\tilde{J},\tilde{h})$ defines a hermitian foliation $\tilde{\mathcal{F}}$ on M of the same dimension as $\mathcal{F}$. Indeed, we first note that the leaves of $\tilde{\mathcal{F}}$ are given by the connected components of $f^{-1}(\mathcal{L})$ for $\mathcal{L} \subset \mathcal{F}$. Clearly $L_V\tilde{h} = 0$ for all $V \in \Gamma(\tilde{\mathcal{F}})$. Hence $\tilde{h}$ is a holonomy invariant hermitian metric with respect to $\tilde{J}$. Finally $\tilde{g} := g_{\mathcal{F}} + \tilde{h}$ is a bundle-like riemannian metric on M. We call $\tilde{\mathcal{F}}$ the *conformal change of $\mathcal{F}$ by f*.

REMARK. An example of a transversal conformal mapping is the following. Let $\mathcal{F}$ be a hermitian foliation of codimension two. Then an arbitrary holonomy invariant hermitian metric on $\mathcal{H}^{\mathbf{C}}$ can be locally written as $h = \lambda dz d\bar{z}$, $\lambda > 0$, for a local complex isothermal coordinate system (z). Let $(x_\alpha^1, \ldots, x_\alpha^p)$ be a local coordinate system along the leaves of $\mathcal{F}$. Let $f_{\alpha\beta} : M \to M$ be a local coordinate change defined by $(x_\alpha^i, z) \to (x_\beta^i = x_\alpha^i, w = f_{\alpha\beta}(z))$, which induces a conformal change of h constant along the leaves. Thus by the cocycle condition we have a global diffeomorphism f on M transversally conformal.

By the same arguments of [KMP], we have immediately the following lemmas.

LEMMA 2. *B is invariant for any conformal change of a hermitian foliation $\mathcal{F}$ of codimension $2n \geq 4$.*

LEMMA 3. *Let $\mathcal{F}$ be a kähler foliation of codimension $2n \geq 6$.*

(a) *the transversal Ricci contraction S_B of B is parallel if and only if the transversal Ricci tensor S^D is parellel,*
(b) *B vanishes everywhere if and only if $\mathcal{F}$ is of constant transversal holomorphic sectional curvature.*

PROPOSITION 4. *Let $\mathcal{F}$ be a kähler foliation of codimension $2n \geq 6$. Then B is parallel if and only if $\mathcal{F}$ is transversally symmetric in the sense of Tondeur-Vanhecke ([TV]).*

Proof. By (1.8), a direct computation gives for $X \in \Gamma(\mathcal{H})$

$$R^D{}_{X(JX)X(JX)} = B_{X(JX)X(JX)} - \frac{2}{n} S^D(X,X)h(X,X) + \frac{(3n+4)c^D}{2n^2(n+1)} h(X,X)^2.$$

If B is parallel then by Lemma 3 the transversal Ricci tensor S^D, so the transversal scalar curvature tensor c^D is parallel. Since D is metrical with respect to h, we have $D_X R^D{}_{X(JX)X(JX)} = 0$, and vice verse.

FACT 5 ([TV]). *If $\mathcal{F}$ is a 1-dimensional bundle -like geodesic, transversally symmetric foliation, the ambient space (M, g) is locally homogeous. If moreover (M, g) is complete and simply-connected, it is a naturally reductive homogeneous space.*

2. Proof of Theorem A

Since $\mathcal{F}$ is regular, the leaf space $(M/\mathcal{F}, J, h, \phi)$ is eqipped with a kähler manifold structure such that $\pi := M \to M/\mathcal{F}$ is a locally trivial fibration. If B is parallel then by Proposition 4 and Fact 5, the ambient space (M, g) is a homogeneous space. Thus the leaf space $(M/\mathcal{F}, J, h, \phi)$ is a compact homogeneous kähler manifold. By the O'Neill curvature formula for submersion, the transversal sectional curvature to $\mathcal{F}$ is positive, which implies the positivity of the transversal bisectional curvature $H^D(p, p')$ by the identity;

$$H^D(p,p') = R^D{}_{XYXY} + R^D{}_{X(JY)X(JY)}.$$

It is well-known ([KO]) that a complex n-dimensional compact homogeneous kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{C}P^n$. Thus (a) is proved. The assertion (b) follows from the Lemma 3.

REMARK. If the sectional curvature K_M of M is strictly negative, M can not admit a 1-dimensional kähler foliation. If K_M is nonpositive, M is a local riemannian product (cf. [Ra]).

3. Proof of Theorem B

Let M be a $(2n+1)$-dimensional fibred riemannian space with Sasakian structure (φ, g, ξ, η). For each point in M, there is a local coordinate system $(x, y^1, \dots, y^{2n})$ such that

$$\eta = dx + \sum\nolimits_{a=1}^{n} (-y^{n+a})\, dy^a,$$

and the orbits of ξ are locally given by $y^\alpha = c^\alpha (c^\alpha$ constants). Then M admits a foliation $\mathcal{F}$ generated by the orbits of ξ. Let $\nu_a := \partial/\partial y^a + (y^{n+a})\xi$ and $\nu_{n+a} := \varphi\partial/\partial y^a$. Then $\{\xi, \nu_a, \nu_{n+a}\}$ forms a local basis with the dual basis $\{\eta, dy^a, dy^{n+a} := dy^a \circ \varphi\}$. Clearly $g(\nu_\alpha, \xi) = \eta(\nu_\alpha) = 0$. Since ξ is a Killing vector field, $g := \eta \otimes \eta + g_{\alpha\beta} dy^\alpha dy^\beta$ is a bundle-like riemannian metric. Let $J := \varphi|_{\mathcal{H}}$ and $h := g|_{\mathcal{H}} = g_{\alpha\beta} dy^\alpha dy^\beta$, where $\mathcal{H} \simeq TM/\mathcal{F}$. Then $h(JX, JY) = h(X, Y)$ for $X, Y \in \Gamma(\mathcal{H})$ and $L_\xi J = L_\xi h = 0$. Let ∇^M (resp. D) be the Levi-Civita (resp. transversal Levi-Civita) connection on M (resp. $\mathcal{H}$). By a direct computation we have for $X, Y, Z \in \Gamma(\mathcal{H})$

$$h((D_X J)Y, Z) = \mathcal{H}g((\nabla^M_X \varphi)Y, Z) = \eta(Y)h(X, Z) - h(X, Y)\eta(Z) = 0.$$

Hence $\mathcal{F}$ is a 1-dimensional regular geodesic kähler foliation. Therefore Theorem B follows from Lemma 3.

REMARK. If the transversal bundle $\mathcal{H}$ is integrable then the leaf space $M/\mathcal{F}$ is the base space N and the transversal Levi-Civita connection projects to the Levi-Civita connection ∇^N on N. Moreover, the contact conformal curvature tensor C_0 defined in [JLOP] restricted to $\mathcal{H}$ coincides with the transversal conformal curvature tensor B. Indeed, note that $\phi(X, Y) = d\eta(X, Y) = -\eta[X, Y] = 0$ for $X, Y \in \Gamma(\mathcal{H})$. Since C_0 is constructed by using the method of Boothby-Wang fibration $\pi : M \to N$, the curvature tensor R^M (resp. R^N) on M (resp. N) with respect to ∇^M (resp. ∇^N) satisfies for $X, Y, Z, W \in \Gamma(\mathcal{H})$

$$R^M{}_{XYZW} = R^N{}_{X_* Y_* Z_* W_*} \circ \pi,$$

where $(\ )_* := \pi_*(\ )$. Thus we have $C_{0,XYZW} = B_{XYZW}$.

References

[BB]. P. Baum and R. Bott, *Singularities of holomorphic foliations,* J. Diff. Geom. **7**(1972), 279–342.

[CW]. L. A. Cordero and R. A. Wolak, *Examples of foliations with foliated geometric structures,* Pac. J. of Math. **142** (1990), 265–276.

[JLOP]. J. C. Jeong, J. D. Lee, G. H. Oh and J. S. Pak, *On the contact conformal curvature tensor,* Bull. Korean Math. Soc. **27** (1990), 133–142.

[Ka]. A. El Kacimi-Alaoui, *Examples of foliations and problems in transverse complex analysis,* PUB. IRMA, LILLE, 1988.

[KMP]. H. Kitahara, K. Matsuo and J. S. Pak, *A conformal curvature tensor field on hermitian manifolds,* J. Korean Math. Soc. **27**(1990), 7-17.

[KO]. S. Kobayashi and T. Ochiai, *Complex homogeneous kähler manifolds with positive tangent bundle,* Differential geometry in honor of K. Yano, Kinokuniya, Tokyo, 1972, 221–232.

[NT]. S. Nishikawa and Ph. Tondeur, *Transversal infinitesimal automorphisms for harmonic kähler foliations,* Tohoku Math. J. **40** (1988), 465–471.

[Ra]. A. Ranjan, *Structural equations and integral formula for foliated manifolds,* Geom. Dedicata **20** (1986), 85–91.

[TV]. Ph. Tondeur and L. Vanhecke, *Transversally symmetric riemannian foliations,* Tohoku Math. J. **42** (1990), 307–317.
[Va]. I. Vaisman, *From the geometry of hermitian foliate manifolds,* Bull. Math. Soc. Sci. Math. R. S. Roumanie **17** (1973), 71–100.

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