Nonlinear Aspects of the Frequency Response of a Gas-filled Bubble Oscillator

기포진동 주파수응답의 비선형적 현상

Dong Hyuk Kim,* Jeung Tae Kim,* Yong Bong Lee*

김 동 혁,* 김 정 태,* 이 용 봉*

ABSTRACT

A numerical analysis is carried out for the nonlinear phenomena of the bubble oscillator. The model is based on the Keller's formulation for the bubble dynamics. Interpretation of the bubble interior is based on the formulation by Prosperetti. His formulation adopts the energy equation for the analysis of the bubble interior. The numerical simulation Shows typical nonlinear phenomena in its frequency response. Among such nonlinear aspects are the jump phenomenon, the shift of natural frequency of the system, and the appearance of superharmonic resonances. It is deduced that the nonlinear frequency response is dependent upon the initial condition of the bubble oscillator and some multi-valued frequency region can appear in the response curve. Nonlinear phenomena appeared in the bubble oscillator is compared with those of the Duffing equation and it may be said that the bubble dynamic equation has similar nonlinear aspects to the Duffing equation.

요 약

기포 진동 시스템에 대한 수치해석이 수행되었다. 수학적 모델은 기포역학에 대해서는 Keller의 식을, 기포내부 해석을 위해서는 Prosperetti의 식을 채택하였다. Prosperetti는 기포내부 해석을 위해 에너지 방정식을 도입하였으며 매우 정확한 해석을 가능케 하였다. 수치해석결과 기포진동의 주파수 응답곡선에 있어 전형적인 비선형 현상들을 볼 수 있었다. 이러한 비선형 현상들에는 점프현상 (jump phenomena), 공진주파수의 변화, 그리고 superharmonic 공진점의 발생등이 있다.

비선형 주파수 응답은 기포진동 시스템의 초기조건에 따라 달라지는데 이에 의해 어느 가진 주파수 대역에서는 두 개 이상의 해가 존재할 수 있게 된다.

기포진동 시스템에서 발생하는 비선형 진동현상은 Duffing 방정식과 비교가 되는데 두 시스템은 비슷한 비선형 현상들을 가지고 있다고 볼 수 있다.

*Acoustics and Vibration Laboratory Korea Standards Research Institute

Ⅰ. Introduction

It is Rayleigh that firstly made attempt to analyze the problem in cavitation and the bubble dynamics. He assumed that the gas filling the bubble undergoes isothermal compression. Neglecting surface tension and liquid viscosity and assuming liquid incompressibility, he established the famous "Raleigh equation".

$$R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{p(R)-p_\infty}{\rho_L} \tag{1}$$

Here, ρ_L is the liquid density, p_∞ is the pressure in the liquid at a large distance from the bubble, and $p(R)$ is the pressure in the liquid at the bubble surface. But Rayleigh's formulation is applicable only for small amplitude oscillation of a bubble because of the isothermal assumption. For large amplitude of bubble oscillation, internal temperature and pressure of a bubble change so much that the isothermal assumption fails. These internal quantities are very importnat to determine the dynamics of a gas bubble. Although these quantities should be determined based on the solution of the conservation equation of continuum mechanics inside and outside the bubble joined togerther by suitable boundary conditions, it was customary to use a polytropic relation of the form

$$p=p_0(R_0/R)^{3\kappa} \tag{2}$$

where κ is the polytropic index and subscript zero indicates equilibrium values. This polytropic relation was firstly used by Minneart[1]. Since then most researchers such as Noltingk and Neppiras[2], Flynn[3], Apfel[4], and Lauterborn[5] have used the relation.

Although this relation is simple to use, it has many problems due to its neglecting thermal effect in the bubble which is very important one in the large amplitude oscillation of a gas bubble. The polytropic index can be from 1(isothermal) to the ratio of specific heats γ(adiabatic). But it is extremely difficult to find an appropriate criteria for the proper choice of value for large amplitude oscillation of a bubble.

An accurate mathematical formulation was proposed by Prosperetti[6] for calculation of high amplitude bubble oscillation. He intoduced the energy equation to dectermine the interior quantities of a bubble in addition to the conservation of mass and momentum.

This paper investigates some nonlinear characteristics of a single bubble oscillating with large amplitude based on the formulation by Prosperetti. Of many nonlinear phenomena, the frequency response is of special interest in this paper. Frequency response for different amplitude of exciting pressure are calculated and the hysteresis effect, the jump phenomenon, the change of natural frequency and the appearance of subharmonics and superharmonics are examined.

Ⅱ. Mathematical Formulation

The bubble dynamic equation proposed by Rayleigh in 1917 did not consider liquid viscosity and surface tension. When the compressibility of the liquid is the only thing to be neglected, the motion of the bubble boundary is governed by the well known Rayleigh-Plesset equation,

$$\rho_L(R\ddot{R}+\frac{3}{2}\dot{R}^2)=p-p_\infty-p_s(t)-\frac{2\sigma}{R}-4\mu_L\frac{\dot{R}}{R} \tag{3}$$

where $p_s(t)$ is a nonconstant ambient pressure component such as a sound field. The surface tension and viscosity are denoted by σ and μ_L, respectively.

This equation does not consider the liquid compressibility. Keller and coworkers formulated a new equation which included the compressibility effects is

$$(1-\frac{\dot{R}}{c})R\ddot{R}+\frac{3}{2}\dot{R}^2(1-\frac{\dot{R}}{3c})$$
$$=(1+\frac{\dot{R}}{c})\frac{1}{\rho_L}[p_B(t)-p_s(t+\frac{R}{C})-p_\infty]+\frac{R}{\rho_L c}\frac{dp_B(t)}{dt}, \qquad (4)$$

where c is the speed of sound in the liquid and $p_B(t)$ is the liquid pressure on the external side of the bubble wall. This pressure $p_B(t)$ is related to the internal bubble pressure $p(t)$ by

$$p(t)=p_B(R,t)+\frac{2\sigma}{R}+4\mu_L\frac{\dot{R}}{R} \qquad (5)$$

Since the Keller's formulation is the most complete one, it is used as the basis of our numerical model to examine the nonlinear frequency response of the bubble oscillation. As can be seen in Eq. (5), the form of the equation has high nonlinearity. Up to now, it seems to be impossible to solve the nonlinear equation analytically. The only thing we can do is to linearize the equation for small amplitude of exciting pressure. Moreover, the pressure at the bubble surface, $p_B(t)$, is closely related to the internal pressure which creates a very complex nonlinear system. Hence the first step is describing the bubble interior accurately. Prosperetti introduced the energy equation in order to interpret the bubble interior. For small amplitude of pressure disturbance, the temperature effects are negligible. However the temperature effects such as conduction become more important as the exciting pressure amplitude becomes stronger. So it is inevitable to use Prosperetti's formulation to govern the bubble interior for large amplitude of bubble oscillation. The governing equations for bubble interior are

$$\frac{d\rho_G}{dt}+\rho_G\nabla\cdot v \;=\; 0 \qquad (6)$$

$$\rho_G\frac{dv}{dt}+\frac{\partial p}{\partial r} \;=\; 0 \qquad (7)$$

$$\rho_G C_p\frac{dT}{dt}+\frac{T}{\rho_G}\left(\frac{\partial\rho_G}{\partial T}\right)_p\frac{\partial p}{\partial t} \;=\; \nabla\cdot(K\nabla T), \qquad (8)$$

whrer v, C_p, T denotes the velocity field in the bubble, specific heat of the gas and the temperature field in the bubble, respectively.

Along with the equations from(6) to (8), we need boundary conditions at the bubble surface. The correct boundary conditions are continuity of temperature and heat flux. From the continuity of heat flux, it is easily deduced that the temperature variation of the surrounding liquid is negligible compared to that inside the bubble. The continuity of temperature gives

$$T(R,t)=T_\infty \qquad (9)$$

The dynamics of a bubble can be solved by the numerical method based on th equation (4) to (9). For more details, it is recommended to refer to the paper by Prosperetti[6].

In the dynamic equation(4), $p_s(t)$ denotes the fluctuating pressure component on the ambient pressure. It is customary to set $p_s(t)$ using harmonic function as

$$p_s(t)=\varepsilon\cos\omega t, \qquad (10)$$

where ω is the angular frequency, ε the amplitude of pressure fluctuation.

There are many ways of indicating the level of bubble oscillation. One is to take the ratio of the maximum radius of the bubble to the minimum during an oscillating cycle in the steady-state. Another one is to take the pressure instead of the radius of a bubble. In this paper, it is adopted to take the quantity of work done during a cycle. The method is based on the fact that the work done by a bubble to the surrounding fluid is proportional to the level of bubble oscillation. The work done by a bubble during a cycle can be expressed as

$$\mathcal{P} = \frac{\omega}{2\pi}\int_{t}^{t+2\pi/\omega} 4\pi R^2 \dot{R} p_s(t)dt. \tag{11}$$

The integration is done by using a numerical method based on the quadrature formula.

The next step is to nondimensionalize the above equations which govern the dynamics of a bubble motion after nondimensionalization, a predictor-corrector method is adapteu to solve the system of partial differential equations. The method is best fitted for highly nonlinear equations such as those related to the bubble dynamics. Details of these manipulations can be found in reference[6].

Ⅲ. Characteristics of nonlinear equations

In general, it is impossible to find the solution of a nonlinear equation analytically. The solutions can be obtained for some special cases. Among these are the "Van der Pole" equation and "Duffing" equation[7]. Of these two, the Duffing equation has similar characteristics as the bubble dynamic equation. The Duffing equation can be mathematically expressed as

$$\ddot{u} + \omega_0^2 u = \epsilon f(u, \dot{u}) + E, \tag{12}$$

whrer $\epsilon \ll 1$ and E is an externally applied, generalized force called the excitation. Although the Duffing equation and the Rayleigh-Plesset equation are different in their style, the frequency responses of these two equations have many similar aspects. The solution of Eq. (12) has the following form

$$u = a\cos(\Omega t - \gamma) + o(\epsilon). \tag{13}$$

Here, a and γ are constants that depend on the amplitude and frequency of the excitation and the initial conditions. The state plane for this equation is shown in Fig. 1. As can be seen in Fig. 1, there exist three solutions. The trajectories show how the reponse progresses toward a steady state solution from any initial conditions. For all the initial conditions lying in the shaded area, the high-amplitude steady state will develop, while for all the initial conditions lying in the unshaded area, the low-amplitude steady state will develop. We note that the two inward bound separatrices for the saddle point, P_2 separate the domains of attraction for the stable steady states. This means that the

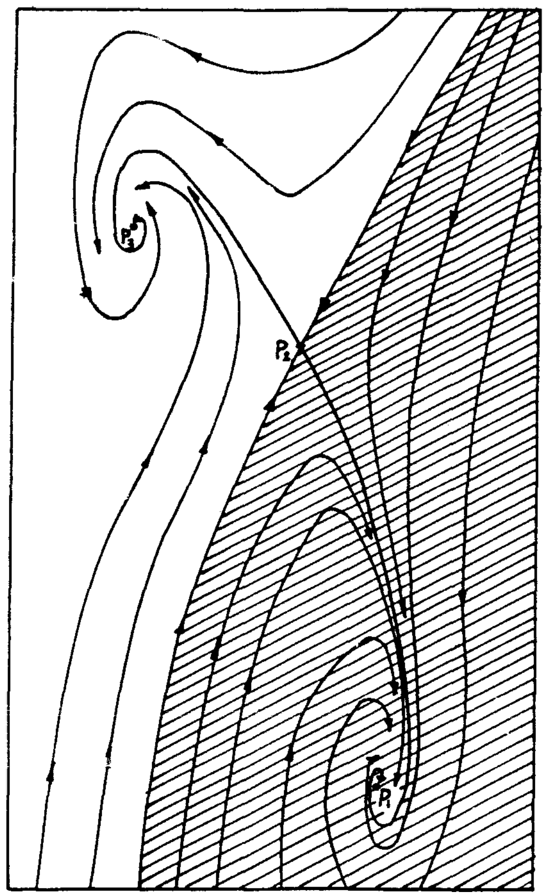

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Fig. 1. State-plane for the Duffing equation when three steady-state solution exist : P_1 is the upperbranch stable focus, P_2 is the saddle point, and P_3 is the lowerbranch stable focus. (from Nayfeh & Mook)

steady state solution can depend on the initial conditions, which is typical in nonlinear systems.

Other nonlinear characteristics of the frequency response of the Duffing equation are as follows.

- The excitation changes the natural frequency of the system. The natural frequency increases for hardening spring system and decreases for softening spring system.
- There appear many subharmonics and superharmonics in the frequency response curve and the magnitude of the harmonics gets bigger as the excitation gets stronger.

Ⅳ. Numerical solution of the bubble oscillator and discussions

As explained in the previous section, the initial conditions are crucial to reach a solution for non-linear systems. In the frequency response of a nonlinear system, there mav exist multiple solutions in some frequency band. The solution depends on the initial conditions.

To verify these characteristics in the bubble oscillator, some numerical simulations are conducted. The exciting frequency is slowly increased or decreased so that the steady-state oscillation of a bubble is reached. When the system reaches to the steady-state the value of the work done by the oscillation is taken. The frequecny and the work done are dimensionless valves. Here the absolute value of the work done does not have any meaning. Rather, it may be understood as relative value. Dimensionless frequency is defined as the exciting frequency divided by the natural frequency of the bubble oscillator. The frequency increment is set to 0.01, which is small enough to observe the nonlinear phenomena of the frequency response in the bubble oscillator. As the exciting frequency moves to another frequecny by the increment 0.01, the final condition at the

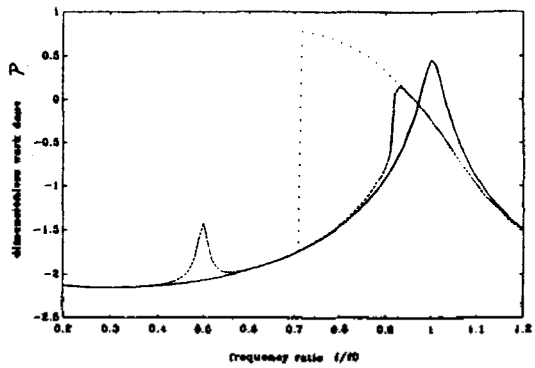

Fig. 2. Frequency response curves of the excited bubble of 1 mm radius. The dimensionless amplitude of the exciation ε is 0.1. The solid line represents linearized solution.

previous exciting frequency becomes the initial condition of the next one. It gives unique "path-dependent" solutions in the frequency response curve. Fig. 2 shows the frequecny response curves of the bubble oscillator with its radius of 1 mm. The amplitude of excitation is 0.1, which is dimensionless value defined as the pressure divided by ambient pressure P. The solid line represents the linear solution. The thick dotted-line is the sloution which is obtained by solwly increasing the exciting frequency and the thin dotted-line the solution by slowly decreasing the exciting frequency. From the figure it is easily seen that the natural frequency is changed to a lower frequency either for "increasing" case or "decreasing" case. In the light of the Duffing equation, we can say that the bubble oscillator is a "softening" system as the natural frequency changes to a lower one. The values of work done for "increasing" case vary slowly as frequency goes up and show a superharmonic resonance around $f/f_0=$ 0.5. And it makes a jump around $f/f_0=0.9$. This jump phenomenon is typical to the nonlinear oscillator system as bubble. The jump phenomenon also occurs for "decreasing" case at a lower frequency than for the "increasing" case so that there exist a multi valued region in the frequecny range between 0.7 and 0.9. In this frequency range, we [illegible] conditions. These two solutions are stable foci in the state plane, which is the same as the case of the Duffing equation. Another solution exists between the two solutions and that is the saddle point in the state-plane. However this solution is so unstable that it is almost impossible to find the exact value by the numerical method. It is because the initial conditions for that solution is extremely difficult to find.

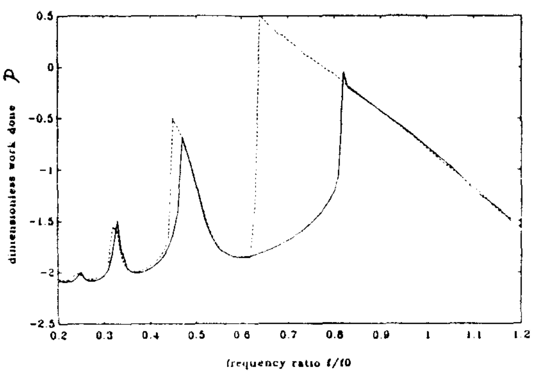

Fig. 3. Frequency reponse curves of the excited bubble of 1mm radius. The dimensionless amplitude of the excitation ϵ is 0.3.

Fig. 3 shows the frequency response curve for the increased exciting pressure amplitude. The amplitude has been changed from 0.1 to 0.3 and the result shows higher nonlinearity in the response curve. The solid line denotes the "increasing" case and the dotted line the "decreasing" case. It is observed that more superharmonic resonances occur at the dimensionless frequencies of 0.25, 0.33, and around 0.5. There appear another regions of multi-valued solutions. And the multi-valued regions become wider than for $\epsilon=0.1$ case. All these aspects can be explained by "higher nonlinearity".

The next example is for the bubble radius of 0.05 mm with the exciting amplitude of $\epsilon=0.1$. The results are displayed in Fig. 4. Still we have small superharmonic resonance at $f/f_0=0.5$ but its magnitude becomes very small compared with the case for 1 mm radius. And the "increasing" and "decreasing" cases do not differ so much. Based on the result, it may be deduced that the smaller bubble has less nonlinearity in its frequency response, because the smaller bubble has higher stiffness than the larger one.

Final example, Fig.5, shows the nonlinear frequency response for $\epsilon=0.3$ with the bubble radius of 0.05 mm. Although the smaller bubble has less nonlinearity, it still gives highly nonlinear pheno

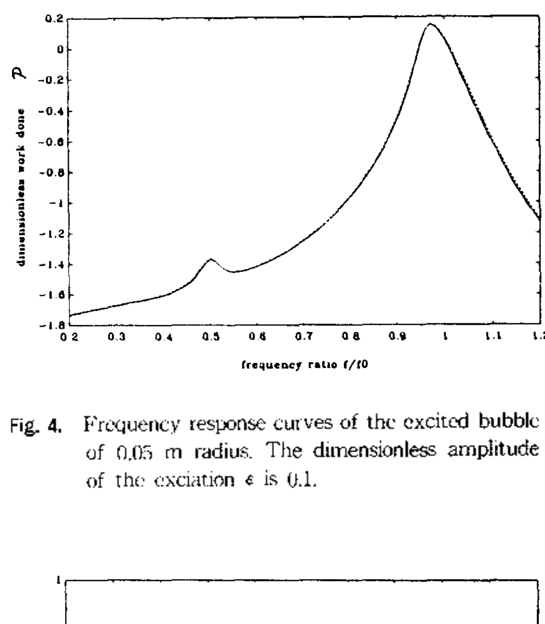

Fig. 4. Frequency response curves of the excited bubble of 0.05 m radius. The dimensionless amplitude of the exciation ϵ is 0.1.

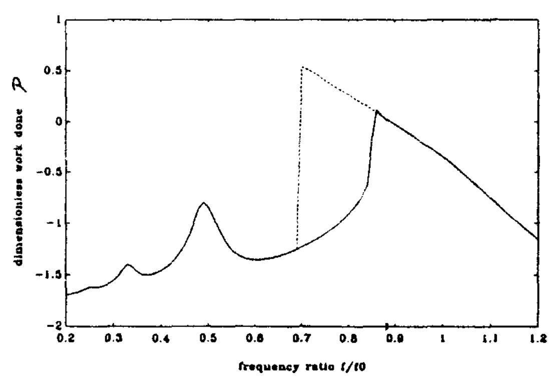

Fig. 5. Frequency response curves of the excited bubble of 0.05 mm radius. The dimensionless amplitude of the excitation ϵ is 0.3.

ments for increased pressure amplitude. In Fig. 5, we can clearly observe the superharmonic resonances, the jump phenomena, and the multi-valued regions.

\ . Conclusions

The bubble oscillator is a very complex system. A bubble model by Keller and Prosperetti is adapted to solve the nonlinear oscillation of a bubble. This formulation leads to accurate results since the energy equation is introduced to the governing equations for the bubble interior. Numerical simulations are conducted to get the frequency response curve of the excited bubble. The numerical results show some interesting nonlinear phenomena for the bubble oscillator. The excitation changes the natural frequency of the bubble and makes some superharmonic resonances at $f/f_0=1/2, 1/3, \cdots$.

It is found that there exist multi-valued regions in the frequency response curve due to the effect of the initial conditions. In addition, the frequency response curve jumps up or down at a certain frequecny. The smaller bubble has less nonlinearity than the bigger one and this can be explained by the stiffness of the bubble. .

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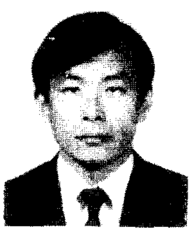
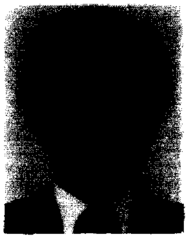


▲Dong-Hyuk Kim
- 1976 : Seoul National University (B.S in Mechanical Eng.)
- 1984 : New Jersey Institute of Technology (M.S.M.E)
- 1988 : The Johns Hopkins University (Ph.D)
- 1976~1982 : Agency for Defense Development
- 1983~Present : Korea Standards Research Institute (Acoustics & Vibration Lab.)

▲Jeung-Tae Kim
- 1976 : Seoul National University (B.S)
- 1978 : Korea Advanced Institute of Science(M.S)
- 1987 : M.I.T. (Ph. D)
- 1981-1986 : Lesturer, Han-Yang University
- 1987-1989 : Research Staff, General Electric Company, U.S.A.
- 1989~Present : Senior Researcher, Acoustics and Vibration Lab, Korea Standards Research Institute.



▲Yong-Bong Lee

- 1982 : Inha University (B.S)
- 1987 : Korea Advanced Institute of Science and Technology (M.S)
- 1987 – 1990 : Researcher, R & D Center, Korea Machinery Company
- 1990 – Present : Researcher, Acoustics and Vibration Lab, Korea Standards Research Institute.