FINITELY GENERATED MODULES

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Introduction

In this paper, unless otherwise indicated, we shall not assume that our rings are commutative, but we shall always assume that every ring has an identity element. By a module, we shall always mean a unitary left module.

We provide a characterization of non-zero finitely generated Noetherian modules, some properties of finitely generated Noetherian and Artinian modules, and the localization $E_S$ of a finitely generated module $E$ over a commutative ring $R$ with respect to a multiplicatively closed subset $S$ of $R$ not containing 0.

Finally, this paper deals with aspects of the identification of the maximal submodules of a finitely generated module over a commutative ring $R$. It shows an analogy between this set of submodules and the spectrum of $R$.

1. Finitely generated Noetherian and Artinian modules

The Cohen theorem [C50] says that if every prime ideal in a commutative ring $R$ is finitely generated, then $R$ is Noetherian. We first generalize this result.

Let $E$ be an $R$-module. A submodule $M$ of $E$ is said to be a maximal submodule of $E$ if (i) $M$ is a proper submodule of $E$ and (ii) there is no proper submodule of $E$ strictly containing $M$.

It is well known [SV72] that every non-zero finitely generated $R$-module possesses a maximal submodule.

DEFINITION. Let $E$ be an $R$-module. Then a submodule $P$ of $E$ is said to be a prime submodule of $E$ if (a) $P$ is proper and (b) whenever $re \in P$ ($r \in R$, $e \in E$), then either $e \in P$ or $rE \subseteq P$.

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LEMMA 1. Let $R$ be a commutative ring and $A$ a simple $R$-module. Then every zero-divisor on $A$ is an annihilator of $A$.

Proof. Let $r$ be an arbitrary zero-divisor on $A$. Then there exists $e \in A$, $e \neq 0$ such that $re = 0$. Since $A$ is a simple $R$-module, the submodule of $A$ generated by $e$ must be $A$ itself. Hence

$$rA = r(Re) = (rR)e = (Rr)e = R(re) = 0$$

and so $r$ is an annihilator of $A$.

PROPOSITION 2. Let $R$ be a commutative ring and $A$ an $R$-module. Then every maximal submodule of $A$ is prime.

Proof. Let $M$ be an arbitrary maximal submodule of $A$. Then $M$ is proper. Replacing $A$ by $A/M$, we can assume that $A$ is a simple $R$-module and $M = 0$. It suffices to show that every zero-divisor on $A$ is an annihilator of $A$. But, this follows from Lemma 1.

Note that for every $R$-module $E$, the annihilator, denoted by $\text{Ann}_RE$, of $E$ is a two-sided ideal of $R$. Let $E$ be an $R$-module and $P$ a submodule of $E$. Let $p$ denote $\text{Ann}_R(E/P)$. Then if $P$ is prime, then $p$ is a prime ideal of $R$ by the definitions. Further, if $P$ is maximal, then $p$ is a maximal ideal of $R$. In fact, $E/P$ is a simple $R$-module and is $R$-isomorphic to $R/m$ for some maximal left ideal $m$ of $R$. This implies that

$$p = \text{Ann}_R(E/P) = \text{Ann}_R(R/m) = m,$$

which becomes a maximal ideal of $R$. Hence we have the following result.

PROPOSITION 3. Let $E$ be an $R$-module and $P$ a submodule of $A$. Let $p$ denote $\text{Ann}_R(E/P)$. Then:

(i) $P$ is prime if and only if the factor module $E/P$, as an $R/p$-module, is torsion-free;

(ii) If $P$ is maximal, then $E/P$, as an $R/p$-module, is divisible. In particular, when $E$ is cyclic over a commutative ring $R$, $P$ is maximal if and only if $R/p$ is a field.

If $N$ is an $R$-submodule of an $R$-module $E$ and $a$ an ideal of $R$, we define $N : E a$ to be the $R$-submodule of $E$ consisting of all $x \in E$ such that $ax \subseteq N$. 
LEMMA 4. Let $N$ be a submodule of an $R$-module $E$ and $r$ an element of $R$. If $N + rE$ and $N :_E rR$ are finitely generated, then $N$ is also finitely generated.

Proof. Adapt the proof of [N62, (3.3), p.8].

It is well known [N62] that every finitely generated module over a Noetherian ring is a Noetherian module. Note that any submodule of a finitely generated module is not necessarily finitely generated. The following result is a characterization of finitely generated Noetherian modules and is also a generalization of the Cohen theorem.

THEOREM 5. A non-zero finitely generated $R$-module $E$ is Noetherian if and only if every prime submodule of $E$ is finitely generated.

Proof. The only if part is a consequence of [N62, (3.1), p.7]. Use Zorn's lemma and Lemma 4 to prove the if part (cf. [N62, (3.4), p.8]).

The Formanek theorem [F73] says that if $R$ is a commutative ring and $M = Rm_1 + \cdots + Rm_k$ is a faithful finitely generated $R$-module which satisfies the ascending chain condition (ACC) on "extended submodules" $IM$, where $I$ is an ideal in $R$, then $M$ is a Noetherian $R$-module and hence $R$ is a Noetherian ring. In the remainder of this section we discuss under what conditions 'faithful' can be replaced.

LEMMA 6. Let $M = Rm_1 + \cdots + Rm_k$ be a finitely generated module over a commutative ring $R$. Suppose that, for every ideal $I$ in $R$ and for each $i$, $Im_i$ is equal to $Rm_i \cap IM$. Then $M$ satisfies ACC (resp. DCC) on extended submodules if and only if each $Rm_i$, satisfies ACC (resp. DCC) on extended submodules.

Proof. We consider only the ACC case since the proof of the DCC case is similar.

Assume that $M$ satisfies ACC on extended submodules. We show only the case of $i = 1$ since the proof of the other case is similar. Consider the ascending chain

$$I_1m_1 \subseteq I_2m_1 \subseteq I_3m_1 \subseteq \cdots$$

of extended submodules of $Rm_1$. This gives an ascending chain

$$\text{Ann}_R(Rm_1/I_1m_1) \subseteq \text{Ann}_R(Rm_1/I_2m_1) \subseteq \text{Ann}_R(Rm_1/I_3m_1) \subseteq \cdots$$
of ideals of $R$. But, each $\text{Ann}_R(Rm_1/I_m1)$ is equal to the sum $I_i + \text{Ann}_R(m_1)$. We get an ascending chain

$$(I_1 + \text{Ann}_Rm_1)M \subseteq (I_2 + \text{Ann}_Rm_1)M \subseteq (I_3 + \text{Ann}_Rm_1)M \subseteq \cdots$$

of extended submodules of $M$. By our assumption, there exists a positive integer $s$ such that $(I_n + \text{Ann}_Rm_1)M = (I_s + \text{Ann}_Rm_1)M$ for all $n \geq s$. Thus,

$$I_nm_1 = Rm_1 \cap I_nM$$
$$\subseteq Rm_1 \cap (I_n + \text{Ann}_Rm_1)M$$
$$= Rm_1 \cap (I_s + \text{Ann}_Rm_1)M$$
$$= (I_s + \text{Ann}_Rm_1)m_1$$
$$= I_sm_1$$

for all $n \geq s$. Also, it is clear that $I_nm_1 \supseteq I_sm_1$ for all $n \geq s$. Therefore the given ascending chain terminates.

Conversely, assume that each $Rm_i$ satisfies ACC on extended submodules. Consider the ascending chain

$$I_1M \subseteq I_2M \subseteq I_3M \subseteq \cdots$$

of extended submodules of $M$. This gives ascending chains

$$Rm_i \cap I_1M \subseteq Rm_i \cap I_2M \subseteq Rm_i \cap I_3M \subseteq \cdots, \quad i = 1, \ldots, k.$$

But, $Rm_i \cap I_nM = I_nm_i$ for each $1 \leq i \leq k$ and for each $n \geq 1$. By our assumption, for each $1 \leq i \leq k$ there exists a positive integer $s_i$ such that $I_nm_i = I_sm_i$ for all $n \geq s_i$. Take $s = \max\{s_1, s_2, \ldots, s_k\}$. Then $I_nm_i = I_sm_i$ for all $n \geq s$. Hence

$$I_nM = I_nm_1 + I_nm_2 + \cdots + I_nm_k = I_sm_1 + I_sm_2 + \cdots + I_sm_k = I_sM$$

for all $n \geq s$. Thus, the given ascending chain terminates.
Theorem 7. Let $M$ be as in Lemma 6. If, for every ideal $I$ in $R$ and for each $i$, $Im_i$ is equal to $Rm_i \cap IM$, and $M$ satisfies ACC (resp. DCC) on extended submodules, then $M$ is a Noetherian (resp. Artinian) $R$-module and hence $R/\text{Ann}_R M$ is a Noetherian (resp. Artinian) ring.

Proof. We consider only the Noetherian case, since the proof of the Artinian case is similar. By our hypothesis and Lemma 6 each $Rm_i$ in $M$ satisfies ACC on extended submodules. Since each $Rm_i$ is $R$-isomorphic to $R/\text{Ann}_R m_i$, it is Noetherian. By [SV72, Proposition 1.18, p.18] $Rm_1 \oplus \cdots \oplus Rm_k$ is Noetherian. Define a mapping $f : Rm_1 \oplus \cdots \oplus Rm_k \to Rm_1 + \cdots + Rm_k$ by $f(r_1 m_1, \ldots , r_k m_k) = r_1 m_1 + \cdots + r_k m_k$, where $r_i \in R$. Then $f$ is an epimorphism. This gives an exact sequence

$$0 \to \text{Ker} f \to Rm_1 \oplus \cdots \oplus Rm_k \to M \to 0$$

of $R$-modules. Hence $M$ is Noetherian.

Now define a mapping $g : R \to Rm_1 \oplus \cdots \oplus Rm_k$ by $g(r) = (rm_1, \ldots , rm_k)$, where $r \in R$. Then $g$ is an $R$-homomorphism and $\text{Ker} g = \text{Ann}_R M$. So, $R/\text{Ann}_R M$ can be regarded as an $R$-submodule of $Rm_1 \oplus \cdots \oplus Rm_k$. Hence since $Rm_1 \oplus \cdots \oplus Rm_k$ is Noetherian, so is $R/\text{Ann}_R M$.

Proposition 8. Let $R$ be a commutative domain and $M$ as in Lemma 6. If $M \neq 0$ is divisible, then $M$ is faithful. Moreover, the converse holds if $M$ is simple.

Proof. $M$ is faithful if and only if for each non-zero $r \in R$ there is at least one of $m_1, \ldots , m_k$ (depending on $r$) such that $rm_i \neq 0$. This latter property is inductive. The remainder of the proof is obvious.

It is well known [SV72, Proposition 2.6, p.33] that every injective module is divisible. Of course, every torsion-free divisible module over a commutative domain is injective [SV72, Proposition 2.7, p.34]. Hence the following proposition follows from Proposition 8 and the Formanek theorem.

Proposition 9. Let $R$ be a commutative domain and $M$ as in Lemma 6. If $M \neq 0$ is injective and satisfies ACC on extended submodules, then $M$ is a Noetherian $R$-module and hence $R$ is a Noetherian domain.
2. Localization

In this section we discuss the localization $E_S$ of a finitely generated module $E$ over a commutative ring $R$ with respect to a multiplicatively closed subset $S$ of $R$ not containing $0$. Specifically, if $P$ is a prime submodule of $E$, then we will take $S = R \setminus \text{Ann}_R(E/P)$ and consider the corresponding localization of $E$.

**Proposition 10.** Let $E$ be a non-zero finitely generated module over the commutative ring $R$, $S$ a multiplicatively closed subset of $R$ not containing $0$, and $P$ a prime $R$-submodule of $E$. Let $p$ denote $\text{Ann}_R(E/P)$. Then:

(i) when $S \cap p$ is non-empty, then $P \otimes_R R_S = E \otimes_R R_S$;

(ii) when $S \cap p$ is empty, then $P \otimes_R R_S$ is a prime $R_S$-submodule of the finitely generated $R_S$-module $E \otimes_R R_S$.

Hence there is a one-to-one order-preserving correspondence between the prime $R_S$-submodules of $E \otimes_R R_S$ and the prime $R$-submodules $Q$ of $E$ such that $S \cap \text{Ann}_R(E/Q)$ is empty.

**Proof.** Note that $E \otimes_R R_S = E_S$ and $P \otimes_R R_S = P_S$. Let $E = R e_1 + \cdots + R e_n$. Then

(i) when $s \in S \cap p$, then, for $1 \leq i \leq n$, $e_i/1 = se_i/s \in P_S$; hence $P_S = E_S$.

(ii) Since $E$ is finitely generated over $R$, $E_S$ is finitely generated over $R_S$.

Assume $S \cap p$ is empty. Then $E_S$ is non-zero. For, if not, then $e_i/1 = 0$ for $1 \leq i \leq n$; hence there exists $\sigma$ in $S$ such that $\sigma e_i = 0$, which belongs to $P$ and so $\sigma \in p$, a contradiction. Clearly $P_S \neq E_S$.

Now let

$$(a/s)(e/t) \in P_S, \quad e/t \notin P_S,$$

where $a \in R, s, t \in S$, and $e \in E$. Then $ae/st = p/u$ for some $u \in S$ and $p \in P$; hence there is $\sigma$ in $S$ such that $\sigma((ua)e - (st)p) = 0$, which implies $(\sigma u a)e \in P$. Hence since $e \notin P$ and $P$ is prime in $E$, we have $\sigma u a \notin p$. Since $\sigma u \notin p$ we have $a \in p$. This implies $a/s \in pR_S$. It is well known [N76, p.41] that $\text{Ann}_R(E/P)R_S = \text{Ann}_{R_S}(E_S/P_S)$. Thus $a/s \in \text{Ann}_{R_S}(E_S/P_S)$. Therefore every zero-divisor on $E_S/P_S$ is an annihilator of $E_S/P_S$.
COROLLARY. Let $E$ be a non-zero finitely generated module over a commutative ring and $P$ a prime $R$-submodule of $E$. Let $\mathfrak{p}$ denote $\text{Ann}_R(E/P)$. Then the prime $R_{\mathfrak{p}}$-submodules of the non-zero finitely generated $R_{\mathfrak{p}}$-module $E \otimes_R R_{\mathfrak{p}}$ are in one-to-one order-preserving correspondence with the prime $R$-submodules $Q$ of $E$ such that $\text{Ann}_R(E/Q) \subseteq \mathfrak{p}$.

3. Spectra of finitely generated modules

This section deals with aspects of the identification of the maximal submodules of a finitely generated module over a commutative ring $R$. It shows an analogy between this set of submodules and the spectrum of $R$.

If $E$ is an $R$-module, then the radical of $E$, denoted by $J(E)$, is defined to be the intersection of all maximal submodules of $E$, that is,

$$J(E) = \bigcap_{M \in \Omega_E} M,$$

where $\Omega_E$ is the collection of all maximal submodules of $E$. From now on we call $\Omega_E$ the maximal spectrum of $E$.

The following proposition is concerned with a relation between the radical $J(E)$ of a finitely generated $R$-module $E$ and the Jacobson radical $J(R)$ of $R$.

PROPOSITION 11. Let $E$ be a non-zero finitely generated $R$-module and $\mathfrak{a}$ an ideal of $R$ contained in the Jacobson radical $J(R)$ of $R$. Then $\mathfrak{a} \subseteq \text{Ann}_R(E/J(E))$. In particular, $J(R) \subseteq \text{Ann}_R(E/J(E))$.

Proof. Let $\Omega_E$ denote the maximal spectrum of $E$. Then $\Omega_E$ is non-empty. For every $M$ in $\Omega_E$, $\text{Ann}_R(E/M)$ is a maximal ideal of $R$. Hence by hypothesis

$$(3.1) \quad \mathfrak{a} \subseteq \bigcap_{M \in \Omega_E} \text{Ann}_R(E/M).$$

Moreover, it is easy to show that

$$\bigcap_{M \in \Omega_E} \text{Ann}_R(E/M) = \text{Ann}_R(E/J(E)).$$
Therefore the proof is complete.

Note that (3.1) can also be proved by using Nakayama's lemma [AM69, Proposition 2.6, p.21].

Following the most general definition [CE56, p.147, M58, p.516, and SV72, p.63] we will call a ring $R$ a quasi-local ring if the set of non-units of $R$ forms a two-sided ideal, or equivalently, if $R$ has only one maximal two-sided ideal.

**Definition.** An $R$-module $M$ is said to be a quasi-local module if $M$ has the equivalent properties:

(a) $M$ has a unique maximal submodule;

(b) $M/J(M)$ is simple.

Let $E$ be a finitely generated $R$-module. Then the spectrum of $E$, denoted by $\text{Spec}_R(E)$, is defined to be the collection of all prime $R$-submodules of $E$. Thus, for any ring $R$, $\text{Spec}_R(R)$ is the ordinary spectrum of $R$. Let $\bar{R}$ denote $R/\text{Ann}_RE$ and define a mapping $f : \text{Spec}_R(E) \to \text{Spec}_R(\bar{R})$ by $f(P) = \text{Ann}_R(E/P)$, where $P \in \text{Spec}_R(E)$. Then $f$ is surjective. In fact, for any prime ideal $\mathfrak{p}$ of $R$ containing $\text{Ann}_R(E)$, $\mathfrak{p}E$ is a prime $R$-submodule of $E$ and $\text{Ann}_R(E/pE) = \mathfrak{p}$. The image of its restriction $f|_{\Omega_E} : \Omega_E \to \text{Spec}_R(\bar{R})$ to the maximal spectrum $\Omega_E$ of $E$ is $\Omega_R$. The mapping $f$ is not always injective since $f|_{\Omega_E}$ is not. The example is given as follows:

**Example.** Note that the ring $\mathbb{Z}$ of integers is a faithful $\mathbb{Z}$-module. Consider the ring $\mathbb{Z}[i]$ of Gaussian integers, where $i = \sqrt{-1}$. Then since $i$ is integral over $\mathbb{Z}$, $\mathbb{Z}[i]$ is a finitely generated $\mathbb{Z}$-module. It is trivial that, for any prime number $p$ of $\mathbb{Z}$, $p\mathbb{Z} + i\mathbb{Z}$ and $\mathbb{Z} + i(p\mathbb{Z})$ are distinct maximal $\mathbb{Z}$-submodules of $\mathbb{Z}[i]$. Moreover,

$$\text{Ann}_{\mathbb{Z}}(\mathbb{Z}[i]/(p\mathbb{Z} + i\mathbb{Z})) = \text{Ann}_{\mathbb{Z}}(\mathbb{Z}[i]/(\mathbb{Z} + i(p\mathbb{Z}))) = p\mathbb{Z}.$$ 

Thus the mapping $f : \text{Spec}_{\mathbb{Z}}(\mathbb{Z}[i]) \to \text{Spec}_{\mathbb{Z}}(\mathbb{Z})$ is not injective.

If $E$ is cyclic, then the mapping $f : \text{Spec}_R(E) \to \text{Spec}_R(\bar{R})$ is bijective. Hence we have the following lemma.

**Lemma 12.** Let $E$ be a non-zero cyclic $R$-module. Then every prime $R$-submodule of $E$ is of the form $pE$, where $p$ is a prime ideal of $R$ containing $\text{Ann}_RE$. 
THEOREM 13. Let E be a non-zero cyclic R-module and P a prime R-submodule of E. Let \( p \) denote \( \text{Ann}_R(E/P) \). Then the non-zero \( R_p \)-module \( E \otimes_R R_p \) is a quasi-local \( R_p \)-module with unique maximal \( R_p \)-submodule \( P \otimes_R R_p = p R_p(E \otimes_R R_p) \).

Proof. Note that \( E_p = E \otimes_R R_p \) and \( P_p = P \otimes_R R_p \). Since E is cyclic, so is \( E_p \). Hence \( E_p \) is \( R_p \)-isomorphic to \( R_p/\text{Ann}_{R_p}(E_p) \). Since \( R_p \) is quasi-local with unique maximal ideal \( pR_p \), \( E_p \) is quasi-local with unique maximal submodule \( (pR_p)E_p \) by Lemma 12.

This theorem can also be proved directly. In fact, since \( E \) is cyclic, \( E \cong R/a \) for some ideal \( a \) of R. It follows that \( P \cong q/a \) for some prime ideal \( q \) of \( R \) containing \( a \). Hence

\[\text{Ann}_R(E/P) = \text{Ann}_R((R/a)/(q/a)) = \text{Ann}_R(R/q) = q.\]

We need to show that \( E_q \) is a quasi-local \( R_q \)-module.

\[
0 \rightarrow a \rightarrow R \rightarrow R/a \rightarrow 0 \quad \text{is exact so}
\]

\[
0 \rightarrow aR_q \rightarrow R_q \rightarrow (R/a)_q \rightarrow 0 \quad \text{is exact [AM69, Proposition 3.3].}
\]

Thus \( R_q/aR_q \cong (R/a)_q \). In order to show that \( R_q/aR_q \) is a quasi-local \( R_q \)-module it is sufficient to prove that the ring \( R_q/aR_q \) is a quasi-local ring. But by using Proposition 3.1 of [AM69] it is easy to see that

\[R_q/aR_q \cong (R/a)_{q/a}\]

which implies that \( R_q/aR_q \) is a quasi-local ring [AM69, Example 1, p.38].

**Corollary.** Let \( R \) be a commutative quasi-local ring with unique maximal ideal \( m \). Let \( E \) be a non-zero finitely generated \( R \)-module. Then \( E \) is quasi-local if and only if \( E \) is cyclic.

Proof. By Nakayama's lemma [AM69, Proposition 2.6, p.21], \( mE \neq E \). This means that \( E/mE \) is non-zero, or equivalently that \( \text{Ann}_R(E/mE) \neq R \). Also, \( m \subseteq \text{Ann}_R(E/mE) \). Hence \( m = \text{Ann}_R(E/mE) \). Note that \( R_m = R \). Then if \( E \) is cyclic, then \( E \) is a quasi-local \( R \)-module with unique maximal submodule \( mE \) (Theorem 13).
Conversely, assume that $E$ is quasi-local with unique maximal submodule $M$. Take $e \in E \setminus M$. Then $e$ generates $E$. For, otherwise, $Re$ is a proper submodule of $E$. By [SV72, Proposition 1.6, p.7] $Re \subseteq M$, so $e \in M$, which contradicts.

Unless the finitely generated module $E$ is cyclic Theorem 13 does not hold in general because the mapping $f : \text{Spec}_R(E) \to \text{Spec}_R(R)$ is not always injective.

Let $R$ be a commutative quasi-local ring with unique maximal ideal $m$. Let $E$ be a finitely generated $R$-module. $E/mE$ is annihilated by $m$, hence is naturally an $R/m$-module, i.e., a vector space over the field $R/m$, and as such is finite-dimensional. If $E/mE$ is zero-dimensional, then $mE = E$. This implies that $E$ is zero by Nakayama's lemma [AM69, Proposition 2.6, p.21]. Therefore we have the following result.

**Proposition 14.** Let $R$ be a commutative quasi-local ring with unique maximal ideal $m$. Let $E$ be a non-zero finitely generated $R$-module. Then $E$ has at least $n$ distinct maximal $R$-submodules, where $n$ is the dimension of the vector space $E/mE$ over the field $R/m$.

**Proof.** Let $n = \dim_{R/m}(E/mE)$. As we have already observed, we have $1 \leq n < \infty$. Let $e_i \ (1 \leq i \leq n)$ be elements of $E$ whose images $\bar{e}_i$ in $E/mE$ form a basis of this vector space. Then the $e_i$ generate $E$ [AM69, Proposition 2.8, p.22]. Now let $M_i = me_i + \sum_{j \neq i} Re_j$, $1 \leq i \leq n$. Then we shall show that these are distinct maximal submodules of $E$.

Since $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis for the space $E/mE$ and $1 \notin m$ it follows that the $M_i$ are distinct. In order to show that these are maximal, it is sufficient to prove that each $E/M_i$ is a simple $R$-module. Again, to show this, it suffices to prove that $E/M_i$ is a simple $R/m$-module.

$M_i = \sum_{j \neq i} Re_j + mE$, so each $E/M_i$ is annihilated by $m$, hence is naturally an $R/m$-module, i.e., a vector space over the field $R/m$. Further, each $E/M_i$ is $R/m$-isomorphic to $(E/mE)/(M_i/mE)$ [SV72, Proposition 1.9 Corollary 2, p.11]. Hence to show that each $E/M_i$ is a simple $R/m$-module, it suffices to prove that each subspace $M_i/mE$ of the space $E/mE$ is a hyperspace in the space $E/mE$. But this follows immediately from the fact that each set $\{\bar{e}_1, \ldots, \bar{e}_{i-1}, \bar{e}_{i+1}, \ldots, \bar{e}_n\}$ forms a basis for the subspace $M_i/mE$. 

If \( E \) is a non-zero finitely generated module over a commutative quasi-local ring \( R \) with unique maximal ideal \( \mathfrak{m} \), then the proposition implies that

\[
\text{Card}(\text{Spec}_R(E)) \geq \text{Card}(\Omega_E) \geq \dim_{R/\mathfrak{m}}(E/\mathfrak{m}E),
\]

where \( \text{Card} A \) means the cardinality of a set \( A \).

References


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