ON THE BOUNDARY BEHAVIOR AND TAYLOR
COEFFICIENTS FOR MIXED NORM SPACES $D^{p,q}$

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1. Introduction
Let $U = \{ z : |z| < 1 \}$ and $T = [-\pi, \pi]$. For $0 < p < \infty$, and $1 \leq q \leq \infty$, the spaces $H^q$ and $D^{p,q}$ are defined to consist of those $f$ holomorphic in $U$, respectively for which

$$\|f\|_q := \sup_{0 \leq r < 1} M_q(r, f) < \infty$$

and

$$\int_0^1 (1 - r)^{p - p/q} M_q(r, f')^p dr < \infty,$$

where

$$M_q(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^\pi |f(re^{it})|^q dt \right)^{1/q},$$

$$M_\infty(r, f) = \sup_{t \in T} |f(re^{it})|.$$ 

By the theorem of Hardy and Littlewood (Theorem 5.11 in [2]), if $0 < p < q$,

$$D^{p,q} = \{ f : \int_0^1 (1 - r)^{-p/q} M_q(r, f)^p dr < \infty \}.$$ 

For $0 < s, t \leq \infty$, $l(s, t)$ denotes the space of those sequences $\{a_k\}_{k=0}^\infty$ for which

$$\left\{ \sum_{k \in I_m} |a_k|^s \right\}_{m=0}^\infty \in l^t \quad (s < \infty).$$

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and

\[ \{ \sup_{k \in \mathcal{I}_m} |a_k| \}^\infty_{m=0} \in l^s \quad (s = \infty), \]

where \( \mathcal{I}_m = \{ k : 2^m \leq k < 2^{m+1} \} \) (\( m = 1, 2, \cdots \)) and \( \mathcal{I}_0 = \{ 0 \} \) (See [5]).

In [1], P. Ahern and M. Jevtic defined mixed norm spaces \( DP, q \) and showed that when \( q = 2 \) these are exactly the spaces \( D^p \) introduced by F. Holland and B. Twomey [4]. They also investigated the dual space and multipliers for \( DP, q \).

This note is concerned with the tangential boundary behavior and Taylor coefficient conditions of holomorphic functions in connection with \( DP, q \). We list some of the known properties of \( DP, q \) in the following. Here and throughout this note \( \frac{1}{q} + \frac{1}{q'} = 1 \) whenever \( 1 \leq q \leq \infty \).

**Proposition.**

1. If \( p < q \), then \( H^p \subset DP, q \).
2. If \( p > q \), then \( H^p \supset DP, q \).
3. If \( q \leq 2 \), then \( H^q \supset D^{q, q} \).
4. If \( 2 \leq q < \infty \), then \( H^q \subset D^{q, q} \).
5. If \( p_1 \leq p_2 \), then \( DP_1, q \supset DP_2, q \).
6. If \( q_1 \leq q_2 \), then \( DP, q_1 \subset DP, q_2 \).
7. If \( 1 \leq q \leq 2 \) and if \( f(z) = \sum_{n=1}^{\infty} a_n z^n \in DP, q \), then

\[ \{ n^{1/q-1/p} a_n \}^\infty_{n=1} \in l(q', p). \]

**Proof.** See [1. Theorem 6] for (1), (2), (3) and (4). If \( f \in DP, q \), then \( M_q(r, f') \leq C(1-r)^{1/q-1/p-1} \). Hence the proof of (5) is complete. (6) follows from the fact that

\[ M_{q_2}(r, f') \leq C(1-r)^{1/q_2-1/q_1} M_{q_1}(r, f'). \]

See [6. p.48(4')] for (7).

2. **Representation and tangential boundary behavior**

For \( 0 \leq r < 1 \), and \( \beta > 0 \), let us define

\[ G^\beta_r(t) = (1 - re^{it})^{-\beta}, \quad t \in T. \]
THEOREM 1. Let $0 < p \leq 2$ and $1 \leq q \leq 2 < s < \infty$. If $f'(z) \in D^{p,q}$, then there is an $F(t) \in L^q(T)$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) G_\beta^\theta (\theta - t) dt$$

$$= (F * G_\beta^\theta)(\theta), \quad z = re^{i\theta} \in U,$$

where $\beta = 1/p - 1/s$.

Proof. Let $0 < p \leq 2$, $1 \leq q \leq 2$ and let

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \in D^{p,q}.$$ 

Then $f'(z) \in D^{p,2}$ by (6) of Proposition. So

$$\{n^{3/2-1/p} a_n\}_{n=1}^{\infty} \in l(2,p) \subset l(q',p)$$

by (7) of Proposition. If we fix $s; 2 \leq s < \infty$, then it follows from Hölder's inequality that

$$\sum_{k \in I_n} |k^{1-\beta} a_k|^{s'} \leq (\sum_{k \in I_n} |k^{3/2-1/p} a_k|^{q'})^{s'/q'} (\sum_{k \in I_n} \frac{1}{k})^{1-s'/q}.$$ 

But since $\sum_{k \in I_n} \frac{1}{k}$ is bounded independently on $n$, we conclude that

$$\sum_{k \in I_n} \frac{1}{k}$$

(2.1)

$$\{n^{1-\beta} a_n\}_{n=1}^{\infty} \in l(s',p),$$

where $\beta = 1/p - 1/s$. Next, set

$$b_n = \Gamma(\beta)\Gamma(n+1)a_n/\Gamma(n+\beta) \quad n = 1, 2, \ldots.$$ 

Then since $\Gamma(\beta)\Gamma(n+1)/\Gamma(n+\beta) = 0(n^{1-\beta})$ and $p \leq 2$, we obtain from (2.1)

$$\{b_n\}_{n=1}^{\infty} \in l(s',p) \subset l(s',2).$$
Finally let

\( F(t) = \sum_{n=1}^{\infty} b_n e^{int}, \quad t \in T. \)  

Then it follows from (2.2), (2.3), and the Kellog's version of the Hausdorff-Young theorem [5] that

\( F(t) \in L^S(T). \)

On the other hand, termwise integration gives that

\( f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)(1 - e^{-it}z)^{-\beta} dt, \quad z \in U. \)

Here termwise integration is justified because \( F(t) \in L^S \) and the series expansion of \( (1 - e^{-it}z)^{-\beta} \) is uniformly convergent whenever \( z \in U \) is fixed.

From (2.4) and (2.5) the proof is complete.

**Remark.** If we suppose (2.0) for some \( F(t) \in L^S(T) \), then by Young's inequality after differentiation of \( f(z) \) we get

\( M_q(r, f'') \leq C \|F\|_{L^S} \|G_r^{\beta+2}\|_{L^1}, \)

where \( 1/q = 1/s + 1/t - 1 \). Since \( \|G_r^{\beta+2}\|_{L^1} = O(1 - r)^{-2-\beta+1/t} \) ([2. p.65]), by (2.6) we conclude that (2.0) implies

\( M_q(r, f'')^p = O((1 - r)^{-1-p+p/q}). \)

If we compare (2.7) with the definition of \( f' \in D^{p,q} \) we see that our exponent \( \beta = 1/p - 1/s \) is best possible.

**Theorem 2.**

(1) If \( f'(z) \in D^{1,q} \) for some \( q \), then \( f(z) \in BMOA. \) (See [3] for BMOA).

(2) If \( f'(z) \in D^{1,q} \) for some \( q : q \leq 2 \), then \( f(z) \) is continuous in \( \{z : |z| \leq 1\} \).

(3) If \( 2/3 < p < 1 \), \( q \leq 2 \) and if \( f'(z) \in D^{p,q} \), then the limit of \( f(z) \) as \( z \to e^{i\theta} \) within \( \Omega_\gamma \) exists almost everywhere on \( T \), where \( \Omega_\gamma = \{re^{it} : 1 - r > |\sin \frac{\theta-1}{2}| \gamma \} \) and \( \gamma = \frac{1}{2(p-1)}. \)
Proof. (1) Let $f'(z) \in D^{1,\infty}$. Then $M_{\infty}(r, f') = O(1 - r)^{-1}$ from the very definition of $D^{1,\infty}$. Hence

$$
\int_0^1 (1 - r)M_{\infty}(r, f')^2 dr < \infty.
$$

Thus $(1 - |z|)|f'(z)|^2 dxdy$ is a Carleson measure, whence $f \in BMOA$. ([3. P.240]).

(2) Let $f'(z) = \sum_{n=0}^{\infty} n a_n z^n \in D^{1,2}$. Then $\{n^{1/2}a_n\} \in l(2,1)$ by (7) of Proposition, so that

$$
\{a_n\}_{n=1}^{\infty} \in l(1,1)
$$

by Hölder's inequality. From (2.8) and the Weierstrass $M$-test $g(t) = \sum_{n=0}^{\infty} a_n e^{int}$ converges uniformly and becomes a continuous function on $T$. On the other hand, by Abel’s theorem [9, p.229],

$$
\lim_{r \to 1} f(re^{it}) = g(t)
$$

for every $t \in T$. Hence $f(z)$ is the Poisson integral of the continuous function $g(t)$. Whence $f(z)$ is continuous on $\{z : |z| \leq 1\}$.

(3) Let $2/3 < p < 1$, $q \leq 2$, and $f'(z) \in D^{p,q}$. Take $\gamma = \frac{1}{2(1/p - 1)}$. Then by Theorem 1,

$$
f(z) = (F * G^\beta_r)(t), \quad z = re^{it}
$$

for some $F \in L^S$, $\beta = 1/p - 1/s$. Now the existence of the $\Omega \gamma$-limit follows from [8. Theorem A-(a)].

3. Taylor coefficients

THEOREM 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $U$. Suppose that

$$
\sum_{n=0}^{N} n^2 |a_n|^2 = O(N^\alpha)
$$

for some $\alpha$. Then $f(z)$ is in $BMOA$.
for some $\alpha \geq 0$. If $q \leq 2$, then $f(z) \in D^{p,q}$ for all $p$ with $1/p > 1/q + \alpha/2 - 1$.

Proof. If $q \leq 2$,

\begin{equation}
M_q(r, f') \leq M_2(r, f') = \left(\sum n^2 |a_n|^2 r^{2n}\right)^{1/2}.
\end{equation}

It follows from summation by parts that

\begin{equation}
\sum_{n=0}^{\infty} n^2 |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} k^2 |a_k|^2 (r^{2n} - r^{2n+2}) + \lim_{N \to \infty} \sum_{k=0}^{N} k^2 |a_k|^2 r^{2N}.
\end{equation}

The last term of (3.2) is 0 by (3.0). Thus from (3.1) and (3.2)

\begin{equation}
M_q(r, f')^p \leq C \left(\sum_{n=0}^{\infty} n^\alpha (r^{2n} - r^{2n+2})\right)^{p/2} \leq C (1 - r)^{-\alpha p/2}.
\end{equation}

Hence

\[ \int_0^1 (1 - r)^{p - p/q} M_q(r, f')^p dr < \infty \]

for all $p$ with $p - p/q - \alpha p/2 > -1$.

Corollary. Let $f(z) = \sum a_n z^n$ be analytic in $U$. If (3.0) holds for some $\alpha; 0 < \alpha < 2$, then $f \in H^2$.

Remark. A routine calculation gives that (3.0) is equivalent to the condition

\begin{equation}
\{ n^{1-\alpha/2} a_n \}_{n=1}^{\infty} \in l(2, \infty).
\end{equation}

But by (7) of Proposition $f(z) = \sum a_n z^n \in D^{p,q}$, $1 \leq q \leq 2$, should satisfy

\begin{equation}
\{ n^{1/q - 1/p} a_n \}_{n=1}^{\infty} \in l(q', p).
\end{equation}

If we take $a_n = n^{(\alpha-3)/2}$ for $n = 2^m$, $m = 0, 1, 2, \ldots$ and $a_n = 0$ otherwise, then $\{a_n\}$ satisfies (3.3) but not (3.4) when $1/p = 1/q + \alpha/2 - 1$. Therefore the exponent $p$ in the result of Theorem 3 is best possible.
**Theorem 4.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and if
\[
\{n^{1/q-1/p} a_n\}_{n=1}^{\infty} \in l(1,p),
\]
then \( f \in D^{p,q} \).

**Proof.**

\[
(3.5) \quad \int_0^1 (1 - r)^{p-p/q} M_q(r, f')^p \, dr \\
\leq \int_0^1 (1 - r)^{p-p/q} M_{\infty}(r, f')^p \, dr \\
\leq \int_0^1 (1 - r)^{p-p/q} \left( \sum_{n} n|a_n| r^n \right)^p \, dr.
\]

If we apply [7. Theorem A] the last integral of (3.5) is at most a constant times
\[
\sum_{0}^{\infty} \left( \sum_{k \in I_n} k^{1/q-1/p} |a_k| \right)^p,
\]
whence completes the proof.

**References**


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