1. Introduction

Let $V^n$ and $M^m$ be smooth manifolds of dimension $n$ and $m$, respectively, with $3n + 3 \leq 2m$ throughout the paper unless it is said otherwise. We also assume that $V$ is closed. In [6], Haefliger reduces the problem of homotoping a map from $V$ into $M$ to a smooth embedding to the construction of an equivariant map between two proper spaces, where the construction of such a map can be considered as a homotopy theoretic problem. There are a few applications of Haefliger’s result, especially when $M = \mathbb{R}^m$ ([1], [3], [4], [10]).

In this paper, we study embedding and immersion problems when $V$ and $M$ are \(((2n - m + 1)/2) + 1\)-connected and $6 \leq 2n - m$. The main tools are the handle subtraction (Proposition 2 of [5]) and the handle attaching (Lemma 1 of this paper) on generic maps (see [5] for the definition of generic map).

Following the notation of [5], if $f : V \to M$ is a generic map, let $\Delta(f)$ denote the closure of the double point set, $S'(f) = \partial \Delta(f)$, the set of singular values of $f$, $D(f) = f^{-1}(\Delta(f))$ and $S(f) = f^{-1}(S'(f))$. All these sets are smooth manifolds and if one is contained in the other it is contained as a submanifold, and the dimension of $\Delta(f)$ is equal to $2n - m$. Let $H$ be a $(p - r)$-handle, $-1 \leq r \leq p$, in $\Delta(f)$ relative to $S'(f)$, where $p = 2n - m - 1$. We use $p$ for this number in the rest of the paper. Now $H$ determines uniquely an element $\theta(H)$ in $\pi_{p-r+1}(f)$ up to sign (see 5.3 how this is defined). We say that $H$ can be subtracted from $f$ if and only if $\theta(H) = 0$. This definition is motivated from the main result (Proposition 2) of [5] which says that given a null homotopy of $\theta(H)$, one can construct a generic map $f'$ homotopic to $f$ such that $\Delta(f')$ is diffeomorphic to the closure of $\Delta(f) - H$. Therefore, it follows that if $f$ is $(p + 2)$-connected, then all the handles in $\Delta(f)$ can be subtracted in any handle decomposition of
(Δ(f), S′(f)), thus f is homotopic to an embedding. We call f′ the result of a handle subtraction on f. Note that f′ depends on the null homotopy. By reversing the handle subtraction, Δ(f) can be viewed as being obtained from Δ(f′) by attaching an (r + 1)-handle to Δ(f′) along S′(f′).

This raises the following question. Given an embedding h : Sr × Dp−r → S′(f), −1 ≤ r ≤ p, is there a generic map f′ homotopic to f such that Δ(f′) is diffeomorphic to Δ(f) with an (r + 1)-handle attached using h? The main result (Lemma 1) of this paper shows that handle attaching can be done with some restrictions on h and r. The handle attaching lemma is proved in Section 2 following the steps of handle subtraction in [5] with time (τ) inverted. But handle attaching requires more work (see 2.1-2.5) than handle subtraction. A second proof of the lemma was suggested in [9]. Using handle attaching, we show that surgeries can be done (Lemma 2) on the interior of Δ(f) under proper conditions. It follows (2.7-2.9) that if V and M are ([(p + 1)/2] + 1)-connected, then it is possible to subtract or attach handles, or do surgeries on Δ(f) below the middle dimension with one exception.

We now give some result obtained by applying these operations.
If 3n < 2m, then every map from V into M can be approximated by a generic map and an immersion can be approximated by an immersion which is generic. We will call such an immersion a generic immersion and an immersion will mean a generic immersion in this paper.

DEFINITION. An immersion from V into M is orientable if its double point set is orientable.

Note that every immersion is orientable if m − n is even.

DEFINITION. A generic map f is nice if Δ(f) is orientable and S(f) is two-sided in D(f). Nice homotopy is defined the same way.

DEFINITION. A generic map f is a pseudo-embedding if Δ(f) ≅ D2n−m (≡ means “is diffeomorphic to”).

Note that every embedding is homotopic to a pseudo-embedding by Lemma 1.

THEOREM 1. Let V and M be ([(p + 1)/2] + 1)-connected and let f : V → M be a map. Then the following are equivalent to each other.
(a) $f$ is homotopic to an orientable immersion.
(b) $f$ is homotopic to a nice map.
(c) $f$ is homotopic to a pseudo-embedding.

The theorem suggests that there are two stages of obstructions to homotoping a map to an embedding under the connectivity assumption on $V$ and $M$ in Theorem 1. The first is to homotoping a map to a pseudo-embedding and the second is to homotoping a pseudo-embedding to an embedding. Next theorem identifies the second obstruction.

Given a map $f : V \to M$, let $\pi_i(f) = \pi_i(Z(f), V)$, where $Z(f)$ denotes the mapping cylinder of $f$, and let $\pi_i(f)^+ = \pi_i(f)/\sim$, where $a \sim b$ if $a = \pm b$.

**Theorem 2.** Let $V$ and $M$ be $\left\lceil \left(\frac{p+2}{2}\right) + 1 \right\rceil$-connected and let $f : V \to M$ be a pseudo-embedding. Then there exists an element $\Gamma(f) \in \pi_{p+2}(f)^+$ such that $f$ is nicely homotopic to an embedding if and only if $\Gamma(f) = [0]$.

A similar result was obtained in [3] by assuming that the immersion has a non-trivial normal vector field with a stronger connectivity condition on $M$.

We now describe the first obstruction in a form of a test. Under the connectivity condition on $V$ and $M$, if $m - n$ is even, the obstruction is a geometric approach to the immersion classification theorem of [7]. It may also be regarded as a partial converse of the embedding theorem of [5]: the embedding theorem was proved by showing that a relative $r$-handle in $(\Delta(f), S'(f))$ of a generic map $f$ can be subtracted if the element of $\pi_{r+1}(f)$ determined by the handle is trivial. If the element is not trivial, then the handle can not be subtracted by the method of [5]. But it is possible that $f$ may still be homotopic to an embedding. Next theorem shows that if a generic map $g$ homotopic to $f$ is chosen properly, then the failure of a handle subtraction from $(\Delta(g), S'(g))$ implies that $g$ can not be homotopic to a pseudo-embedding, thus $f$ can never be homotopic to an embedding.

The test works as follows. Suppose that $V$ and $M$ are $\left\lceil \left(\frac{p+2}{2}\right)+1 \right\rceil$-connected and we are given a map from $V$ to $M$. We first approximate the map by a generic map $f$. If $f$ is nice, then the test ends. So assume that $f$ is not nice. Then by a sequence of handle attaching and
subtraction on $f$, we can find a generic map (not nice) $g$ homotopic to $f$ such that $\pi_1(S'(g)) \cong \pi_1(\Delta(g)) \cong \mathbb{Z}_2$, and $\Delta(g)$ and $S'(g)$ are $([p+1]/2) - 1$ and $([p]/2) - 1$-connected, respectively (see 5.1).

**Theorem 3.** Under the above notation, $f$ is homotopic to a nice generic map if and only if we can subtract all the handles except for a top dimensional one in any handle decomposition of $\Delta(g)$ relative to $S'(g)$.

The handle attaching lemma is proved in (2.1-2.5), the surgery lemma is proved in (2.6), and the possibility of killing homotopy elements of $\Delta(f)$ and $S'(f)$ is studied in (2.7-2.9). Theorem 1, 2 and 3 are proved in Section 3, 4 and 5, respectively.

2. Handle attaching lemma

Let $f : V \to M$ be a generic map and $h : S^r \times D^{p-r} \to S'(f)$, $-1 \leq r \leq p$, be an embedding. Suppose that $f' : V \to M$ is a generic map homotopic to $f$ such that $\Delta(f') \cong \mathcal{H}(\Delta(f), h) = \Delta(f) \cup_h D^{r+1} \times D^{p-r}$.

We call $f'$ the result of a handle attaching on $f$ using $h$. If such a handle attaching is possible, it follows that $f^{-1}h$ is contractible in $V$. So the contractibility is a necessary condition for a handle attaching. But this condition is not sufficient to attach a handle in general. We may need to modify $h$ before we can attach a handle.

Let $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ (the stable homotopy group), $r \geq -1$, be the homomorphism induced by the inclusion map. If $r = -1$, both groups are understood to be trivial and if $r = 0$, they must be replaced by $\pi_0(O_p)$ and $\pi_0(O)$, respectively. The following discussions are made for $r \geq 1$ but with proper modification they hold true for $r = -1, 0$.

Given $\lambda \in \pi_r(SO_{p-r})$ define new embedding $h_\lambda$ by $h_\lambda(x, v) = h(x, \lambda(x)v)$, $(x, v) \in S^r \times D^{p-r}$.

**Lemma 1.** Let $f : V \to M$ be a generic map, and let $h : S^r \times D^{p-r} \to S'(f)$ be an embedding such that $f^{-1}(h[S^r \times \{0\}])$ is null homotopic in $V$. Assume that $S(f)$ is two-sided in $D(f)$ over the image of $f^{-1}h$ if $r = 1$. If $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ is onto, then there exists an element $\lambda \in \pi_r(SO_{p-r})$ and a generic map $g$ homotopic to $f$ such that $\Delta(g)$ is diffeomorphic to $\mathcal{H}(\Delta(f), h_\lambda)$. Furthermore, $\lambda$ can be replaced with any element in the coset $\text{Ker}(s_*) + \lambda$ in $\pi_r(SO_{p-r})$. 
We prove the lemma by following the steps and notation of the proof of Proposition 2 of [5].

2.1. Let $g_\tau$, $0 \leq \tau \leq 1$, be the family of maps from

$$\mathbb{R}^n(x_1, x_2, \cdots, x_{m-n}, u_1, \cdots, u_{p-r}, v_1, \cdots, v_{r+1})$$

to

$$\mathbb{R}^m(X_1, \cdots, X_{m-n}, Y_1, \cdots, Y_{m-n}, U_1, \cdots, U_{p-r}, V_1, \cdots, V_{r+1})$$

defined in 4.4 of [5]. We repeat the definition for the convenience of readers.

Let $u^2 = \sum u_i^2$, $v^2 = \sum v_j^2$, $w^2 = u^2 + v^2$, $R^2 = U^2 + V^2$ and

$$p^2 = \sum_{1<i\leq m-n} x_i^2.$$ Let $\gamma$ be an even function with variable $x$ so that $\gamma(x) = 1$ for $|x| \leq 1$, $\gamma(x) = 0$ for $|x| \geq 2$, and it increases for $x < 0$. Define $\theta(w^2) = \theta_0^2 \gamma(w^2/\theta_0^2)$, where $\theta_0$ is a real number, $0 < \theta_0 \leq 1$. Let $\alpha$ be an even function in $\rho$ such that $0 \leq \alpha(\rho) \leq 1$, $\alpha(0) = 1$ and $\alpha(\rho) = 0$ for $\rho > \varepsilon$, where $\varepsilon$ is a positive real number. Then $g_\tau$ is defined as follows.

$$g_\tau : \begin{cases} X_1 = x_1(1 - 2Y_1) \\ Y_1 = A(u^2, v^2, \rho^2, \tau)\gamma(x_1)/(1 + x_1^2) \\ X_i = x_i, Y_i = x_1 x_i, 1 < i \leq m-n \\ U_j = u_j, V_j = v_j, \end{cases}$$

where $A(u^2, v^2, \rho^2, \tau) = [1 + \theta(w^2)(1 - 2\alpha(\rho)\tau + 2v^2)]/(1 + w^2)$. It can be checked that $g_\tau$ is a generic map for $\tau \neq 1/2$.

Let $K_\varepsilon$ be the subset of $\mathbb{R}^n$ defined by $|x_1| \leq 2$, $\rho^2 \leq \varepsilon^2$ and $w \leq 2\theta_0$, and let $K'_\varepsilon$ be the subset of $\mathbb{R}^m$ defined by $|X_1| \leq 2$, $\sum_{i>1} X_i^2 \leq \varepsilon^2$, $0 \leq Y_1 \leq 1$, $\sum_{i>1} Y_i^2 \leq 4\varepsilon^2$ and $R \leq 2\theta_0$. Then $g_\tau^{-1}(K'_\varepsilon) = K_\varepsilon$ for all $\tau$ and $g_\tau$ is fixed outside of $K_\varepsilon$ in $\tau$.

Let $K$ denote $K_\varepsilon(x_i = 0, i > 1)$, i.e., $K$ is the intersection of $K_\varepsilon$ with the plane $x_i = 0, i > 1$, and $K' = K'_\varepsilon(X_i = Y_i = 0, i > 1)$.

Let $H^p_{\tau+r+1}$ be the $(p-r+1)$-dimensional upper half disk consisted of points in $(p-r+1)$-disk whose last coordinate is non-negative and let $H^p_{\tau+r+1}$ be the lower half. Let $D^p_{\tau+r}$ be the subset of points in $H^p_{\tau+r+1}$ whose last coordinate is equal to 0. Regard $D^p_{\tau+r+1}$ as the
union of $H^{p-r+1}_-\cap H^{p-r+1}_-$ identified along $D^{p-r}_0$. We use 1 for $\theta_0$ from now on.

Let $S^r_0$ and $S^r_1$ be the $r$-spheres contained in $K_e$ and $K'_e$, respectively, satisfying the following equations.

\[ S^r_0; x_i = 0, \ u_j = 0 \quad \text{and} \quad v^2 = 1. \]
\[ S^r_1; X_i = 0, \ Y_1 = 1/2, \ Y_j = 0, \ 1 < j, \ U_k = 0, \ V^2 = 1. \]

We also let $D^{r+1}_0 = K_e(x_i = 0, \ u_j = 0, \ v^2 \leq 1)$ and $D^{r+1}_1 = K'_e(X_i = 0, \ Y_1 = 1/2, \ Y_j = 0, \ 1 < j, \ U_k = 0, \ V^2 \leq 1)$.

From the description of $\Delta(g_r)$ in 4.4 of [5], $\Delta(g_1) \cap K'_e$ can be naturally identified with $S^r_1 \times H^{p-r+1}_+$, $S^r_1 \cap K'_e$ with $S^r_1 \times D^{p-r}_0$, $D(g_1) \cap K_e$ with $S^r_0 \times D^{p-r+1}_0$ and $S(g_1)$ with $S^r_0 \times D^{p-r}_0$. These identifications can be done so that $\partial/\partial U_1, \ldots, \partial/\partial U_{p-r}$ and $v'_0$ are independent normal vector fields to $S^r_1 \times \{0\}$ in $S^r_1 \times H^{p-r+1}_+$ (0 denotes the center of a disk), where $v'_0$, a linear combination of $\partial/\partial V_1, \ldots, \partial/\partial V_{r+1}$, is a normal vector field to $S^r_1 \times D^{p-r}_0$ in $S^r_1 \times H^{p-r+1}_+$ and $\partial/\partial U_1, \ldots, \partial/\partial U_{p-r}$ and $\partial/\partial x_1$ are independent normal vector fields to $S^r_0 \times \{0\}$ in $S^r_0 \times D^{p-r+1}_0$. We may further assume that $g_1(v, y) = (v, y) \in S^r_1 \times H^{p-r+1}_+$ for $(v, y) \in S^r_0 \times H^{p-r+1}_+$ and $g_1(v, y') = (v, y) \in S^r_1 \times H^{p-r+1}_+$ for $(v, y') \in S^r_0 \times H^{p-r+1}_+$, where the last coordinate of $y$ is equal to $-(last$ coordinate of $y')$ and the other coordinates of $y$ and $y'$ are the same.

Now $\Delta(g_0) \cap K_e$ is diffeomorphic to $\Delta(g_1) \cap K'_e$ with an ambient $(r+1)$-handle attached in $K'_e$ using a small thickening of $S^r_1$ in $S^r_1 \times D^{p-r}_0$, where the core of the handle is $D^{r+1}_1$.

Given a generic map $f : V \to M$ and an embedding $h : S^r \times D^{p-r} \to S'(f)$ with $f^{-1}(h(S^r \times \{0\}))$ null homotopic in $V$, if $s_r : \pi_r(SO_{p-r}) \to \pi_r(SO)$ is onto, then we construct diffeomorphisms $H : K_e \to V$ and $H' : K'_e \to M$ such that

1. $H'g_1 = fH$ over $K_e$
2. $f^{-1}H'(K'_e) = H(K_e)$
3. $H'|S^r_1 \times D^{p-r}_0 = h_\lambda$ for some $\lambda \in \pi_r(SO_{p-r})$.

If generic map $g$ is defined by $g = f$ outside of $H(K_e)$ and $g = H'g_0H^{-1}$ over $H(K_e)$, then $g$ satisfies the conclusions of Lemma 1. We construct $H$ and $H'$ in several steps as in 4.5 through 4.6 of [5].

2.2. Over $(D(g_1) \cap K_e) \cup (a$ neighborhood of $S(g_1) \cap K_e$ in $K_e)$ and $(\Delta(g_1) \cap K'_e) \cup (a$ neighborhood of $S'(g_1) \cap K'_e$ in $K'_e)$. 
Define $H'_1: S'_1 \times D^{p-r+1}_0 \to M$ such that $H'_1|S'_1 \times D^{p-r}_0 = h$ and $H'_1$ gives a collar neighborhood of $H'_1(S'_1 \times D^{p-r}_0)$ in $\Delta(f)$. Then we can find a diffeomorphism $H_1: S'_0 \times D^{p-r+1}_0 \to V$ so that the conditions (1), (2) and (3) hold. Here we use the assumption that $S(f)$ is two-sided in $D(f)$ over $f^{-1}(\text{Im}(h))$ when $r = 1$.

Let $e_i = \partial/\partial x_i$, $1 \leq i \leq m-n$ and $e'_i = \partial/\partial X_i$, $e''_i = \partial/\partial Y_i$, $2 \leq i \leq m-n$. Since $2(2n-m) < n$ and $H_1|S'_0 \times D^{p-r}_0$ is null homotopic in $V$, there exist independent normal vector fields $\xi_1, \ldots, \xi_{m-n}, \nu$ to $H_1(S'_0 \times D^{p-r}_0)$ in $H_1(S'_0 \times D^{p-r+1}_0)$. Define an isomorphism $dH_2: T(K_\varepsilon)|S'_0 \times D^{p-r}_0 \to T(V)|H_1(S'_0 \times D^{p-r}_0)$ by $dH_2(e_i) = \xi_i$, $1 \leq i \leq m-n$, $dH_2(v_0) = \nu$, where $v_0$ is the unit normal vector field to $S'_0 \times D^{p-r+1}_0$ in $K$, thus $v_0$ is a linear combination of $\partial/\partial v_1, \ldots, \partial/\partial v_{r+1}$. By 3.2 and 3.3 of [5], there exist diffeomorphisms $H_2$ from $(D(g_1) \cap K_\varepsilon) \cup$ (a neighborhood of $S'(g_1) \cap K_\varepsilon$ in $K_\varepsilon$) into $V$ and $H'_2$ from $(\Delta(g_1) \cap K'_\varepsilon) \cup$ (a neighborhood of $S'(g_1) \cap K'_\varepsilon$ in $K'_\varepsilon$) into $M$ such that $H_2$ and $H'_2$ extend $H_1$ and $H'_1$, respectively, and (1), (2) and (3) hold.

Before further extensions of $H_2$ and $H'_2$ are considered, we study the possibility of choosing $H_2$ and $H'_2$ differently, which is crucial in the proof of the lemma.

2.3. Let $\lambda_1 \in \pi_r(SO_{p-r})$, $\lambda_2 \in \pi_r(SO_{m-n-1}) \cong \pi_r(SO)$, $u = (u_1, \ldots, u_{p-r})$, $v = (v_1, \ldots, v_{r+1})$, $U = (U_1, \ldots, U_{p-r})$ and $V = (V_1, \ldots, V_{r+1})$. Define diffeomorphisms $J(\lambda_1, \lambda_2)$ of $R^n(|v| \neq 0)$ onto itself and $J'(\lambda_1, \lambda_2)$ of $R^n(|v| \neq 0)$ onto itself by $J(\lambda_1, \lambda_2)(x_1, x_2, \ldots, x_{m-n}, u, v) = (x_1, \lambda_2(v/|v|)(x_2, \ldots, x_{m-n}), \lambda_1(v/|v|)(u, v))$ and $J'(\lambda_1, \lambda_2)(x_1, x_2, \ldots, x_{m-n}, Y_1, Y_2, \ldots, Y_{m-n}, U, V) = (X_1, \lambda_2(V/|V|)(X_2, \ldots, X_{m-n}), Y_1, \lambda_2(V/|V|)(Y_2, \ldots, Y_{m-n}), \lambda_1(V/|V|)(U, V))$. Then $g_1J(\lambda_1, \lambda_2) = J'(\lambda_1, \lambda_2)g_1$ over $R^n(|v| \neq 0)$, $J(\lambda_1, \lambda_2)(K_\varepsilon, |v| \neq 0) = (K_\varepsilon, |v| \neq 0)$, $J(\lambda_1, \lambda_2)(D(g_1) \cap K_\varepsilon) = D(g_1) \cap K_\varepsilon$, $J(\lambda_1, \lambda_2)|S'_0 = \text{id}$ of $S'_0$, $J'(\lambda_1, \lambda_2)(K'_\varepsilon, |v| \neq 0) = (K'_\varepsilon, |v| \neq 0)$, $J'(\lambda_1, \lambda_2)(\Delta(g_1) \cap K'_\varepsilon) = \Delta(g_1) \cap K'_\varepsilon$ and $J'(\lambda_1, \lambda_2)|S'_1 = \text{id}$ of $S'_1$.

By choosing proper domains, $H_2$ and $H'_2$ can be replaced by $H_2J(\lambda_1, \lambda_2)$ and $H'_2J'(\lambda_1, \lambda_2)$, respectively. Clearly $H'_2J'(\lambda_1, \lambda_2)|S'_1 \times D^{p-r}_0 = h_{\lambda_1}$.

If $dJ(\lambda_1, \lambda_2)$ is restricted over $T(K_\varepsilon)|S(g_1)\cap K_\varepsilon$, then $dJ(\lambda_1, \lambda_2)(e_i) = e_1$, $dJ(\lambda_1, \lambda_2)(v_0) = v_0$ and $dJ(\lambda_1, \lambda_2)(e_i) = \lambda_2(v/|v|)(e_i)$, $2 \leq i \leq m-n$.

If $dJ(\lambda_1, \lambda_2)$ is restricted over $T(K_\varepsilon)|S(g_1)\cap K_\varepsilon$, then $dJ(\lambda_1, \lambda_2)(e_i) = e_1$, $dJ(\lambda_1, \lambda_2)(v_0) = v_0$ and $dJ(\lambda_1, \lambda_2)(e_i) = \lambda_2(v/|v|)(e_i)$, $2 \leq i \leq m-n$.
If \( dJ'(\lambda_1, \lambda_2) \) is restricted over \( T(K'_e) | S'(g_1) \cap K' \), then \( dJ'(\lambda_1, \lambda_2) \)
\((\partial/\partial x_1, \partial/\partial Y_1) = (\partial/\partial x_1, \partial/\partial Y_1) \), \( dJ'(\lambda_1, \lambda_2)(e_i) = \lambda_2(V/|V|)(e_i') \),
\( 2 \leq i \leq m - n \), \( dJ'(\lambda_1, \lambda_2)(e_i) = \lambda_2(V/|V|)(e_i') \), \( 2 \leq i \leq m - n \), and
\( dJ'(\lambda_1, \lambda_2)(\nu_0) = \nu_0 \), where \( \nu_0 \) is defined in 2.1.

**Remark.** Suppose that \( \bar{H}_2 \) and \( \bar{H}'_2 \) are two diffeomorphisms that
can replace \( H_2 \) and \( H'_2 \), respectively, in 2.2. If we assume that \( H_2 = \bar{H}_2 \over S_0' \) and \( H'_2 = \bar{H}'_2 \over S'_1 \), then \( H_{2}^{-1} \bar{H}_2 \) and \( H'_{2}^{-1} \bar{H}'_2 \) are
diffeomorphisms of small neighborhoods of \( S_0' \) and \( S'_1 \) into themselves,
respectively. It can be seen that if there exist \( \lambda_1 \in \pi_r(SO_{m-n-1}) \) such that \( d(H_{2}^{-1} \bar{H}_2) = dJ(\lambda_1, \lambda_2) \) on the tangent
subspace generated by \( e_i, 1 < i < m - n \) and \( \partial/\partial u_1, \ldots, \partial/\partial u_{m-n} \) along
\( S_0' \), then \( d(H'_{2}^{-1} \bar{H}'_2) = dJ'(\lambda_1, \lambda_2) \) on the tangent subspace generated
by \( e'_i, e''_i, 1 < i < m - n \) and \( \partial/\partial u_1, \ldots, \partial/\partial u_{m-n} \) along \( S'_1 \).

2.4. Over \( K \) and \( K' \).

As in 4.6 of [5], we can find diffeomorphisms \( H_3 : K \to V \) and
\( H'_3 : K' \to M \) satisfying (1), (2) and (3) since \( H_2 | S_0' \) is null homotopic
in \( V \). Notice that \( H'_3 | S'_1 \times D_0^{-r} = h \). Furthermore, \( H_3 \) and \( H'_3 \) agree
with \( H_2 \) and \( H'_2 \) in the neighborhoods of \( S(g_1) \cap K_\varepsilon \) in \( K_\varepsilon \) and
\( S'(g_1) \cap K'_\varepsilon \) in \( K'_\varepsilon \), respectively. The argument for the construction of
\( H_3 \) and \( H'_3 \) is easier here than that of 4.6 of [5] because \( g_1(K) \) is a
deformation retract of \( K' \) in our case.

2.5. Over \( K_\varepsilon \) and \( K'_\varepsilon \).

To extend \( H_3 \) and \( H'_3 \) over \( K_\varepsilon \) and \( K'_\varepsilon \), it suffices to construct vector
bundle isomorphisms \( dH \) from \( N(K_\varepsilon, K) \) (the normal bundle of
\( K \) in \( K_\varepsilon \)) to \( N(V, H_3(K)) \) covering \( H_3 \) and \( dH' \) from \( N(K'_\varepsilon, K') \) to
\( N(M, H'(K')) \) covering \( H'_3 \) so that \( dH'dg_1 = dfdH \) and they agree
with \( dH_2 \) and \( dH'_2 \) over \( S(g_1) \cap K_\varepsilon \) and \( S'(g_1) \cap K'_\varepsilon \), respectively.

Observe that \( dH_2 \) defines a bundle isomorphism from \( N(K_\varepsilon, K) \cap S'_0 \)
to \( N(V, H_3(K)) \cap H_3(S'_0) \) and it gives an element \( \lambda \in \pi_r(SO_{m-n-1}) \) by
using a fixed trivialization of \( N(V, H_3(K)) \).

Replace \( H_2 \) and \( H'_2 \) with \( \tilde{H}_2 = H_2J(0, -\lambda) \) and \( \tilde{H}'_2 = H'_2J'(0, -\lambda) \),
respectively, where \( J \) and \( J' \) are defined in 2.3. Construct \( \tilde{H}_3 \) and \( \tilde{H}'_3 \)
from \( \tilde{H}_2 \) and \( \tilde{H}'_2 \). Then \( d\tilde{H}_2 = dH_2dJ(0, -\lambda) \) as bundle maps from
\( N(K_\varepsilon, K) \cap S'_0 \to N(V, \tilde{H}_3(K)) \cap \tilde{H}_3(S'_0) \). Since we may assume that
\( \tilde{H}_3 = H_3 \) over \( K \), \( d\tilde{H}_2 = \lambda - \lambda = 0 \) in \( \pi_r(SO_{m-n-1}) \) in terms of the above trivialization of \( N(V, \tilde{H}_3(K)) \).

Since \( S(g_1) \cap K_e \) and \( K \) deformation retract to \( S_0 \) and \( D_0^{r+1} \), respectively, the bundle isomorphism \( d\tilde{H}_2 \) from \( N(K_e, K) \) to \( N(V, \tilde{H}_3(K)) \) extends to an isomorphism \( dH \) from \( N(K_e, K) \) to \( N(V, \tilde{H}_3(K)) \) covering \( \tilde{H}_3 \).

**Remark.** If we replace \( \tilde{H}_2 \) and \( \tilde{H}_3 \) in the above with \( \tilde{H}_2 J(\lambda_1, \lambda_2) \) and \( \tilde{H}_2 J'(\lambda_1, \lambda_2) \), respectively, where \( \lambda_1 \oplus \lambda_2 = 0 \) in \( \pi_r(SO_{p-r+m-n-1}) \), we can still construct \( H_3, \tilde{H}_3 \) and \( dH \).

We assume that \( H_2 \) and \( H_3 \) have been chosen so that \( H_3, \tilde{H}_3 \) and \( dH \) are constructed as required. Let \( \xi_i'' \), \( 1 < i \), be the independent vector fields of \( N(M, H_3'(K')) \) defined over \( H_3'(\Delta(g_1) \cap K') \) by

\[
\xi_i'' = dH_2'(e_i'') \text{ over } H_3'(S'(g_1) \cap K'), \quad i > 1, \text{ and } \\
\xi_i''(d) = [dfH(e_i(d)) - dfH(e_i(d_2))]/[x_1(d_1) - x_1(d_2)],
\]

where \( d \in H_3'(\Delta(g_1) \cap K') \) and \( d = fH_3(d_1) = fH_3(d_2) \). If \( \xi_i'' \), \( 1 < i \), can be extended over \( H_3'(K') \) so that \( dfH(e_i), \xi_i'' \), \( 1 < i \), are independent at every point in \( H_3'(g_1(K) - \Delta(g_1)) \), then \( dH' \) can be constructed as in 4.7 of [5] with the desired properties.

Let \( N_0 \) be the subvector bundle of \( N(M, H_3'(K')) | H_3 g_1(D_0^{r+1}) \) generated by \( \{dfH(e_i)\}_1<i \) and let \( N_1 \) be the complementary bundle of \( N_0 \). Then \( N_1 \) is generated by \( \xi_i'' = dH_2'(e_i'') \), \( 1 < i \), over \( H_3'(S_1') \) = \( H_3'(S_1') \). If these vector fields can be extended to independent vector fields of \( N_1 \), then \( \xi_i'' \), \( 1 < i \), can be extended over \( N(M, H_3'(K')) \) as required, since \( H_3' g_1(D_0^{r+1}) \cup H_3' (\Delta(g_1) \cap K') \) is a deformation retract of \( H_3'(K') \).

For a fixed trivialization of \( N_1, dH_2'(e_i'') \), \( 1 < i \), determine an element \( \lambda \in \pi_r(SO_{m-n-1}) \). We may regard \( \lambda \) as an element of \( \pi_r(SO_{p-r}) \) by the assumption that \( s_* \) is onto. Replace \( H_2 \) and \( H_3' \) with \( \tilde{H}_2 = H_2 J(\lambda, -\lambda) \) and \( \tilde{H}_3 = H_2 J'(\lambda, -\lambda) \), respectively. By the above remark, \( \tilde{H}_3, \tilde{H}_3' \) and \( d\tilde{H} \) can be constructed. Choose an \((n+1)\)-dimensional submanifold \( Q \) of \( M \) containing \( L = H_3' \) (a tubular neighborhood of \( g_1(D_0^{r+1}) \) in \( K' \)) so that \( N(Q, L) | H_3' g_1(D_0^{r+1}) = N_0 \) and \( N(M, Q) | H_3' g_1(D_0^{r+1}) = N_1 \).
Let \( N'_0 \) be subbundle of \( N(M, \bar{H}'_3(K')) | \bar{H}'_3 g_1(D_0^{r+1}) \) generated by \( \{ df d\bar{H}(e_i) \}_{1 < i} \), and let \( N'_1 \) be its complementary bundle. Then \( \bar{H}_3, \bar{H}'_3 \) and \( d\bar{H} \) can be constructed so that \( N'_1 = N(M, Q) | \bar{H}'_3 g_1(D_0^{r+1}) \) and \( H'_3 g_1(D_0^{r+1}) = \bar{H}'_3 g_1(D_0^{r+1}) \). Hence the trivialization of \( N_1 \) induces a trivialization of \( N'_1 \) and with respect to this trivialization, \( d\bar{H}_2(e_i) \), \( 1 < i \), represent the trivial element of \( r_r(SO_{m-n-1}) \). Therefore, we have \( \bar{H}_3, \bar{H}'_3, d\bar{H} \) and \( d\bar{H}' \) satisfying the requirements given at the beginning of this subsection.

Let \( H \) and \( H' \) be the diffeomorphisms covered by \( d\bar{H} \) and \( d\bar{H}' \). Then \( H' | S'_r \times D_0^{p-r} = h_\lambda \) for some \( \lambda \in \pi_r(SO_{p-r}) \). If \( g \) is constructed as in 2.1, then \( \Delta(g) \) is diffeomorphic to \( \mathcal{H}(\Delta(f), h_\lambda) \). Furthermore, we may replace \( \lambda \) with any element of \( \text{Ker}(s_\ast) + \lambda \). This completes the proof of Lemma 1.

**REMARK.** In the above construction, \( g \) depends on the null homotopy of \( f^{-1}(h | S^r \times \{0\}) \).

**2.6.** The handle attaching lemma implies the following surgery lemma. Given an embedding \( h : S^r \times D^{p-r+1} \rightarrow \text{Int}(\Delta(f)) \), let \( \chi(\Delta(f), h) \) denote the result of a surgery on \( \Delta(f) \) using \( h \).

**LEMMA 2.** Let \( f : V \rightarrow M \) be a generic map and let \( h : S^r \times D^{p-r+1} \rightarrow \text{Int}(\Delta(f)) \) be an embedding, \( r \neq 1 \). Suppose that \( \pi_0(\Delta(f), S'_r(f)) = 0, \pi_1(f) = \pi_2(f) = \pi_r(V) = \pi_{r+1}(f) = 0 \), and \( s_\ast : \pi_r(SO_{p-r}) \rightarrow \pi_r(SO) \) is onto, then there exists \( \lambda \in \pi_r(SO_{p-r}) \) and a generic map \( g \) homotopic to \( f \) such that \( \Delta(g) \) is diffeomorphic to \( \chi(\Delta(f), h_\lambda) \), where \( s_\ast(\lambda) \in \pi_r(SO_{p-r+1}) \).

Regarding \( \text{Im}(h) = h(D^r_+ \times D^{p-r+1}) \cup h(D^r_- \times D^{p-r+1}) \) by [5], we can subtract the 0-handle \( h(D^r_+ \times D^{p-r+1}) \) by [5]. This can be done since \( \pi_1(f) = 0 \). Then \( h(D^r_- \times D^{p-r+1}) \) is an \( r \)-handle in \( Cl(\Delta(f) - h(D^r_+ \times D^{p-r+1})) \) and it can be subtracted since \( \pi_{r+1}(f) = 0 \). Therefore, there exists a generic map \( f' \) homotopic to \( f \) such that \( \Delta(f') = Cl(\Delta(f) - \text{Im}(h)) \).

Define \( h' \) as the composition: \( S^r \times D^{p-r} = S^r \times D^{p-r} \subset S^r \times S^{p-r} = S^r \times \partial D^{p-r+1} \rightarrow h(S^r \times \partial D^{p-r+1}) \subset \Delta(f') \). Then \( h' \) is an embedding of \( S^r \times D^{p-r} \) into \( S'(f') \) and \( f'^{-1}(h | S^r \times \{0\}) \) is null homotopic in \( V \). By Lemma 1, there exist \( \lambda \in \pi_r(SO_{p-r}) \) and a generic map \( f'' \) homotopic to \( f' \) such that \( \Delta(f'') \) is diffeomorphic to \( \mathcal{H}(\Delta(f'), h'_\lambda) \).
We claim that $\Delta(f'')$ is diffeomorphic to $\chi(\Delta(f), h_s(\lambda))$ with the interior of an embedded $(p + 1)$-disk in the interior of $\chi(\Delta(f), h_s(\lambda))$ removed.

$$\Delta(f'') = \Delta(f') \cup (D^{r+1} \times D^{p-r}) \text{ (identified by } h'_\lambda)$$

$$\chi(\Delta(f), h_s(\lambda_1)) = \Delta(f') \cup (D^{r+1} \times S^{p-r}) \text{ (identified by } h_s(\lambda_1))$$

$$= \Delta(f') \cup [(D^{r+1} \times D^{p-r}_+) \cup (D^{r+1} \times D^{p-r}_-)] \text{ (identified by } h_s(\lambda))$$

To see the claim, remove the interior of $D^{r+1} \times D^{p-r}_-$ from $\chi(\Delta(f), h_s(\lambda))$.

Now $S'(f'') = S^p \cup S'(f)$. Find an embedded 1-handle $D^1 \times D^p$ in $\Delta(f'')$ such that $(D^1 \times D^p) \cap S'(f'') = S^0 \times D^p$, $\{1\} \times D^p \subset S^p$ and $\{-1\} \times D^p \subset S'(f)$. Since $\pi_2(f) = 0$, this handle can be subtracted, which implies that there exists a generic map $g$ homotopic to $f$ such that $\Delta(g)$ is diffeomorphic to $\chi(\Delta(f), h_s(\lambda))$, thus completing the proof of Lemma 2.

In the following subsections, we study the possibility of killing the homotopy groups of $\Delta(f)$ and $S'(f)$.

2.7. Let $f : V \to M$ be a generic map and let $i : S^r \to S'(f)$ be an embedding such that $f^{-1}i$ is null homotopic in $V$. We show that if $S(f)$ is two-sided in $D(f)$ over $f^{-1}i(S^r)$, then the normal bundle of $i$ is stably trivial. Therefore, if $2r < p$, then the normal bundle of $i$ is trivial. Note that $S(f)$ is always two-sided over $f^{-1}i(S^r)$ if $r \neq 1$.

Denote the normal bundle of $i$ by $N(S'(f), i)$ or $N(S'(f), i(S^r))$. Push the embedding $i$ into the interior of $\Delta(f)$ using a collar structure of $S'(f)$ in $\Delta(f)$. Let $i_1$ be the resulting embedding. Now $f^{-1}i_1$ can be regarded as two embeddings $i'$ and $i''$ of $S^r$ into $V$ and both are null homotopic. The extra assumption is used here when $r = 1$. Clearly, $N(\Delta(f), i_1)$ is isomorphic to $N(S'(f), i) \oplus \varepsilon^1$, where $\varepsilon^1$ is the trivial line bundle, and $N(M, \Delta(f))| i_1(S^r)$ is isomorphic to $N(V, D(f))| i'(S^r) \oplus N(V, D(f))| i''(S^r)$. Observe that the following
bundles are stably trivial.

\[ N(M, i_1) = N(\Delta(f), i_1) \oplus N(M, \Delta(f)) | i_1(S'^r), \]
\[ N(D(f), i') \oplus N(V, D(f)) | i'(S'^r) \quad \text{and} \]
\[ N(D(f), i'') \oplus N(V, D(f)) | i''(S'^r). \]

Therefore, \( N(\Delta(f), i_1) \oplus N(V, D(f)) | i'(S'^r) \oplus N(V, D(f)) | i''(S'^r) \)
is stably trivial and \( N(\Delta(f), i_1) \) is stably equivalent to \( N(D(f), i') \oplus N(D(f), i'') \). But \( N(\Delta(f), i_1), N(D(f), i') \) and \( N(D(f), i'') \) are isomorphic bundles. This implies that \( N(\Delta(f), i_1) \) is stably trivial and so is \( N(S'f(f), i) \). The above argument also shows that the normal bundle of any embedding \( i \) of \( S'^r \) into the interior of \( \Delta(f) \) is stably trivial if \( r \geq 2 \) and \( \pi_r(V) = 0. \)

2.8. By [2] and [8], it can be shown that the inclusion homomorphism \( s_*: \pi_r(SO_{r+j}) \to \pi_r(SO) \), \( 1 \leq r \) and \( -1 \leq j \), is onto in any one of the following cases.

(1) \( j \geq 1 \)
(2) \( j = 0 \) or \( -1 \) and \( r \neq 1, 3, 7. \)

2.9. The discussions above imply that if \( V \) and \( M \) are \([p + 1]/2\]-connected and \( f: V \to M \) is generic, then there is no difficulty in killing the homotopy groups of \( \Delta(f) \) and \( S'(f) \) below the middle dimension except for the fundamental group. If \( f \) is nice, then the fundamental group can also be killed. Finally, if \( f' \) is the result of a handle subtraction or attaching on a nice map \( f \), then the construction can be done such that \( f' \) is again nice.

3. Proof of Theorem 1

Clearly, (a) implies (b) by the definition, and (c) implies (a) by Theorem 1 of [9].

It suffices to show that (b) implies (c).

Suppose that \( f \) is homotopic to a nice map \( g \). If \( S'(g) \) is empty, subtract a 0-handle from \( g \) to find a nice map \( g' \) homotopic to \( g \) such that \( S'(g') \cong S^p \). To save notation, we will use \( g \) for \( g' \). By subtracting or attaching 1-handles, make \( \Delta(g) \) and \( S'(g) \) connected and \( (\Delta(g), S'(g)) \) 1-connected. We may assume that \( g \) is a nice map. Represent the
generators of \( \pi_1(S'(g)) \) by disjointly embedded circles. They have trivial normal bundle by 2.7. Attach 2-handles to kill the generators by Lemma 1 and 2.8, thus making \( \Delta(g) \) and \( S'(g) \) simply connected. By repeating the corresponding steps in higher dimensions, and by doing surgery on \( \Delta(g) \), we can make \( \Delta(g) \) and \( S'(g) \) \(((p + 1)/2) - 1\) and \(((p/2) - 1)\)-connected, respectively. We now consider two cases.

(1) \( p + 1 = 2k \).

We have \( H_i(\Delta(g), S'(g)) \cong 0 \) if \( i \neq k, 2k, \) and \( H_k(\Delta(g), S'(g)) \cong H^k(\Delta(g)) \cong \text{Hom}(H_k(\Delta(g)), \mathbb{Z}) \) is a finitely generated free abelian group since \( \Delta(g) \) and \( S'(g) \) are \((k - 1)\) and \((k - 2)\)-connected, respectively, where the homology groups are taken over \( \mathbb{Z} \). Now \( \pi_k(\Delta(g), S'(g)) \cong H_k(\Delta(g), S'(g)) \) by Hurewicz isomorphism theorem. Represent elements of a basis of \( H_k(\Delta(g), S'(g)) \) by disjointly embedded \( k \)-handles relative to \( S'(g) \) by the method of §4 of [12] and subtract them using that \( \pi_{k+1}(g) = 0 \). Then \( \Delta(g) \) is contractible and \( S'(g) \) is a homotopy sphere. It follows that \( \Delta(g) \) is diffeomorphic to the \((2n - m)\)-disk by the \( h \)-cobordism theorem ([11]).

(2) \( p + 1 = 2k + 1 \)

In this case, \( \Delta(g) \) and \( S'(g) \) are \((k - 1)\)-connected. Subtract \( k \)-handles relative to \( S'(g) \) from \( \Delta(g) \) to make \( (\Delta(g), S'(g)) \) \( k \)-connected. Then \( \Delta(g) \) and \( S'(g) \) are still \((k - 1)\)-connected. Hence we have a short exact sequence of free abelian groups.

\[
0 \to H_{k+1}(\Delta(g), S'(g)) \overset{\partial}{\to} H_k(S'(g)) \to H_k(\Delta(g)) \to 0.
\]

We again use the technique in §4 of [12] to represent elements of a basis of \( \text{Im}(\partial) \) by disjoint embeddings of \( S^k \times D^k \) into \( S'(g) \), and attach \((k + 1)\)-handles by Lemma 1 and 2.8 if \( k \neq 3, 7 \). If \( k = 3, 7 \), we use Lemma 3 (mirror handle attaching lemma) of [9] to attach \((k + 1)\)-handles. Now \( S'(g) \) is a homotopy sphere, and \( H_i(\Delta(g)) \cong 0 \) for \( i \neq k, k + 1, 0 \), and \( H_k(\Delta(g)) \cong H_{k+1}(\Delta(g)) \) is free abelian. We do surgery to eliminate \( H_k(\Delta(g)) \) using Lemma 2, 2.7 and 2.8 to make \( \Delta(g) \) contractible. Finally, apply the \( h \)-cobordism theorem to finish the proof.
4. Proof of Theorem 2

4.1. Let $f$ be an orientable immersion. By Theorem 1 and 2.9, $f$ is nicely homotopic to a pseudo-embedding $g$. Define an element $\theta(g) \in \pi_{p+2}(g)$ as follows. Let $x_0 \in V$ be a base point and regard $D^{p+2}_+ = D^{p+1}_+ \times I/(x,t) = (x,0)$ for $x \in \partial D^{p+1}_+$, where $D^{p+1}_+$ is the upper hemisphere of $\partial D^{p+2} = S^{p+1}$. There exists a commutative diagram unique up to orientation and isotopy,

$$
\begin{array}{ccc}
S^{p+1} & \overset{e}{\longrightarrow} & D^{p+2} \\
\downarrow i & & \downarrow j \\
V & \overset{g}{\longrightarrow} & M
\end{array}
$$

where $e$ is an inclusion, $j|D^{p+1}_+$ is a diffeomorphism onto $\Delta(g)$, $j(x,t) = j(x,0)$ for each $t$, and $i$ is a diffeomorphism onto $D(g)$. Let $\theta(g)$ be the above diagram. Then $\theta(g)$ represents an element of $\pi_{p+2}(g)$ up to sign.

Let $F$ be a nice homotopy from $f$ to $g$, i.e., $F|V \times \{0\} = f$ and $F|V \times \{1\} = g$. Let $u : Z(f) \rightarrow Z(F)$ be the inclusion map. We regard $\left(\{x_0\} \times I\right)*\theta(g)$ as an element of $\pi_{p+2}(F)$, where $*$ denotes the action of a path on a homotopy element. Define $\Gamma(f) = [u^{-1}\left(\left(\{x_0\} \times I\right)*\theta(g)\right)] \in \pi_{p+2}(F^+)$.  

4.2. $\Gamma(f)$ is well defined. Let $F'$ be a nice homotopy from $f$ to another pseudo-embedding $g'$. Define nice homotopy $G = F \cup (-F')$ from $g'$ to $g$ by

$$
G(x,t) = F'(x,1-2t), \quad 0 \leq t \leq 1/2 \\
G(x,t) = F(x,2t-1), \quad 1/2 \leq t \leq 1
$$

As in the proof of Theorem 1, we can find a generic map $G'$ homotopic to $G$ relative to $V \times \{0\} \cup V \times \{1\}$ such that $\Delta(G')$ is a $h$-cobordism. Here we use $3n + 3 < 2m$ and $([p+2]/2)+1$-connectedness of $V$ and $M$. By the $h$-cobordism theorem, $\Delta(G') \cong \Delta(g') \times I$. Hence $\theta(g)$ and $\theta(g')$ are freely homotopic in $(Z(G'), V \times I)$. This implies that $\Gamma(f)$ is well defined since $V$ is simply connected.
4.3. Suppose that $f$ is nicely homotopic to an embedding by a nice homotopy $F'$. Let $F$ be a nice homotopy from $f$ to a pseudo-embedding $g$. We show that $\theta(g) = 0$ in $\pi_{p+2}(g)$, which implies that $\Gamma(f) = [0]$.

Let $G = F' \cup (-F)$. By attaching and subtracting handles on $G$ as in the proof of Theorem 1, we can find a generic map $G'$ homotopic to $G$ relative to $V \times \{0\} \cup V \times \{1\}$ such that there exists a diffeomorphism $h$ from $\Delta(G')$ to $D^{p+2}$ with corners along $S^p$, where $S^{p+1} = \partial D^{p+2} = D^{p+1}_+ \cup S, D^{p+1}_-$, and $h|\Delta(g)$ is a diffeomorphism onto $D^{p+1}_+$.

Let $\theta(g)$ be represented by a diagram as in 4.1. Then $(G')^{-1}h^{-1}(D^{p+2})$ is diffeomorphic to $D^{p+2}$, and its boundary is equal to $\text{Im}(i)$. Hence $i$ is null homotopic in $\pi_1$. Furthermore, there exists a map $h': D^{p+3} \to V \times I$ such that $h'(D^{-p+2}_-) \subset V \times \{0\}$, $h'|S^{p+1} = i$ and $h'|D^{p+2}_-$ is a diffeomorphism onto $(G')^{-1}h^{-1}(D^{p+2})$, where $S^{p+2} = \partial D^{p+3} = D^{p+2}_+ \cup S, D^{p+2}_-$. Now $G'(h'|D^{p+2}_-)$ can be regarded as a map of $S^{p+2}$ into $M \times \{0\}$ and $PG'h'$ is a null homotopy of this map, where $P: M \times I \to M \times \{0\}$ is the projection. This shows that $\theta(g)$ is trivial in $\pi_{p+2}(g)$.

To show the converse, suppose that $\Gamma(f) = [0]$, then $f$ is nicely homotopic to an embedding by a handle subtraction on a pseudo-embedding which is nicely homotopic to $f$.

5. Proof of Theorem 3

5.1. Suppose that $f$ is a generic map that is not nice. Without loss of generality, we assume that $\Delta(f)$ and $S'(f)$ are connected and $(\Delta(f), S'(f))$ is 1-connected.

Define a homomorphism $\phi : \pi_1(S'(f)) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as follows. For $\alpha \in \pi_1(S'(f))$, represent $\alpha$ by an embedded circle $S$ in $S'(f)$. Then $\pi(\alpha) = (a, b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $a = 1$ if $S(f)$ is not two-sided in $D(f)$ over $f^{-1}(S)$ and $a = 0$ otherwise, and $b = 1$ if the normal bundle of $S$ in $S'(f)$ is non-trivial and $b = 0$ otherwise. Since $f$ is not nice, $\phi$ is a non-trivial homomorphism. Furthermore, $\text{Im}(\phi) = \mathbb{Z}_2 \oplus \{0\}$ if $m - n$ is even, and $\text{Im}(\phi)$ is the subgroup generated by $(1,1)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $m - n$ is odd by 4.11 of [5].

By Lemma 1 and 2.8, kill the kernel of $\phi$ thus making the inclusion homomorphism $\pi_1(S'(f)) \to \pi_1(\Delta(f))$ an isomorphism with both groups isomorphic to $\mathbb{Z}_2$. We can also kill the homotopy groups below the middle dimension as in the simply connected case. Hence after
finite steps we can find a generic map $g$ homotopic to $f$ such that $\pi_1(S'(g)) \cong \pi_1(\Delta(g)) \cong \mathbb{Z}_2$, $\tilde{\Delta}(g)$ and $\tilde{S}'(g)$ are $([p+1]/2) - 1$ and $(p/2) - 1$-connected, respectively, where $\tilde{\Delta}(g)$ denotes the universal covering space of $\Delta(g)$.

5.2. Suppose that $f$ is homotopic to a pseudo-embedding $g'$. To prove the theorem we must produce a sequence of null homotopies for the handles in any given handle decomposition of $\Delta(g)$ except for a top dimensional handle.

Let $F : V \times I \to M \times I$ be a generic homotopy (not necessarily level preserving) from $g$ to $g'$. We will homotopy $F$ relative to $V \times \{0\} \cup V \times \{1\}$ so that $\Delta(F)$ becomes a $h$-cobordism between $\Delta(g)$ and $S'(F) \cup \Delta(g')$.

By attaching and subtracting handles on $F$ away from 0 and 1-level, make $\Delta(F)$ and $S'(F)$ connected and $(\Delta(F), S'(F))$ 1-connected. Attach 2-handles along $S'(F)$ to make $\Delta(F)$ and $S'(F)$ $(p+1)$- and $(p/2) - 1$-connected, respectively, where $\Delta(F)$ denotes the universal covering space of $\Delta(g)$.

We now divide the argument into two cases.

(1) $p = 2k$.

As in 5.1, make $\Delta(F)$ $k$-connected and $S'(F)$ $(k-1)$-connected. Now $\partial \Delta(F) = D^{p+1} \cup S'(F) \cup \Delta(g)$. Let $N = D^{p+1} \cup S'(F) = Cl(\partial \Delta(F) - \Delta(g))$. In this section, the homology of a manifold or a pair is understood to be the homology of the universal covering space with coefficients $\mathbb{Z}$, i.e., $H_i(N)$ denotes actually $H_i(\tilde{N})$, where $\tilde{N}$ is the universal cover of $N$. This group can be considered as a $\mathbb{Z}\pi_1(N)$-module. Similarly, the cohomology of a manifold $N$ (or a pair) is the cohomology induced from the cellular chain complex of $\mathbb{Z}\pi_1(N)$-modules of the universal covering space of $N$ with coefficients $\mathbb{Z}\pi_1(N)$ unless it is said otherwise. (See [12] for more information.)
From the connectivities of $\Delta(g)$, $\Delta(F)$ and $S'(F)$, $H_i(\Delta(F), N) \cong 0$ for all $i$ except for $i = k + 1$ by duality. Also $H^{k+2}(\Delta(F), N) = H_k(\Delta(F), \Delta(g)) = 0$. (This is true for every $\mathbb{Z}\mathbb{Z}_2$-module as coefficients.) By Lemma 2.3 of [12], $H_{k+1}(\Delta(F), N)$ is a finitely generated stably free $\mathbb{Z}\mathbb{Z}_2$-module. By attaching trivial $(k+1)$-handles along $S'(F)$, make $H_{k+1}(\Delta(F), N)$ a free $\mathbb{Z}\mathbb{Z}_2$-module. Now represent, a set of basis elements of $H_{k+1}(\Delta(F), N)$ by disjoint $(k+1)$-handles relative to $S'(F)$, and subtract the handles.

It follows that $\Delta(F')$ is an $h$-cobordism between $\Delta(g)$ and $S'(F)$. Hence $\Delta(F)$ is diffeomorphic to $\Delta(g) \times I$ since $Wh(\mathbb{Z}_2) = 0$.

We now need a lemma whose proof is postponed until 5.3.

**Definition.** Let $C$ be a compact manifold with a non-empty boundary, and let $A$ be a compact codimension 0 submanifold of $\partial C$ or empty. Define the wedge $W(C, A)$ of $C$ relative to $A$ as the manifold with corners obtained by rotating $C$ by $90^\circ$ about $A$. If $A = \emptyset$, then $W(C, A)$ is defined to be $C \times I$. Let $W_0(C, A)$ be the copy of $C$ at $0^\circ$ in $W(C, A)$ and $W_1(C, A)$ the copy of $C$ at $90^\circ$. Note that $W(C, A)$ contains naturally a copy of $A$ and we denote it by $A$ again. Observe that $W(C, A)$ can be regarded as $C \times I$ with $A \times I$ collapsed to $A \times \{0\}$.

**Remark.** $S^r+1(r \geq 0)$ may be regarded as the union of four copies of $W(D^r, S^{r-1})$.

**Lemma 3.** Let $F : V \times I \rightarrow M \times I$ be a generic map such that $F(V \times \{0\}) \subset M \times \{0\}$ and $F(V \times \{1\}) \subset M \times \{1\}$. Let $g$ be $F|V \times \{0\}$. Let $C$ be a compact connected manifold with non-empty boundary, and let $A$ be a compact submanifold of $\partial C$ of codimension 0 or empty. Suppose that there is an embedding $h : W(C, A) \rightarrow \Delta(F)$ such that $h(W(C, A)_0) \subset \Delta(g)$, $h(A) = S'(g) \cap h(W(C, A)_0)$, $h(W(C, A)_1) \subset S'(F)$ and $W(W(C, A)) \cap M \times \{1\} = \emptyset$. Then there exists a generic map $F' : V \times I \rightarrow M \times I$ such that $\Delta(F') = Cl(\Delta(F) - Im(h))$, $g' = F'|V \times \{0\}$ is a generic map obtained from $g$ by a sequence of handle subtractions, $\Delta(g) = Cl(\Delta(g) - h(W(C, A)_0))$ and $F'|V \times \{1\} = F|V \times \{1\}$.

Using the lemma, we finish the proof of the first case. Given a handle decomposition of $\Delta(g)$ relative to $S'(g)$ with a top dimensional handle $H$ (this is a $(2n - m)$-disk and there is one in any handle decomposition), define a trivialization $T : W(\Delta(g), S'(g)) \rightarrow \Delta(F)$ by
the $h$-cobordism theorem such that $T|W_0(\Delta(g), S'(g))$ is the identity and $T(W_1(H, \emptyset)) = \Delta(g') \times \{1 - \varepsilon\} \subset M \times \{1 - \varepsilon\}$ for some small $\varepsilon > 0$, where we assume that $F$ is a product for $t \geq 1 - 2\varepsilon$, $t \in I$ and $W(H, \emptyset)$ is regarded as a subset of $W(\Delta(g), S'(g))$.

Let $C = Cl(\Delta(g) - H)$, $A = S'(g) \subset \partial C$ and $h : W(C, A) \to \Delta(F)$ be an embedding defined by $h = T|W(C, A)$, where $W(C, A)$ is regarded as a subset of $W(\Delta(g), S'(g))$. By Lemma 3 we can subtract all the handles except for $H$.

(2) $p = 2k + 1$.

After the initial modification $S'(g)$ is $(k - 1)$-connected, and $\Delta(g)$ $k$-connected. We may also assume that $S'(F)$ and $\Delta(F)$ are $k$-connected. Let $N = Cl(\partial \Delta(F) - \Delta(g))$. By subtracting $(k + 1)$-handles in $\Delta(F)$ relative to $S'(F)$, make $(\Delta(F), N)$ $(k + 1)$-connected with the same connectivity conditions on $\Delta(F)$ and $(\Delta(F), \Delta(g))$.

Since $H_i(\Delta(F), \Delta(g)) = 0$, $i \leq k$, $H_i(\Delta(F), \Delta(g)) \cong H^{2k+3-i}(\Delta(F), N) \cong 0$, $i \geq k + 2$, and $H^{k+2}(\Delta(F), \Delta(g)) \cong H_{k+1}(\Delta(F), N) \cong 0$ (with any $\mathbb{Z} \mathbb{Z}_2$-module as coefficients), $H_{k+1}(\Delta(F), \Delta(g))$ is a finitely generated stably free $\mathbb{Z} \mathbb{Z}_2$-module. By attaching trivial $(k + 1)$-handles along $S'(F)$, make $H_{k+1}(\Delta(F), \Delta(g))$ a free $\mathbb{Z} \mathbb{Z}_2$-module. Represent the basis elements by a disjoint union $H$ of $(k + 1)$-handles in $\Delta(F)$ relative to $\Delta(g)$. Let $W = Cl(\Delta(F) - H)$ and $N' = Cl(\partial W - N)$. From the long exact sequence of the triple $(\Delta(F), H \cup \Delta(g), \Delta(g))$, it can be seen that $(W; N', N)$ is an $h$-cobordism. Since $Wh(\mathbb{Z}_2) = 0$, there exists a diffeomorphism $T : W(N', \partial N') \to W$ such that $T|W_0(N', \partial N')$ is the identity and $(T|W_1(N', \partial N'))^{-1}(\Delta(g')) \subset W_1(N' - \Delta(g), \phi)$. Furthermore, we may assume that $H \cap \Delta(g)$ is contained in a top dimensional handle of any handle decomposition of $\Delta(g)$ relative to $S'(g)$ since $\Delta(g)$ is $k$-connected.

Let $C = Cl(\Delta(g)$-the above top dimensional handle). Then we have $T(W(C, S'(g))) \subset \Delta(F)$, $T|W_0(C, S'(g))$ is the identity, and $T(W_1(C, S'(g))) \subset S'(F)$. As in the first case apply Lemma 3 to subtract all the handles in $C$. This completes the proof.

5.3. Proof of lemma 3.

It suffices to prove the lemma when $C$ is an $r$-handle. Suppose that $H = D^r \times D^{2n-m-r}$ is an $r$-handle of $\Delta(g)$ relative to $S'(g)$ with $S^{r-1} \times D^{2n-m-r}$ as $A$. Put $q = 2n - m$. Let $h : W(D^r \times D^{q-r}, S^{r-1} \times$
Define reflection $\alpha : M \times [-1, 1] \to M \times [-1, 1]$ by $\alpha(x, t) = (x, -t)$, $(x, t) \in M \times [-1, 1]$. Now construct generic map $G : V \times [-1, 1] \to M \times [-1, 1]$ by $G[V \times [0, 1] = F$ and $(G[V \times [-1, 0])(x, t) = \alpha(F(x, -t))$.

There is a natural $(r + 1)$-handle in $\Delta(G)$ relative to $S'(G)$ obtained from $h$ as follows. Regard $D^{r+1} \times D^{q-r}$ as the union of two copies $W_1$ and $W_2$ of $W$ with $W_0^1$ identified with $W_0^2$ ($W_0$ is the 0-level of $W$.) Define $(r + 1)$-handle $h' : (D^{r+1} \times D^{q-r}, S^r \times D^{q-r}) \to (\Delta(G), S'(G))$ by $h'|W^1 = h$ and $h'|W^2 = \alpha h$.

Then $h'$ determines a commutative diagram $\theta(h')$ as in 4.1.

$$\begin{array}{ccc}
S^{r+1} & \xrightarrow{e} & D^{r+2} \\
\downarrow i & & \downarrow j \\
V \times [-1, 1] & \xrightarrow{g} & M \times [-1, 1]
\end{array}$$

In the above diagram, $e$ is an inclusion, $j|D^{r+1}_+^1$ is a diffeomorphism onto $h'(D^{r+1} \times \{0\})$ and $i$ is a diffeomorphism onto $G^{-1}(h'(D^{r+1} \times \{0\}))$.

Using the symmetry of $G$ in $V \times \{0\}$, it is easy to produce a null homotopy of $\theta(h')$ which restricts to a null homotopy of $\theta(H)$ in 0-level. We subtract the handle $h'$ from $G$ to get a generic map $G'$. The construction of $G_\tau$, $0 \leq \tau \leq 1$ in §4 of [5] can be done so that $G_\tau$ preserves 0-level for all time (see §5 of [5]). Let $F' = G'|V \times [0, 1]$. Then $F'$ has the desired properties of the lemma.

References


Department of Mathematics
University of Wisconsin-Parkside
Kenosha, WI 53141
U.S.A.