THE GROUP OF BOUNDED ELEMENTS IN A LIE GROUP

WOON GAB JEONG AND JAIHAN YOON

Let $G$ be a real analytic group, $A$ an arbitrary, not necessarily connected subgroup of $\text{Aut}(G)$. By $B(G,A)$, we mean the group of all elements $x \in G$ such that $Ax$ is relatively compact.

In [1] it is shown that $B(G,A)$ is closed when $S$ is faithfully representable, and in [2] the result is is generalized to the case when $R \cap S$ is finite where $R, S$ denote the solvable radical and a Levi factor of $G$, respectively.

The purpose of this note is to prove the following theorem, which generalizes the above result.

**THEOREM.** Let $G$ be an analytic group. Then $B(G,A)$ is closed for every subgroup $A$ of $\text{Aut}(G)$.

In proving the theorem we shall use the following two propositions in [2].

**PROPOSITION 1.** Let $G$ be an analytic group such that $R \cap S$ is finite, where $R, S$ denote the solvable radical and a Levi factor of $G$, respectively. Then $B(G,A)$ is closed.

**PROPOSITION 2.** Let $\pi : G' \rightarrow G$ be a covering homomorphism of semi-simple analytic groups. Let $A'$ be an arbitrary subgroup of $\text{Aut}(G')$. Then, for each $x \in G'$, $A'x$ is relatively compact if and only if $\pi(A'x)$ is relatively compact.

Let $G$ be a given analytic group and let $\phi : \tilde{G} \rightarrow G$ be a universal covering group of $G$. Let $\tilde{G} = \tilde{R}\tilde{S}$ be a Levi decomposition of $\tilde{G}$. Then $G = RS$ is a Levi decomposition of $G$, where $R = \phi(\tilde{S})$, $S = \phi(\tilde{S})$. Let $G' = R \times_{\sigma} \tilde{S}$ be a semidirect product, where $\sigma : \tilde{S} \rightarrow \text{Aut}(R)$ is a continuous homomorphism defined by $\sigma(\tilde{s})(r) = \phi(\tilde{s})r(\phi(\tilde{s}))^{-1}$, $r \in R$, $\tilde{s} \in \tilde{S}$.

Received December 26, 1990.
For $\alpha \in \text{Aut}(G)$, choose $\tilde{\alpha} \in \text{Aut}(\tilde{G})$ such that $\phi \circ \tilde{\alpha} = \alpha \circ \phi$ and define $\alpha' : G' \rightarrow G'$ by $\alpha'(r, \tilde{s}) = (\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s}))), p_2(\tilde{\alpha}(\tilde{s})))$, where $p_1 : \tilde{G} \rightarrow \tilde{R}$ and $p_2 : \tilde{G} \rightarrow \tilde{S}$ are the projections to the first and second factor of $\tilde{G} = \tilde{R}\tilde{S}$, respectively $i.e.$, $p_1(\tilde{r}\tilde{s}) = \tilde{r}$ and $p_2(\tilde{r}\tilde{s}) = \tilde{s}$, for $\tilde{s} \in \tilde{R}$, $\tilde{s} \in \tilde{S}$.

**Lemma 3.** Let $G$ be an analytic group and let $\pi : G' \rightarrow G$ be a mapping defined by $\pi(r, \tilde{s}) = r\phi(\tilde{s})$. Then $\pi$ is a covering homomorphism such that $\pi \circ \alpha' = \alpha \circ \pi$ and $\alpha' \in \text{Aut}(G')$ for each $\alpha \in \text{Aut}(G)$.

**Proof.** It is clear that $\pi$ is a continuous open epimorphism with discrete kernel, and hence it is a covering homomorphism. By definition, it is clear that $\alpha'$ is continuous and direct calculation shows that $\pi \circ \alpha' = \alpha \circ \pi$. Therefore we have $(\alpha\beta)' = \alpha' \circ \beta'$ for $\alpha, \beta \in \text{Aut}(G)$ and $(\text{id}_G)' = \text{id}_G$, by the unique lifting property. For $x \in G'$, consider the mapping $f : G' \rightarrow G'$ defined by $f(y) = (\alpha'(x))^{-1}\alpha'(xy)$, $y \in G'$. The unique lifting property shows that $f = \alpha'$, proving the Lemma.

**Proposition 4.** Let $\mathcal{A}' = \{\alpha' : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is an arbitrary subgroup of $\text{Aut}(G)$, and let $x = (r, \tilde{s}) \in G'$. Then $\mathcal{A}'x$ is relatively compact if and only if $\pi(\mathcal{A}'x)$ is relatively compact, where $\pi$ is the covering homomorphism defined in Lemma 3.

**Proof.** The only if part is trivial. To prove the converse let $\iota : \tilde{S} \rightarrow \tilde{G}$ be the inclusion and let $\psi : G \rightarrow G/R$ be the canonical epimorphism. Then $\psi \circ \phi \circ \iota : \tilde{s} \rightarrow G/R$ is a covering homomorphism and a simple calculation shows that $\{p_2 \circ \tilde{\alpha} \circ \iota : \alpha \in \mathcal{A}\}$ is a subgroup of $\text{Aut}(\tilde{S})$. Since $\{((\psi \circ \phi \circ \iota)(p_2 \circ \tilde{\alpha} \circ \iota(\tilde{s})) : \alpha \in \mathcal{A}\} = \psi(\pi(\mathcal{A}'x))$ is relatively compact by assumption, we see that $\{p_2 \circ \tilde{\alpha} \circ \iota(\tilde{s}) : \alpha \in \mathcal{A}\}$ is relatively compact by Proposition 2. Therefore the relative compactness of $\{\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s})) : \alpha \in \mathcal{A}\}$ follows from the relative compactness of $\{\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s}))\phi(p_2 \circ \tilde{\alpha}(\tilde{s})) : \alpha \in \mathcal{A}\}$. This shows that $\mathcal{A}'x$ is relatively compact, which completes the proof.

**Theorem 5.** $B(G, \mathcal{A})$ is closed for every analytic group $G$ and a subgroup $\mathcal{A}$ of $\text{Aut}(G)$.

**Proof.** Proposition 4 shows that $B(G', \mathcal{A}') = \pi^{-1}(B(G, \mathcal{A}))$. Since $BG', \mathcal{A}')$ is closed by Proposition 1, it follows that $B(G, \mathcal{A})$ is closed.
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References


Department of Mathematics
Suwon University
Suwon 445-743, Korea

and

Department of Mathematics
Seoul National University
Seoul 151-742, Korea