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Real Hypersurfaces with Infinitesimal Invariant Ricci Tensor of a Complex Projective Space*

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0. Introduction

In the study of real hypersurfaces of a complex projective space P_nC, Takagi [10] showed that all homogeneous real hypersurfaces could be divided into six types. Moreover, he [9] verified that if a real hypersurfaces M of P_nC has two or three distinct constant principal curvatures, then M is locally congruent to one of the following homogeneous ones ;

(A_1) a geodesic hypersphere,

(A_2) a tube over a totally geodesic P_kC $(1\leqq k\leqq n-2)$,

(B) a tube over a complex quadric Q_{n-1} .

In what follows the induced almost contact metric structure of the real hypersurface M of P_nC is denoted by (ϕ, g, ξ, η). The structure vector ξ is said to be principal if $A\xi=\alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha=\eta(A\xi)$.

By making use of this notion, Kimura [3] proved that M has constant principal curvatures and ξ is principal if and only if M is locally congruent to a homogeneous hypersurface.

The Ricci tensor of type (1, 1) of the hypersurface is denoted by S. We say that the structure vector ξ is Ricci-principal if $S\xi=\nu\xi$, where $\nu=g(S\xi, \xi)$. It is clear that if ξ is principal, then it is Ricci-principal.

Many subjects for real hypersurfaces of a complex projective space were investigated from various of view [1], [2], [4], [5], [6], [8], ets. One of which done by Maeda and Udagawa [6] asserts that the real hypersurface of P_nC is of type A_1 or A_2 if and only if the structure tensor ϕ is invariant under the infinitesimal transformation with respect to ξ.

In the present paper, we shall verify the following ;

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Theorem. Let M be a real hypersurface of a complex projective space P_nC. Then M is of type A_1 or A_2 if and only if the Ricci tensor S is invariant under the local one parameter group of transformations generated by the structure vector field ξ and ξ is Ricci-principal with corresponding positive Ricci curvature.

1. Preliminaries

We begin with recalling basic formulas on real hypersurfaces of a Kaehlerian manifold. Let $\widetilde{M}$ be a 2n-dimensional Kaehlerian manifold equipped with Kaehlerian structure (F, G). Let M be a real hypersurface of $\widetilde{M}$ covered by a system of coordinate neighborhoods $\{U ; X^h\}$ and immersed isometically in $\widetilde{M}$ by the immersion $i : M \to \widetilde{M}$.
When the argument is local, we may identify M with i(M). We represent the immersion i locally by

$$y^A=y^A(x^1, \cdots , x^{2n-1}), \ (A=1, \cdots , 2n)$$

and put $B_j{}^A=\partial_j y^A$, $(\partial_j=\partial / \partial x^j)$, then $B_J=(B_J{}^A)$ are $(2n-1)-$ lineary independent local tangent vector fields of M. A unit normal C to M may then be choosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji}=G(B_j, B_i)$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhood ;

$$(1.1) \qquad FB_i=\sum_h \phi_i{}^h B_h+\eta_i C, \qquad FC=-\sum_i \xi^i B_i,$$

where we have put $\phi_{ji}=G(FB_j, B_i)$ and $\eta_i=G(FB_i, C)$, ξ^h being components of a vector field ξ associated with η_i and $\phi_{ji}=\sum_r \phi_j{}^r g_{ri}$.
Here and throughout this paper, the indices h, i, j, $\cdots$ run over the range $\{1, 2, \cdots , n-1\}$ and the summation convention will be used those indices. By the properties of the almost Hermitian structure F, it is evident that ϕ_{ji} is skew-symmetric. A tensor field of type (1.1) with components $\phi_j{}^h$ will be denoted by ϕ.

By the properties of the almost complex structure F, the following relations are then given

$$\phi_i{}^r \phi_r{}^h=-\delta_i^h+\eta_i\xi^h, \ \xi^r \phi_r{}^h=0, \ \eta_r \phi_i{}^r=0, \ \eta_i\xi^i=1,$$

that is, the set (ϕ, g, ξ) defines an almost contact metric structure. We write ξ_i instead of

η_i .

Denoting by ∇_j the Van der Waerden–Bortolotii covariant differentiation formed with g_{ji}, the equations of Gauss and Weingarten for M are respectively obtained ;

$$(1.2)\qquad \nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j^r B_r,$$

where A_{ji} are components of the second fundamental form σ, $A=(A_j^h)$ which is related by $A_{ji} = A_j^r g_{ri}$ being the shape operator derived from C. By means of (1.1) and (1.2) the covariant derivatives of the structure tensors are yielded ;

$$(1.3)\qquad \nabla_j \phi_{ih} = -A_{ji}\xi_h + A_{jh}\xi_i, \qquad \nabla_j \xi_i = -A_{jr}\phi_i^{\ r}.$$

In the sequel, the ambient space M is assumed to be of constant holomorphic sectional curvature 4, which is called a complex projective space and denoted by P_nC. Then the equations of Gauss and Codazzi for M are respectively given ;

$$(1.4)\qquad R_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih} + A_{kh}A_{ji} - A_{jh}A_{ki},$$

$$(1.5)\qquad \nabla_k A_{ji} - \nabla_j A_{ki} = \xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj},$$

where R_{kjih} are the components of the Riemannian curvature tensor R of M. From (1.4) we see that the Ricci tensor S of M is given by

$$(1.6)\qquad S_{ji} = (2n+1)g_{ji} - 3\xi_j\xi_i + hA_{ji} - A_{ji}^2,$$

where $A_{ji}^2 = A_{jr}A_i^r$, S_{ji} denotes components of S and h is the trace of the shape operator A.

2. Real hypersurfaces with $L_\xi S=0$

The Lie derivative $L_\xi S$ of the Ricci tensor S with respect to ξ is given by

$$(2.1)\qquad L_\xi S_{ji} = \xi^k \nabla_k S_{ji} + (\nabla_j \xi^r) S_{ir} + (\nabla_i \xi^r) S_{jr}.$$

Substituting the second equation of (1.3) into this, we have

$$L_\xi S_{ji} = \xi^k \nabla_k S_{ji} - A_{jt}\phi^{rt} S_{ir} - A_{it}\phi^{rt} S_{jr},$$

which together with (1.6) gives

$$(2.2)\qquad L_\xi S_{ji}=-3(U_j\xi_i+U_i\xi_j)+A_{is}{}^2A_{jr}\phi^{sr}+A_{ir}A_{js}{}^2\phi^{sr}$$
$$-(2n+1)(A_{jr}\phi_i{}^r+A_{ir}\phi_j{}^r)+(\xi^r h_r)A_{ji}$$
$$+h\xi^k\nabla_k A_{ji}-\xi^h(\nabla_k A_{jr})A_i{}^r-\xi^k(\nabla_k A_{ir})A_j{}^r,$$

where $U_j=\xi^r\nabla_r\xi_j$ and $A_{ji}{}^2=A_{jr}A_i{}^r$.

Lemma 1. ([7]) Let M be a real hypersurface of P_nC. If ξ is principal, then we have

$$(2.3)\qquad A_{jr}A_{is}\phi^{sr}=\frac{\alpha}{2}(A_{jr}\phi_i{}^r-A_{ir}\phi_j{}^r)+\phi_{ij}$$

and α is locally constant on M.

From $A_{jr}\xi^r=\alpha\xi_j$, we have

$$(\nabla_j A_{ir})\xi^r=A_{ir}A_{js}\phi^{rs}-\alpha A_{jr}\phi_i{}^r$$

beacuse of (1.3), which together with (1.5) and (2.3) implies that

$$(2.4)\qquad \xi^k\nabla_k A_{ji}=-\frac{\alpha}{2}(A_{jr}\phi_i{}^r+A_{ir}\phi_j{}^r).$$

Proof of Theorem. Now, suppose that M is of type A_1 or A_2, namely $A\phi=\phi A$ ([8]). Then, it is evident that ξ is principal. Thus (2.3) turns out to be $A^2=\alpha A+I-\eta\xi$, I being the unit tensor and hence

$$(2.5)\qquad h_2=\alpha h+2(n-1),$$

where $h_2=A_{ji}A^{ji}$. Thus, it is clear that $h+4(n+1)=0$. By using these facts we can, taking account of (2.2) easily see that $L_\xi S=0$.

Because of (1.6), it is seen that the Ricci curvature ν with respect to the structure vector ξ is given by $\nu=-\alpha^2+h\alpha+2(n-1)$ or using (2.5) we have $\nu=||A_{ji}-\alpha\xi_j\xi_i||^2$. If $\nu=0$ then we have $A_{ji}=\alpha\xi_j\xi_i$, which together with (1.3) gives $\nabla_j\xi_i=0$. This contradicts because of (1.5). Thus, it follows that we have $\nu>0$ on M.

Conversely, we suppose that $L_\xi S=0$ and $S\xi=\nu\xi$, $\nu>0$ hold on the real hypersurface M. Then we have

$$(\nabla_j S_{ir})\xi^r+S_{ir}\nabla_j\xi^r=\nu_j\xi_i+\nu\nabla_j\xi_i,$$

which implies

$$\xi^j(\nabla_j S_{ir})\xi^r + S_{ir}U^r = (\xi^r \nu_r)\xi_i + \nu U_i.$$

Multiplying ξ^i to (2.1) and summing for i, we obtain

$$\xi^k(\nabla_k S_{jr})\xi^r + S_{jr}U^r = 0$$

where, we have used (1.3), $L_\xi S=0$ and $S\xi=\nu\xi$.

From the last two equations, it is evident that $(\xi^r\nu_r)\xi_i+\nu U_i=0$ and hence $U_i=0$ because of $\nu>0$. Thus, we can, using the second equation of (1.3), easily verify that ξ is principal. Accordingly, we have $h^r\xi_r=0$ by means of (2.4). Therefore, (2.2) is reduced to

$$A_{is}{}^2 A_{jr}\phi^{sr} + A_{ir}A_{js}{}^2\phi^{sr} - (2n+1)(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r)$$
$$+ h\xi^k\nabla_k A_{ji} - \xi^k(\nabla_k A_{jr})A_i{}^r - \xi^k(\nabla_k A_{ir})A_j{}^r = 0,$$

which together with (2.3) and (2.4) yields

$$\{h\alpha+4(n+1)\}(A\phi - \phi A)=0.$$

From (1.6) we have $\nu=-\alpha^2+h\alpha+2(n-)$ and hence $h\alpha+4(n+1)\neq 0$ because $\nu>0$. Thise completes the proof of the theorem.

References

[1] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in a complex projective space, *Trans. Amer. Math. Soc.*, *269*(1982), 481~499.

[2] U-H. Ki, Cyclic-parallel real hypersurfaces of a complex space form, *Tsukuba J. Math.*, *12*(1988), 259~268.

[3] M. Kimura, Real hypersurfaces and complex sbumanifold in complex projective space, *Trans. Amer. Math. Soc.*, *296*(1986), 137~149.

[4] M. Kimura, Some real hypersurfaces of a complex projective space, *Saitama Math. J.* *5*(1987), 1~5.

[5] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, perprint.

[6] S. Maeda and S. Udagawa, On real hypersurfaces of a complex projective space Ⅱ, preprint.

[7] Y. Maeda, On real hypersurfaces of a complex projective space, *J. Math. Soc. Japan*, *28*(1976), 529~540.

[8] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.*, *212*(1975), 355~364.

[9] R. Takagi, On homogeneous real hypersurfaces of a complex projective space, *Osaka J. Math.*, *10*(1973), 495~506.

[10] R. Takagi, Real hypersurfaces in a complex projective with constant principal curvatures I, II, *J. Math. Soc. Japan*, 27(1975), 43~53, 507~516.