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On H-Fuzzy Relations on Sets*

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We introduce the category $Rel(H)$ of H-fuzzy relational spaces on sets and the subcategory $Rel_R(H)$ of $Rel(H)$ consisting of fuzzy reflexive relational spaces on sets. We will show that the category $Rel(H)$ ($Rel(H)$, resp.) are topological, cartesian closed and final episinks in $Rel(H)$ ($Rel_R(H)$, resp.) are preserved by pullbacks. Moreover, we will show that the category $Rel_R(H)$ is topological universe over *Set*.

1. Introduction

L.A. Zadeh [9] introduced a concept of a fuzzy relation naturally, as a generalization of crisp relations in fuzzy set theory. The purpose of this paper is to study structures of fuzzy relations on a set in a categorical point of view.

In section 2, we introduce the category $Rel(H)$ of H-fuzzy relational spaces and relation preserving maps. And we will show that $Rel(H)$ is topological, cartesian closed and final episinks in $Rel(H)$ are preserved by pullbacks. In section 3, we introduce the subcategory $Rel_R(H)$ of $Rel(H)$ consisting of H-fuzzy reflexive relational spaces and relation preserving maps. And we will show that $Rel_R(H)$ is topological universe over *Set*.

In this paper, unless mentioned, we will use H as a complete Heyting algebra [4]:

A lattice H is called a ***complete Heyting algebra***,

if (i) H is a complete lattice,

(ii) For any a, $b \in H$, the set $\{x \in H \mid x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$, i.e. $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

For general background for fuzzy set theory, we refer to A. Kauffmann [5] and for general categorical background to H. Herrlich and G.E. Strecker [3].

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2. The category *Rel(H)*

Definition 2.1. *An H—fuzzy relation R* on a set X is H−fuzzy set in $X \times X$. In this case, the pair(X, R) is called an *H—fuzzy relational space over* X. We will denote the function(called the *membership function*) characterizing R by μ_R, where μ_R is a function from $X \times X$ into H.

Definition 2.2. Let(X, R_X) and (Y, R_Y) be any H−fuzzy relational spaces. A map $f:(X, R_X) \to (Y, R_Y)$ is called a *relation preserving map*, if $\mu_{R_X} \leqq \mu_{R_Y} \circ f^2$, where $f^2 = f \times f$.

From the above definitions, we can form a concrete category *Rel(H)* consisting of all H−fuzzy relational spaces and relation preserving maps between them. Each *Rel(H)*−morphism will be called a *Rel(H)−map.*

For any set X, any family $((X_i, R_i))_{i \in I}$ of H−fuzzy relational spaces indexed by a class I, and any family $(f_i : X \to X_i)_{i \in I}$ of maps, let $\mu_R : X \times X \longrightarrow H$ be the function defined by $\mu_R = \bigwedge_I \mu_R^\circ f^2$ Then we can easily see that R is an initial *Rel(H)*−structure of X with respect to $(X, (f_i)_{i \in I}, ((X_i, R_i)) \in I)$. Hence we obtain the following result :

Proposition 2.3. The category *Rel(H)* is topological over *Set*. Moreover, *Rel(H)* is well−powered and co−will−powered.

Theorem 2.4. Final episinks in *Rel(H)* are preserved by pullbacks.

Proof. Let$(g_\lambda : (X_\lambda, R_\lambda) \longrightarrow (Y, R_Y))_{\lambda \in \Lambda}$ be any final episink in *Rel(H)* and $f:(W, R_W) \longrightarrow (Y, R_Y)$ any *Rel(H)−map.* For each $\lambda \in \Lambda$, let $U_\lambda = \{(\omega, x_\lambda) \in W \times X_\lambda \mid f(\omega) = g_\lambda(x_\lambda)\}$, $\mu_{R_{U_\lambda}} = \mu_{R_W \times R_\lambda}|_{U_\lambda \times U_\lambda}$, $e_\lambda : U_\lambda \longrightarrow W$ and $P_\lambda : U_\lambda \longrightarrow X_\lambda$ denote the usual projections of U_λ . Then for each $\lambda \in \Lambda$, $e_\lambda : (U_\lambda, R_{U_\lambda}) \longrightarrow (W, R_W)$ and $P_\lambda : (U_\lambda, R_{U_\lambda}) \longrightarrow (X_\lambda, R_\lambda)$ are *Rel(H)*−maps and the following diagram is a pullback square in *Rel(H)*:

$$\begin{array}{ccc}
(U_\lambda, R_{U_\lambda}) & \xrightarrow{\quad P_\lambda \quad} & (X_\lambda, R_\lambda) \\
{\scriptstyle e_\lambda}\downarrow & & \downarrow{\scriptstyle g_\lambda} \\
(W, R_W) & \xrightarrow[\quad f \quad]{} & (Y, R_Y).
\end{array}$$

We will show that $(e_\lambda : (U_\lambda, R_{U_\lambda}) \longrightarrow (W, R_W))_{\lambda \in \Lambda}$ is a final episink in *Rel(H)*. For any $\omega \in W$, since$(g_\lambda)_{\lambda \in \Lambda}$ is an episink, there exists $\lambda \in \Lambda$ such that $g_\lambda(x_\lambda) = f(\omega)$ for some $x_\lambda \in X_\lambda$. Thus $(\omega, x_\lambda) \in U_\lambda$ and $\omega = e_\lambda(\omega, x_\lambda)$. Hence $(e_\lambda)_{\lambda \in \Lambda}$ is an episink.

Suppose R^* is the final H−fuzzy relation on W with respect to $(e_\lambda)_{\lambda\in\Lambda}$. Then for any $(\omega, \omega')\in W\times W$,

$$\begin{aligned}
&\mu_{R_W}(\omega, \omega')\\
&=\mu_{R_W}(\omega, \omega')\wedge\mu_{R_W}(\omega, \omega')\\
&\leqq\mu_{R_W}(\omega, \omega')\wedge\mu_{R_Y}\circ f^2(\omega, \omega')\\
&=\mu_{R_W}(\omega, \omega')\wedge[\bigvee_{\lambda\in\Lambda}\bigvee_{(x, x')\in g_\lambda^{-1^2}(f(\omega), f(\omega'))}\mu_{R_\lambda}(x_\lambda, x_\lambda')](g_\lambda^{-1^2}=g_\lambda^{-1}\times g_\lambda^{-1})\\
&=\bigvee_{\lambda\in\Lambda}\bigvee_{(x_\lambda, x_\lambda')\in g_\lambda^{-1^2}(f(\omega), f(\omega'))}[\mu_{R_W}(\omega, \omega')\wedge\mu_{R_\lambda}(x_\lambda, x_\lambda')]\\
&=\bigvee_{\lambda\in\Lambda}\bigvee_{((\omega, x_\lambda), (\omega', x_\lambda'))\in e_\lambda^{-1^2}(\omega, \omega')}[\mu_{R_W}(\omega, \omega')\wedge\mu_{R_\lambda}(x_\lambda, x_\lambda')]\\
&=\bigvee_{\lambda\in\Lambda}\bigvee_{((\omega, x_\lambda), (\omega', x_\lambda'))\in e_\lambda^{-1^2}(\omega, \omega')}\mu_{R_\lambda}((\omega, x_\lambda), (\omega', x_\lambda'))
\end{aligned}$$

Thus $\mu_{R_W}(\omega, \omega')\leqq\mu_{R^*}(\omega, \omega')$, i.e., $\mu_{R_W}\leqq\mu_{R^*}$. On the other hand, since $(e_\lambda: (U_\lambda, R_{U_\lambda})\to(W, R^*))$ is final, $1_W: (W, R^*)\to(W, R_W)$ is a *Rel(H)*−map and hence $\mu_{R^*}\leqq\mu_{R_W}$. Hence $R_W=R^*$. This completes our proof. / / /

The category *Rel(H)* is not properly fibred over *Set*, since for any singleton set {a}, the H−fuzzy relation R on {a} is not unique. Hence by proposition 2.3 and 2.4, we can see that the category *Rel(H)* satisfies all the conditions of a topological universe over *Set* in the sence of L.D. Nel [7]. Moreover, we note that proposition 2.3 and theorem 2.4 hold, even if H is a complete lattice.

Theorem 2.5. The category *Rel(H)* is cartesian closed over *Set*.

Proof. Since *Rel(H)* has products by proposition 2.3, it is enough to show that *Rel(H)* has exponential objects.

For any H−fuzzy relational spaces $X=(X, R_X)$ and $Y=(Y, R_Y)$, let Y^X be the set of all maps from X into Y and define $\mu_R: Y^X\times Y^X\to H$ by for all $(f, g)\in Y^X\times Y^X$,

$\mu_R(f, g)=\vee\{h\in H|\ \mu_{R_X}(x, y)\wedge h\leqq\mu_{R_Y}(f(x), g(y))$ for all $(x, y)\in X\times X\}$.

Let $Y^X=(Y^X, R)$. By the definition of R, $\mu_R(x, y)\wedge\mu_R(f, g)\leqq\mu_{R_Y}(f(x), g(y))$, for all $(x, y)\in X\times X$.

Define $e_{X,Y}: X\times Y^X\to Y$ by $e_{X,Y}(x, f)=f(x)$ for all $(x, f)\in X\times Y^X$. Take any $((x, f), (y, g))\in(X\times Y^X)\times(X\times Y^X)$. Then,

$$\begin{aligned}
\mu_{R_X\times R}((x, f), (y, g))&=\mu_{R_X}(x, y)\wedge\mu_R(f, g)\\
&\leqq\mu_{R_Y}(f(x), g(y))
\end{aligned}$$

$$=\mu_{R_Y}(e_{X,Y}(x,\ f),\ e_{X,Y}(y,\ g))$$
$$=\mu_{R_Y}\circ e_{X,Y}((x,\ f),\ (y,\ g)).$$

Hence $e_{X,Y}$: $X\times Y^X\rightarrow Y$ is a *Rel(H)*−map.

For any H−fuzzy relational space $Z=(Z,\ R_Z)$, let h: $X\times Z\rightarrow Y$ be a *Rel(H)*−map. Define $\bar{h}$: $Z\rightarrow Y^X$ by for each $z\in Z$ and each $x\in X$, $[\bar{h}(z)](x)=h(x,\ z)$. For any $z,\ z'\in Z$ and any $x,\ x'\in X$, since h: $X\times Z\rightarrow Y$ is a *Rel(H)*−map,

$$\begin{aligned}\mu_{R_X\times R_Z}((x,\ z),\ (x',\ z'))&=\mu_{R_X}(x,\ x')\wedge\mu_{R_Z}(z,\ z')\\ &\leqq\mu_{R_Y}\circ h^2((x,\ z),\ (x',\ z'))\\ &=\mu_{R_Y}(h(x,\ z),\ h(x',\ z'))\\ &=\mu_{R_Y}([\bar{h}(z)](x),\ [\bar{h}(z')](x')).\end{aligned}$$

Thus by the definition of R, $\mu_{R_Z}(z,\ z')\leqq\mu_R(\bar{h}(z),\ \bar{h}(z')=\mu_R\circ\bar{h}^2(z,\ z')$.
Hence $\bar{h}$: $Z\rightarrow Y^X$ is a *Rel(H)*−map. Moreover, $\bar{h}$ is a unique *Rel(H)*−map such that $e_{X,Y}(1_X\times\bar{h})=h$. This completes our proof. / / /

Remark 2.6. A ***relational space*** is a pair (X, R), where X is a set and $R\subset X\times X$. A map f: $(X,\ R)\rightarrow(Y,\ S)$ is said to be ***relation preserving*** if $(f\times f)(R)\subset S$. We now form a category ***Rel*** of all relational spaces and relation preserving maps.

Define a map E: ***Rel***$\rightarrow$***Rel(H)*** by $E(X,\ R)=(X,\ R)$ and $E(f)=f$. Then E is a full embedding functor. Hence we may consider ***Rel*** as a subcategory of ***Rel(H)***. Moreover, ***Rel*** is a bicoreflective subcategory of ***Rel(H)***: For any $(X,\ R_X)\in$ ***Rel(H)***, let $R=\{(x,\ y)\in X\times X\mid\mu_{R_X}(x,\ y)=1\}$. Then the identity map 1_X: $E(X,\ Y)\rightarrow(X,\ R_X)$ is an E−couniversal map for $(X,\ R_X)$. This fact shows that the notion of H−fuzzy relations is a good extention of that of relations.

3. The category $\boldsymbol{Rel_R(H)}$

Definition 3.1. An H−fuzzy relation R on a set X is said to be ***reflexive*** if $\mu_R(x,\ x)=1$ for all $x\in X$.

The class of all H−fuzzy reflexive relational spaces and *Rel(H)*−maps between them forms a subcategory of *Rel(H)* and we will denote it by $Rel_R(H)$.

We can easily obtain the following results :

Proposition 3.2. (1) The category $\mathit{Rel}_R(H)$ is properly fibred over Set. Moreover, $\mathit{Rel}_R(H)$ is a full isomorphism closed subcategory of $\mathit{Rel}(H)$.
(2) The category $\mathit{Rel}_R(H)$ is closed under the formation of initial sources in $\mathit{Rel}(H)$.

From theorems 2.5 and 2.6 in [6], we obtain the following :

Proposition 3.3. (1) The category $\mathit{Rel}_R(H)$ is a bireflective subcategory of $\mathit{Rel}(H)$.
(2) The category $\mathit{Rel}_R(H)$ is topological over Set.

Theorem 3.4. Final episinks in $\mathit{Rel}_R(H)$ are preserved by pullbacks.

Proof. Let $(g_\lambda : (X_\lambda, R_\lambda) \to (Y, R_Y))_{\lambda\in\wedge}$ be any final episink in $\mathit{Rel}_R(H)$ and $f : (W, R_W) \to (Y, R_Y)$ any $\mathit{Rel}(H)$−map, where (W, R_W) is an H−fuzzy reflexive relational space. For each $\lambda \in \wedge$, let us take U_λ, R_{U_λ}, e_λ and P_λ as in the proof of theorem 2.4. Since $\mathit{Rel}_R(H)$ is closed under the formation of pullbacks in $\mathit{Rel}(H)$ (cf. theorem 2.4. in [6]), it is enough to show that $(e_\lambda)_{\lambda\in\wedge}$ is final :

Suppose R^* is the final H−fuzzy relation on W with respect to $(e_\lambda)_{\lambda\in\wedge}$. Then for any $(\omega, \omega') \in (W \times W - \triangle_W)$, where $\triangle_W = \{(\omega, \omega) \mid \omega \in W\}$,

$$
\begin{aligned}
&\mu_{R_W}(\omega, \omega') \\
&= \mu_{R_W}(\omega, \omega') \wedge \mu_{R_W}(\omega, \omega') \\
&\leq \mu_{R_W}(\omega, \omega') \wedge \mu_{R_Y} \circ f^2(\omega, \omega') \\
&= \mu_{R_W}(\omega, \omega') \wedge \Big[\bigvee_{\lambda\in\wedge} \bigvee_{(x_\lambda, x_\lambda') \in g_\lambda^{-1^2}(f(\omega), f(\omega'))} \mu_{R_\lambda}(x_\lambda, x_\lambda') \Big] \\
&= \bigvee_{\lambda\in\wedge} \bigvee_{(x_\lambda, x_\lambda') \in g_\lambda^{-1^2}(f(\omega), f(\omega'))} [\mu_{R_W}(\omega, \omega') \wedge \mu_{R_\lambda}(x_\lambda, x_\lambda')] \\
&= \bigvee_{\lambda\in\wedge} \bigvee_{((\omega, x_\lambda), (\omega', x_\lambda')) \in e_\lambda^{-1^2}(\omega, \omega')} [\mu_{R_W}(\omega, \omega') \wedge \mu_{R_\lambda}(x_\lambda, x_\lambda')] \\
&= \bigvee_{\lambda\in\wedge} \bigvee_{((\omega, x_\lambda), (\omega', x_\lambda')) \in e_\lambda^{-1^2}(\omega, \omega')} \mu_{R_{U_\lambda}}((\omega, x_\lambda), (\omega', x_\lambda'))
\end{aligned}
$$

Thus $\mu_{R_W}(\omega, \omega') \leq \mu_{R^*}(\omega, \omega')$, i.e., $\mu_{R_W} \leq \mu_{R^*}$. On the other hand, by a similar argument in the proof of theorem 2.4, we have $\mu_{R^*} \leq \mu_{R_W}$ on $W \times W - \triangle_W$. Hence $\mu_{R^*}(\omega, \omega') = \mu_{R_W}(\omega, \omega')$ for any $(\omega, \omega') \in (W \times W - \triangle_W)$. Now take $\omega \in \triangle_W$. Then clearly, $\mu_{R^*}(\omega, \omega) = 1 = \mu_{R_W}(\omega, \omega')$, Hence $R^* = R_W$.

This completes the proof. / / /

From proposition 3.2 (1), 3.3 (2) and theorem 3.4, we obtain the following :

Theorem 3.5. The category $Rel_R(H)$ is a topological universe over *Set* in the sense of L.D. Nel [7]. Hence, $Rel_R(H)$ is a concrete quasitopos in the sense of E.J. Dubuc. [1].

Even if H is a complete lattice, we can see that proposition 3.2, 3.3 and theorems 3.4, 3.5 hold.

Remark 3.6. In [8], Y. Noh obtained exponential objects in $Rel_R(I)$, where $I=[0, 1]$. Now, we show that his construction of an exponential object in $Rel_R(I)$ is applicable to the case of $Rel_R(H)$, in general : For any $X=(X, R_X)$, $Y=(Y, R_Y)\in Ob(Rel_R(H))$, let $Y^X=\hom(X, Y)$ and define an H-fuzzy relation R on Y^X by

$$\mu_R(f, g)=\begin{cases} 1 & \text{if } D(f, g)=\phi \\ \bigvee_{(x, y)\in D(f, g)} \mu_{R_Y}(f(x), g(y)) & \text{if } D(f, g)\neq\phi \end{cases}$$

where $D(f, g)=\{(x, y)\in X\times X \mid \mu_{R_X}(x, y)>\mu_{R_Y}(f(x), g(y))\}$. Then by the definition of R, R is an H-fuzzy reflexive relation on Y^X. Let $Y^X=(Y^X, R)$. Now we define a map $e_{X,Y}: X\times Y^X \to Y$ by $e_{X,Y}((a, f))=f(a)$, for each $(a, f)\in X\times Y^X$.
Take any $(a, f), (b, g) \in X\times Y^X$.
If $D(f, g)=\phi$, Then

$$\begin{aligned}\mu_{R_X\times R}((a, f), (b, g))&=\mu_{R_X}(a, b)\wedge\mu_R(f, g)\leq\mu_{R_Y}(f(a), g(b))\\ &=\mu_{R_Y}\circ e^2_{X,Y}((a, f), (b, g)).\end{aligned}$$

If $D(f, g)\neq\phi$, then

$$\begin{aligned}&\mu_{R_X\times R}((a, f), (b, g))\\ &=\mu_{R_X}(a, b)\wedge[\bigwedge_{(x, y)\in D(f, g)} \mu_{R_Y}(f(x), g(y))]\\ &\leq\mu_{R_Y}(f(a), g(b))\\ &=\mu_{R_Y}\circ e^2_{X,Y}((a, f), (b, g)).\end{aligned}$$

Thus $e_{X,Y}$ is a relation preserving map on $X\times Y^X$. For any $Z=(Z, R_Z)$ and any $Rel_R(H)$-map $h: X\times Z\to Y$, define $\bar{h}: Z\to Y^X$ by $[\bar{h}(c)](a)=h(a, c)$ for each $c\in Z$ and each $a\in X$. For each $c\in Z$ and any $a, b\in X$, $\mu_{R_Y}\circ\bar{h}^2(c)(a, b) = \mu_{R_Y}(\bar{h}(c)(a), \bar{h}(c)(b)) = \mu_{R_Y}(h(a, c), h(b, c)) = \mu_R\circ h^2((a, c), (b, c)) \geq \mu_{R_X\times R_Z}((a, c), (b, c)) = \mu_{R_X}(a, b)\wedge\mu_{R_Z}(c, c) = \mu_{R_X}(a, b)$. Thus $\bar{h}(c): X\to Y$ is a $Rel(H)$-map. Hence $\bar{h}$ is well-defined. Taked any $c, c'\in Z$.

If $D(\bar{h}(c), h(c'))=\phi$, then

$$\mu_R \circ \bar{h}^2(c, c')=\mu_R(\bar{h}(c), \bar{h}(c'))=1 \geqq \mu_{R_Z}(c, c').$$

If $D(h(c), h(c'))\neq\phi$, then

$$\begin{aligned}\mu_R(\bar{h}(c), \bar{h}(c')) &= \bigwedge_{(a, b)\in D(\bar{h}(c), \bar{h}(c'))} \mu_{R_Y}(\bar{h}(c)(a), \bar{h}(c')(b)) \\ &= \bigwedge_{(a, b)\in D(\bar{h}(c), \bar{h}(c'))} \mu_{R_Y}(h(a, c), h(b, c')) \\ &\geqq \bigwedge_{(a, b)\in D(\bar{h}(c), \bar{h}(c'))} [\mu_{R_X}(a, b)\wedge\mu_{R_Z}(c, c')]\end{aligned}$$

Note that for any $(a, b)\in D(\bar{h}(c), \bar{h}(c')$,

$$\begin{aligned}\mu_{R_X}(a, b) &> \mu_{R_Y}(\bar{h}(c)(a), \bar{h}(c')(b)) \\ &= \mu_{R_Y}(h(a, c), h(b, c')) \\ &\geqq \mu_{R_X}(a, b)\wedge\mu_{R_Z}(c, c').\end{aligned}$$

Hence $\mu_{R_X}(a, b)>\mu_{R_Z}(c, c')$. Thus $\mu_R(\bar{h}(c), \bar{h}(c'))\geqq\mu_{R_Z}(c, c')$.
Hence $\bar{h}$ is a *Rel(H)*-map. Moreover, $\bar{h}$ is unique and $e_{X,Y}\circ(1_X \times \bar{h})=h$.
This completes the proof. / / /

Remark 3.7. (1) We note that exponential objects in *Rel*$_R$*(H)* is quite different from those in *Rel(H)* constructed in theorem 2.5.
(2) The category *Rel*$_R$*(H)* has no subobject classifier.

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