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Nearly Periodic Flows

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1. Introduction

K.H. Ahamed and S. Elaydi [1] introduced the concept of near periodicity and investigated some relations between near periodicity and other dynamical properties. In particular, they show that a periodic flow with no rest point is necessarily nearly periodic. In this paper, we get the sufficient conditions for a periodic flow to be nearly periodic (Theorem 3.1, Theorem 3.3) and an equivalent condition for a discrete flow, induced from a given flow, to be nearly periodic (Theorem 3.4).

Throughout this paper we let (X, R, f) denote a flow on a locally compact metric space with a metric d. For any two points x in X and t in R, f(x, t) will denoted by xt. The orbit, orbit closure, limit set, prolongational limit, and prolongation relations on X are denoted, respectively, by C, K, L, J and D. The unilateral versions of these relations carry the appropriate + or − superscript. A point x in X is said to be of characteristic $0^+(0^-)$ if $D^+(x)=K^+(x)(D^-(x)=K^-(x))$ and x is of characteristic 0 if it is of characteristic 0^+ and 0^-. A point x in X is said to be positively (negatively) nearly periodic provided $D^+(x)=L^+(x)(D^-(x)=L^-(x))$ and it is said to be nearly periodic if $D^+(x)=L^+(x)$ and $D^-(x)=L^-(x)$. A point x in X is said to be positively (negatively) Poisson stable if $x\in L^+(x)$ and $x\in L^-(x)$. A point x in X is said to be (positively) (negatively) Liapunov stable if for any $\varepsilon>0$ there exists $\delta<0$ such that $d(x, y)<\delta$ implies $d(xt, yt)<\varepsilon$ for all $t\in(R^+)(R^-)R$. If one of the properties above holds at each point of the phase space X, then the flow on X is said to have that property.

2. Preliminaries

We may consult [2] for the basic dynamical system concepts used herein.

Lemma 2.1. Let (X, R, f) be a flow. Then the set of all nearly periodic points in X is invariant.

Proof. Let $x \in X$ be nearly periodic, $t \in R$ and $y \in D^+(xt)$. Then there exist sequences $\langle x_i \rangle$ in X and $\langle t_i \rangle$ in R^+ such that $x_i \to xt$, $x_i t_i \to y$. And then $x_i(-t) \to x$ and $(x_i(-t))t_i = (x_i t_i)(-t) \to y(-t)$. $y(-t) \in D^+(x) = L^+(x)$ implies $y \in L^+(x)t = L^+(x) = L^+(xt)$. Thus $D^+(xt) = L^+(xt)$. Similarly, one can show that $D^-(xt) = L^-(xt)$. / / /

Lemma 2.2. Let $x \in X$. Then the following are equivalent.

(1) x is positively nearly periodic,

(2) x is of characteristic 0^+ and positively Poisson stable,

(3) x is of characteristic 0^+ and $x \in J^+(x)$,

(4) $J^+(x) = K^+(x)$.

Analogous results hold for the negative version.

Proof. (1) ⇒ (2). Since $D^+(x) = L^+(x) \subset K^+(x)$, x is of characteristic 0^+. And $x \in D^+(x) = L^+(x)$.

(2) ⇒ (3). $L^+(x) \subset J^+(x)$ implies $x \in J^+(x)$.

(3) ⇒ (4). $J^+(x) \subset D^+(x) = K^+(x)$. From $x \in J^+(x)$, $K^+(x) \subset J^+(x)$. Thus $J^+(x) = K^+(x)$.

(4) ⇒ (1). Assume that $x \notin L^+(x)$. Then $C^-(x) \cap L^+(x) = \phi$. Since $x \in K^+(x) = J^+(x)$ and $J^+(x)$ is invariant, $C^-(x) \subset J^+(x) = K^+(x) = C^+(x) \cup L^+(x)$. From $C^-(x) \cap L^+(x) = \phi$, $C^-(x) \subset C^+(x)$. And then x is periodic.
It contradicts to the fact $x \notin L^+(x)$. Thus $x \in L^+(x)$. Since $L^+(x)$ is closed and invariant, $K^+(x) \subset L^+(x)$. And so $D^+(x) = C^+(x) \cup J^+(x) = J^+(x) = K^+(x) \subset L^+(x)$. Therefore x is positively nearly periodic. / / /

It is known that the periodicity of a point does not imply the near periodicity and vice versa.

Lemma 2.3.[1]. A periodic flow with no rest points is necessarly nearly periodic. / / /

However, the above lemma does not true if the flow has rest points.

Example 2.4. Consider a flow defined on $X = \{(r, \theta) \mid 0 \leqq r \leqq 1, 0 \leqq \theta \leqq 2\pi\}$ by the differential equations (polar coordinates)

$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = r(1-r).$$

Then the flow is periodic. But, for each point p in $A=\{(1, \theta)\ 0\leqq\theta\leqq 2\pi\}$, $L^+(p)=\{p\}$ and $D^+(p)=A$. Therefore it is not nearly periodic.

Now we describe a relation between a flow and discrete flows induced from this flow. Let (X, R, f) be a flow. Then it is clear that for each $\tau\in R$, the map $f_\tau : X\times Z\to X$ given by $f_\tau(x, n)=f(x, n\tau)=x(n\tau)$ is continuous, $f_\tau(x, 0)=x$, and $f_\tau(f_\tau(x, m), n)=f_\tau(x, m+n)$ for each x in X and m, n in Z. Consequently (X, Z, f_τ) is a discrete flow, and it will be called the discrete flow induced by τ from (X, R, f). As in the flow (X, R, f), we define maps L_τ^+, L_τ^-, D_τ^+, D_τ^- from X into 2^X by setting for each $x\in X$, $L_\tau^+(x)(L_\tau^-(x))=\{y\in X$: there is a sequence $\langle n_i\rangle$ in $Z^+(Z^-)$ such that $n_i\to+\infty(-\infty)$ and $f_\tau(x, n_i)\to y\}$, $D_\tau^+(x)D_\tau^-(x))=\{y\in X$: there is a sequence $\langle x_i\rangle$ in X and a sequence $\langle n_i\rangle$ in $Z^+(Z^-)$ such that $x_i\to x$ and $f_\tau(x_i, n_i)\to y\}$.

Lemma 2.5. If (X, Z, f_τ) is nearly periodic for $\tau>0$, then (X, R, f) is nearly periodic.

Proof. Let $x\in X$ and $y\in D^+(x)$. Then there exist sequences $\langle x_i\rangle$ in X with $x_i\to x$ and $\langle t_i\rangle$ in R^+ such that $x_it_i\to y$. For each i, there exists n_i in Z^+ such that $t_i=n_i\tau+r_i$ with $0\leqq r_i\leqq\tau$. We may assume that $r_i\to r\in[0, \tau]$. Then $x_i r_i\to xr$ and $f_\tau(x_i r_i, n)=x_i t_i\to y$. Thus $y\in D_\tau^+(xr)$. If $D_\tau^+(xr)=L_\tau^+(xr)$, then $y\in D_\tau^+(xr)=L_\tau^+(xr)\subset L^+(xr)=L^+(x)$ and so x is positively nearly periodic with pespect to (X, R, f).

Now we show that $D_\tau^+(xr)=L_\tau^+(xr)$. For $z\in D_\tau^+(xr)$, there exist sequences $\langle x_i\rangle$ in X with $x_i\to xr$ and $\langle n_i\rangle$ in Z^+ such that $f_\tau(x_i, n_i)\to z$. And so $f_\tau(x_i(-r), n_i)\to z(-r)$. Since $z(-r)=D_\tau^+(x)=L_\tau^+(x)$, $y\in L_\tau^+(x)r=L_\tau^+(xr)$. Similarly x is negatively nearly periodic with respect to (X, R, f).
/ / /

In general, the converse of lemma 2.5 does not hold. We give below an example.

Example 2.6. Consider the differential system defined on $X=\{(r, \theta)\mid r\rangle 1/2, 0\leqq\theta\leqq 2\pi\}$ by the differential equations (polar coordinates)

$$\frac{dr}{dt}=0,\qquad \frac{d\theta}{dt}=r+2\pi-1.$$

Let (X, R, f) be a flow by $f((r, \theta), t)=(r, (r+2\pi-1)t+\theta)$. Then, for each point $p=(r, \theta)$ in X, $L^+(p)=D^+(p)=\{(r, \theta)\mid 0\leqq\theta\leqq 2\pi\}$ and so the flow is nearly periodic. Let $\tau=1/4$ and $p=(1, 0)$ a point in X. Then for the discrete flow $(X, Z, f_{1/4})$, $K_{1/4}^+(p)=\{(1, 0), (1, \frac{\pi}{2}), (1, \pi), (1, \frac{3\pi}{2})\}$.

Let $p_n=(1+\frac{\pi}{4n}, 0)$. Then $p_n \to p$ and

$$f_{1/4}(p_n, 4n)=f((1+\frac{\pi}{4n}, 0), n)=(1+\frac{\pi}{4n}, \frac{\pi}{4}+2n\pi)\to(1, \frac{\pi}{4}).$$

And so $(1, \frac{\pi}{4})\in D^+_{1/4}(p)-L^+_{1/4}$. Therefore the point p is not nearly periodic with respect to the flow $(X, Z, f_{1/4}(p)$.

3. Theorems

Theorem 3.1. If a periodic flow is Liapunov stable, then it is nearly periodic.

Proof. Let $x\in X$, $y\in D^+(x)$ and $\tau>0$ be a period of x. Since x is Liapunov stable, for each $\varepsilon>0$, there exists $\delta>0$ such that $d(x, z)<\delta$ implies $d(xt, zt)<\frac{\varepsilon}{2}$ for all $t\in R^+$. $y\in D^+(x)$ means $B(y, \frac{\varepsilon}{2})\cap B(x, \delta)R^+\neq 0$.

Let $z\in B(x, \delta)$ and $s\in R^+$ with $zs\in B(y, \frac{\varepsilon}{2})$. For every $t\in R^+$ there exists a a positive integer n such that $n\tau>t$. Then $d(x(n\tau+s), y)=d(xs, y)\leq d(xs, zs)+d(zs, y)<\varepsilon$. Therefore for each $t\in R^+$, $B(y, \varepsilon)\cap x[t, \infty)\neq\phi$ and so $y\in L^+(x)$. Similarly we can show that x is negatively nearly periodic. / / /

Theorem 3.2. (INTEGRAL CONTINUITY CONDITION). For any point x in X, any number $T>0$ and any $\varepsilon>0$, there exists a $\delta>0$ such that $d(xt, yt)<\varepsilon$ for all $y\in X$ and $t\in R$ which satisfy the inequalities $d(x, y)<\delta$ and $0\leq t\leq T(-T\leq t\leq 0)$.

Proof. See the Theorem 4.2 of Chapter I in [3]

Theorem 3.3. Let (X, R, f) be a periodic flow and $\tau(x)$ a period of a x for each x in X. If there exists $M>0$ such that $\tau(x)\leq M$ for every x in X, then the flow is nearly periodic.

Proof. Suppose $y\in D^+(x)-L^+(x)$ for some x in X. Since $y\notin L^+(x)$, there exist $\varepsilon>0$ and $T>M$ such that $B(y, \varepsilon)\cap x[T, \infty)=\phi$. By integral continuity condition (Theorem 3.2), there exists a $\delta>0$ such that $d(x, z)<\delta$ implies $d(xt, zt)<\frac{\varepsilon}{2}$ for every $t\in[0, M]$. From $y\in D^+(x)$, $B(y, \frac{\varepsilon}{2})\cap B(x, \delta)R^+\neq\phi$.

Let $x \in B(x, \delta)$ and $s \in R^+$ with $zs \in B(y, \frac{\varepsilon}{2})$. Take m, n in R^+ such that $s = m\tau(z) + r$, $0 \leqq r < \tau(z)$ and $n\tau(x) > T$. Then $d(x(n\tau(x)), z) = d(x, z) < \delta$ implies $d(x(n\tau(x)+r), zs) = d(x(n\tau(x)+r), z(m\tau(z)+r)) = d(xr, zr) < \frac{\varepsilon}{2}$. And so, $d(x(n\tau(x)+r), y) \leqq d(x(n\tau(x)+r), zs) + d(zs, y) < \varepsilon$. Since $n\tau(x)+r > T$, $B(y, \varepsilon) \cap x[T, \infty) \neq \phi$. It leads a contradiction. Therefore x is positively nearly periodic. Similarly $D^-(x) = L^-(x)$. / / /

Lemma 3.4. Let (X, Z, f_τ) be an induced discrete flow from the given flow (X, R, f). If $K_\tau^+(x) = K^+(x)$ for $x \in X$ and $\tau > 0$, then x is positively Poisson stable on (X, Z, f_τ).
Analogous result holds for negative version.

Proof. Suppose $x \notin L_\tau^+(x)$. Then there exist a neighborhood U of x and $m \in Z^+$ such that $n > m$ implies $f_\tau(x, n) \notin U$. Therefore x is not a rest point on (X, R, f). And $x \neq f_\tau(x, i)$ for each $i \in Z^+$. (For otherwise, there exists $k \in Z^+$ such that $ki > m$. And then $f_\tau(x, ki) = x \notin U$. It is a contradiction). $V = U - \{f_\tau(x, k) \mid k = 1, 2, \cdots, m\}$ is a neighborhood of x.

(i). Suppose x is periodic with a period $T > 0$ on (X, R, f). Take r with $0 < r < \min\{T, \tau\}$ and $x[0, r] \subset V$. Since $xr \neq x$, $W = V - \{x\}$ is a neighborhood of xr. And $W \cap C_\tau^+(x) = \phi$. Since $f_\tau(x, n) \notin W$ for all $n \in Z^+$. Then $xr \notin K_\tau^+(x) = K^+(x)$. It is a contradiction.

(ii). Suppose x is not periodic on (X, R, f). Take s with $0 < s < \tau$ and $x[0, s] \subset V$. Since $xs \neq x$, $W = V - \{x\}$ is a neighborhood of xs with $W \cap C_\tau^+(x) = \phi$. Thus $xs \notin K_\tau^+(x) = K^+(x)$. It is a contradiction.

Therefore $x \in L_\tau^+(x)$. / / /

Theorem 3.5. Let for each $x \in X$, $K_\tau^+(x) = K^+(x)$ and $K_\tau^-(x) = K^-(x)$ for $\tau \in R^+$. Then a flow (X, R, f) is nearly periodic if and if the induced discrete flow (X, z, f_τ) is nearly periodic.

Proof. The if part is lemma 2.5. Conversely, let $x \in X$. Since $K^+(x) \subset D^+(x) = L^+(x)$, $D_\tau^+(x) \subset D^+(x) = K^+(x) = K_\tau^+(x)$, that is, x is characteristic 0^+ with respect to the flow (X, Z, f_τ). From lemma 3.4, $x \in L^+(x)$ and from lemma 2.2, x is positively nearly periodic on (X, Z, f_τ). By the analogous process, x is negatively nearly periodic. / / /

References

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