FIXED POINT THEOREMS IN NONARCHIMEDEAN MENGER SPACES

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Nonarchimedean probabilistic metric spaces and some topological preliminaries on them were first studied by Istrătescu and Crivăț [7] (see, also [6]). Some fixed point theorems for mappings on nonarchimedean Menger spaces have been proved by Istrătescu [4,5] as a result of the generalizations of some of the results of Sehgal and Bharucha–Reid [11] and Sherwood [12]. Achari [1] studied the fixed points of quasi–contraction type mappings in nonarchimedean probabilistic metric spaces and generalized the results of Istrătescu [5]. Recently, Singh and Pant [15] have established common fixed point theorems for weakly commuting quasi–contraction pair of mappings on nonarchimedean Menger spaces.

In the present paper we replace the condition of commutativity by that of preorbital commutativity, a condition weaker than commutativity and prove some common fixed point theorems for S–type and F–type quasi–contraction triplets (see definitions 1,4) of self–mappings on nonarchimedean Menger spaces. Extension to uniform spaces and application to product spaces of one of the results are also given.

The following definition is due to Istrătescu [5].

**Definition 1.** Let $F_{u,v}$ denote the value at $u,v \in X \times X$ of the function $F : X \times X \to \mathcal{L}$, the collection of all distribution functions. A nonarchimedean Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a nonarchimedean probabilistic metric space and $t$ is a $t$–norm such that the nonarchimedean triangle inequality

$$F_{u,w}(\max\{x, y\}) \geq t\{F_{u,v}(x), F_{v,w}(y)\}$$

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holds for all $u, v, w \in X$ and $x, y \geq 0$. Hereafter $X$ stands for a nonarchimedean Menger space.

We introduce the following:

**Definition 2.** Three mappings $f, g, h : X \to X$ are called an $S$-type quasi-contraction triplet $(f, g; h)$ ($S$ after Singh [13]) iff there exists a constant $k \in (0, 1)$ such that for every $u, v$ in $X$,

\[(2) \quad F_{fu, fv}(kx) \geq \max\{F_{hu, hv}(x), F_{fu, hu}(x), F_{gu, hu}(x), F_{fu, hv}(x), F_{gu, hv}(x)\}\]

holds for all $x > 0$.

On the lines of Tiwari and Singh [17] we have the following:

**Definition 3.** Assume that a sequence of $O(f, g; hu_0)$ converges to a point $u$ in $h(X)$ and $B_u = \{z : hz = u\}$ which is nonempty. For a positive integer $N$, define $A_N = \{u_n \in A : n \geq N\}$.

Then the mapping $f$ and $h$ will be called $(f, g; hu_0)$ preorbitally commuting if the restrictions of $f$ and $h$ on $A_N \cup B_N$ are commuting for some positive integer $N$.

Now we introduce the following:

**Definition 4.** Three mappings $f, g, h$ on a nonarchimedean Menger space $(X, \mathcal{F}, t)$ are called an $F$-type quasi-contraction triplet $(f; g, h)$ ($F$ after Fisher [3]) iff there exists a constant $k \in (0, 1)$ such that for every $u, v$ in $X$,

\[(3) \quad F_{fu, fv}(kx) \geq \max\{F_{gu, hv}(x), F_{fu, gu}(x), F_{fuv, hv}(x), F_{fu, hv}(x), F_{fv, gu}(x)\}\]

holds for all $x > 0$.

The following definition is also on the lines of Tiwari and Singh [17].

**Definition 5.** Assume that a subsequence $\{fu_n\}$ of $O(fu_0; g, h)$ converges to a point $u$ in $g(X) \cap h(X)$ and $B_u = \{z : gz = u\} \cup \{w : hw = u\}$. Let for a positive integer $N$, $A_N = \{u_n \in A : n \geq N\}$.

The mappings $f$ and $g$ will be called $(fu_0; g, h)$ preorbitally commuting if the restrictions of $f$ and $g$ on $A_N \cup B_u$ are commuting for some positive integer $N$.

**Theorem 1.** Let $(X, \mathcal{F}, t)$ be a nonarchimedean Menger space, where $t$ is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$ and $(f, g; h)$ an
**S-type quasi-contraction triplet of self mappings on X.** If there is a point \( u_0 \) and a sequence \( \{u_n\} \) in \( X \) such that

(i) \( hu_{2n+1} = fu_{2n}, hu_{2n+2} = gu_{2n+1}, \quad n = 0, 1, 2, \ldots; \)

(ii) \( h(X) \) is \((f, g; hu_0)\)-orbitally complete;

(iii) \( h \) is \((f, g; hu_0)\)-preorbitally commuting with \( f \) and \( g \), then \( f, g \) and \( h \) have a unique common fixed point and \( \{hu_n\} \) converges to the fixed point.

**Proof.** By (1), (2) and (i)

\[
F_{fu_{2n+2}}(kx) = F_{fu_{2n+1}}(kx) \\
\geq \max\{F_{fu_{2n+1}}, F_{fu_{2n}}, F_{fu_{2n+1}}, F_{fu_{2n+2}}, F_{fu_{2n}}, F_{fu_{2n+1}}(x), F_{fu_{2n+1}}, F_{fu_{2n}}(x)\} \\
\geq \max\{F_{fu_{2n+1}}, F_{fu_{2n+2}}, F_{fu_{2n+1}}, F_{fu_{2n+2}}, F_{fu_{2n+2}}, F_{fu_{2n+1}}(x), F_{fu_{2n+2}}, F_{fu_{2n+1}}(x)\} \\
\geq \max\{F_{fu_{2n+1}}, F_{fu_{2n+2}}, F_{fu_{2n+1}}, F_{fu_{2n+2}}, F_{fu_{2n+2}}, F_{fu_{2n+1}}(x), F_{fu_{2n+2}}, F_{fu_{2n+1}}(x)\} \\
= F_{fu_{2n+2}}, F_{fu_{2n+1}} \quad (kx).
\]

Therefore, by the lemma (Singh and Pant, [14]), \( \{hu_n\} \) is a Cauchy sequence and by virtue of (ii) converges to a point \( p \) (say) in \( h(X) \). This implies the existence of a point \( z \) in \( X \) such that \( hz = p \). Now, let \( U_{hz}(\varepsilon, \lambda) \) be a neighbourhood of \( hz \). Then for \( \varepsilon, \lambda > 0 \), there exists an integer \( N \) such that

\[
(4) \quad F_{hu_{2n}}, hz(\varepsilon/k) > 1 - \lambda \quad \text{and} \quad F_{hu_{2n+1}}, hu_{2n+1}(\varepsilon/k) > 1 - \lambda
\]

for all \( n \geq N \).

Again by (1), (2) and (i),

\[
F_{fu_{2n+2}}, gz(\varepsilon) = F_{fu_{2n+1}}, gz(\varepsilon) \\
\geq \max\{F_{fu_{2n+1}}, hz(\varepsilon/k), F_{hu_{2n+1}}, hu_{2n}(\varepsilon/k), F_{gz}, hz(\varepsilon/k), F_{hu_{2n+1}}, hz(\varepsilon/k)\} \\
\geq \max\{F_{hu_{2n}}, hz(\varepsilon/k), F_{hu_{2n}}, hz(\varepsilon/k)\} \\
> 1 - \lambda, \text{ by (4)}.
\]

So, \( gz = hz \). Similarly \( fz = hz \). Thus \( fz = gz = hz = p \). Now by (iii),

\[
fhz = hfz = fhz = hhz \quad \text{and} \quad ghz = hgz = ggz = hhz.
\]
Putting \( u = p \) and \( v = z \) in (2), we get
\[
fp = gz = p.
\]

Therefore \( fp = gp = hp = p \).

Uniqueness of \( p \) easily follows from (2).

**Theorem 2.** Let \((X, \mathcal{F}, t)\) be a nonarchimedean Menger space, where \( t \) is continuous and satisfies \( t(x, x) \geq x \) for every \( x \in [0, 1] \) and \((f; g, h)\) an \( F\)-type quasi-contraction triplet of self mappings on \( X \). If there is a point \( u_0 \) and a sequence \( \{u_n\} \) in \( X \) such that

(i) \( gu_{2n+1} = fu_{2n}, hu_{2n+2} = fu_{2n+1}, \quad n = 0, 1, 2, \ldots \);

(ii) \( g(X) \cap h(X) \) is \((fu_0; g, h)\)-orbitally complete;

(iii) \( f \) is \((fu_0; g, h)\)-preorbitally commuting with each of \( g \) and \( h \), then \( f, g, h \) have a unique common fixed point and \( \{fu_n\} \) converges to the fixed point.

**Proof.** It is easy to see that \( \{fu_n\} \) is a Cauchy sequence and in view of (ii), it converges to a point in \( g(X) \cap h(X) \). Call it \( p \). Then there exists a point \( z \) in \( X \) such that \( gz = p \). Since \( \{fu_n\} \) is a Cauchy sequence, there exists an integer \( N = N(\epsilon, \lambda) \) such that
\[
(5) \quad F_{fu_{2n+1},fu_{2n+2}}(\epsilon/k) > 1 - \lambda \quad \text{and} \quad F_{u,fu_{2n+2}}(\epsilon/k) > 1 - \lambda
\]
for all \( n \geq N \).

Now we prove that \( fz = gz \).

Taking \( u = z \) and \( v = u_{2n+2} \) in (3), we have
\[
F_{fz,fu_{2n+2}}(\epsilon) \geq \max \{F_{gz,fu_{2n+1}}(\epsilon/k), F_{fz,gz}(\epsilon/k), F_{fu_{2n+2},fu_{2n+1}}(\epsilon/k), F_{fz,fu_{2n+1}}(\epsilon/k), F_{fu_{2n+2},gz}(\epsilon/k), F_{fz,fu_{2n+2}}(\epsilon/k), F_{fu_{2n+2},fu_{2n+1}}(\epsilon/k), F_{fz,fu_{2n+2}}(\epsilon/k), F_{fu_{2n+2},u}(\epsilon/k), F_{fu_{2n+2},fu_{2n+1}}(\epsilon/k), F_{fz,fu_{2n+2}}(\epsilon/k), F_{fu_{2n+2},u}(\epsilon/k)\} > 1 - \lambda, \quad \text{by (5)}.
\]

So \( fz = gz \) since \( fu_{2n+2} \to p \). Similarly for a \( w \) in \( X \) such that \( hw = p \), we can show that \( hw = fw = p \). Now by (iii), \( fp = fgz = gfz = gu \) and \( fu = fhw = hu \) since \( gz = u \) and \( hw = u \). Also, \( fu = fgz = fp \). Therefore \( fu = gu = hu \). Putting \( u = fu_{2n+1}, v = u \) in (3) and passing
to the limits we get \( fu = u \). So \( fu = gu = hu = u \). Uniqueness follows easily.

**Corollary 1.** Let \( X \) and \( f, g, h \) be as in Theorem 1. If there exists a point \( u_0 \) in \( X \) and a sequence \( \{ u_n : n = 0, 1, \cdots \} \) in \( X \) such that

(i) \( hu_{2n+1} = fu_{2n}, hu_{2n+2} = gu_{2n+1}, n = 0, 1, 2, \cdots; \)
(ii) \( h(X) \) is \((f, g; huo)-orbitally complete; \)
(iii) \( h \) commutes \((f, g; huo)-preorbitally either with \( f \) or \( g \); and \)
(iv) \( h \) is \((f, g; huo)-orbitally continuous. Then the conclusions of Theorem 1 hold.

**Extension to Uniform Spaces**

Let \( D = \{ d_\alpha \} \) be a nonempty collection of pseudometrices on \( X \). It is well known that the uniformity generated by \( D \) is obtained by taking as a subbase of all sets of the form \( U_{\alpha, \epsilon} = \{(u, v) \in X \times X : d_\alpha(u, v) < \epsilon \} \), where \( d_\alpha \in D \) and \( \epsilon > 0 \). In fact, the topology determine by the uniformity has all \( d_\alpha \)-spheres as a subbase. For details one may refer to Kelley [9].

Cain and Kasriel [2] have shown that a collection of pseudo-metrics \( \{ d_\alpha \} \) can be defined which generates the usual structure for Menger spaces. Hence the following result is a direct consequence of Theorem 1.

**Theorem 3.** Suppose \( X \) is a Hausdorff space and \( f, g, h; X \to X \) having the property that for every \( d_\alpha \in D \) there is a constant \( k_\alpha \in (0, 1) \) such that

\[
(6) \quad d_\alpha(fu, gv) \leq k_\alpha \{ \max d_\alpha(hu, hv), d_\alpha(fu, hu), d_\alpha(gv, hv), \\
\quad \quad \quad \quad \quad \quad d_\alpha(fu, hv), d_\alpha(gv, hu) \};
\]

\( h(X) \) is sequentially complete; and \( h \) is commuting with \( f \) and \( g \), then \( f, g \) and \( h \) have a unique common fixed point. In an analogous blend Theorem 2 may also be extended to uniform spaces.

Theorem 3 includes a number of fixed point theorems in metric, Menger and uniform spaces, which may be obtained by choosing \( f, g, h \) suitably. In particular, if \( f = g \) then it presents a nice generalization of Jungck's result [8]. Theorem 4 also includes an interesting result of Khan and Fisher [10] and contains as a special case Theorem 1.1 of Tarafdar [16]. It may be noted that Tarafdar [16] has obtained an exact analogue of Banach contraction mapping principle on a complete Hausdorff space.

**Application to Product Spaces**
We now give an application of Theorem 2 to product spaces.

**Theorem 4.** Let $X$ be a Hausdorff space and $f, g, h : X \times X \to X$. If for every $d_\alpha \in D$, there exists a constant $k_\alpha \in (0, 1)$ such that

$$d_\alpha(f(u,v), g(u',v')) \leq k_\alpha \max\{d_\alpha(h(u,v), h(u', v')), d_\alpha(f(u,v), h(u,v)), d_\alpha(g(u',v'), h(u', v'))\},$$

for all $u,v,u',v' \in X$; $h(X \times \{v\})$ is sequentially complete, and

$$h(f(u,v), v) = f(h(u,v), v)$$
$$h(g(u,v), v) = g(h(u,v), v)$$

for all $u,v \in X$. Then there exists exactly one point $p \in X$ such that

$$f(p,v) = g(p,v) = h(p,v) = p$$

for all $u,v \in X$.

*Proof.* For a fixed $v \in X$ and $v \neq v'$, the inequality (7) corresponds to (6). Therefore in view of the conclusions of Theorem 2 for each $v \in X$ there exists a unique $u(v)$ in $X$ such that

$$f(u(v),v) = g(u(v), v) = h(u(v), v) = u(v).$$

Now for every $v,v' \in X$ and $d_\alpha \in D$ from (7) we obtain

$$d_\alpha(u(v), u(v')) = d_\alpha(f(u(v),v), g(u(v'), v')) \leq k_\alpha d_\alpha(u(v), u(v')).$$

Consequently $u(v) = u(v')$. So $u(\cdot)$ is some constant $p \in X$ and hence the proof.

**References**


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